An approach to the line-free sets problem
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Denote $N = 3^n$, and let $D(n)$ denote the maximal cardinality of a line-free subset of $\mathbb{F}_3^n$, $d(n) = D(n)/N$.

The basic approach: Suppose $A \subset \mathbb{F}_3^n$ is line-free, $|A| = D(n)$. We have the following:

Claim 1 $|\widehat{1_A}(x)| \leq N(d(n-1) - d(n))$ *for all* $0 \neq x \in \mathbb{F}_3^n$.

□

The assumption on A implies

$$|A| = 1_A * 1_A * 1_A(0) = N^{-1} \sum_x |\widehat{1_A}(x)|^3 \tag{1}$$

hence

$$N|A| \geq |A|^3 - \sum_{x \neq 0} |\widehat{1_A}(x)|^3 \quad . \tag{2}$$

If $|A|^2 \leq 2N$ then $d(n) = O(3^{-\frac{n}{2}})$.
Otherwise we get by (2)

$$N^3 d(n)^3 = |A|^3 \ll \sum_{x \neq 0} |\widehat{1_A}(x)|^3 \ . \tag{3}$$

The standard way to proceed is to apply the Parseval identity to get

$$\sum_{x \neq 0} |\widehat{1_A}(x)|^3 \leq \max_{x \neq 0} |\widehat{1_A}(x)| \sum_x |\widehat{1_A}(x)|^2 \leq$$

$$N(d(n-1) - d(n))N|A| = N^3 d(n)((d(n-1) - d(n)) \tag{4}$$

hence $d(n)^2 \ll d(n-1) - d(n)$ and $d(n) = O(1/n)$.
At the moment we cannot get a better bound out of Claim 1 and Eq. (3).
Here is our fruitless approach:

Let $\mathcal{L}_k$ denote the family of all $k-$dimensional flats in $\mathbb{F}_3^n$. For $1 \leq q$ and $1 \leq k \leq n$ let

$$\|f\|_{q,k} = \max_{L \in \mathcal{L}_k} \left(\frac{1}{3^k} \sum_{x \in L} |f(x)|^q\right)^{\frac{1}{q}} \quad .$$

We need a refined Parseval inequality for small sets: Suppose that for any $E \subset \mathbb{F}_3^n$ of size n^γ the following holds:

$$\sum_{x\in E} |\widehat{f}(x)|^2 \ll N^2 \|f\|_{q,n^\alpha}^2 \tag{5}$$

It turns out that if (5) holds for $\gamma \to \infty$ and $\frac{\alpha}{q} \to 1$ then $d(n) = O(n^{-\beta})$ for any β:

Let $f = 1_A$ and suppose $|\widehat{f}(x_1)| \geq \ldots \geq |\widehat{f}(x_N)|$. Let $E = \{x_i : 1 \leq i \leq n^\gamma\}$ and let $2 < r < 3$.
We estimate $\sum_x |\widehat{f}(x)|^r$ as follows:

$$\sum_{x\in E} |\widehat{f}(x)|^r \leq (\sum_{x\in E} |\widehat{f}(x)|^2)^{\frac{r}{2}} \leq$$

$$N^r \|f\|_{q,n^\alpha}^r \leq N^r d(n^\alpha)^{\frac{r}{q}} \tag{6}$$

To bound $\sum_{x\notin E} |\widehat{f}(x)|^r$ we need the following:

Claim 2 *If $a_1 \geq \ldots \geq a_N \geq 0$ and $\sum_i a_i = C$ then for $1 \leq m \leq N$ and $\lambda > 1$*

$$\sum_{i=m}^{N} a_i^\lambda \leq \frac{C^\lambda}{(1-\lambda)m^{\lambda-1}} \quad . \tag{7}$$

□

Taking $m = n^\gamma$,$\lambda = r/2$ and $C = \sum_x |\widehat{f}(x)|^2 = N^2 d(n)$ in Claim 2 we obtain

$$\sum_{x\notin E} |\widehat{f}(x)|^r \ll \frac{N^r d(n)^{\frac{r}{2}}}{n^{\gamma(\frac{r}{2}-1)}} \tag{8}$$

Combining Claim 1 with (3),(6) and (8) we get

$$d(n)^3 \ll (d(n-1) - d(n))^{3-r} (d(n^\alpha)^{\frac{r}{q}} + n^{-\gamma(\frac{r}{2}-1)} d(n)^{\frac{r}{2}}) \quad . \tag{9}$$

Setting $d(n) = n^{-\beta}$ we get

$$n^{-3\beta} \ll n^{-(\beta+1)(3-r)} \max\{n^{-\frac{\beta r \alpha}{q}} \ , \ n^{\gamma(1-\frac{r}{2})-\frac{r}{2}\beta}\} \quad . \tag{10}$$

Hence

$$\beta \geq \min \ \{ \ \frac{3-r}{r(1-\frac{\alpha}{q})} \ , \ \frac{6-2r+\gamma(r-2)}{r} \ \} \tag{11}$$

Remarks on estimates of type (5): Let G be a finite abelian group with character group $\Gamma = \widehat{G}$. For $E \subset \Gamma$ let $C_E = \{f : \text{Supp}\widehat{f} \subset E\}$.
E is a $\Lambda(p)$ -set with constant C if one of the two following equivalent conditions hold:
1. $\|f\|_p \leq C\|f\|_2$ for all $f \in C_E$.
2. $\sum_{\chi \in E} |\widehat{h}(\chi)|^2 \leq C^2|G|^2\|h\|_q^2$ for all h.

Let $G = \mathbb{F}_3^n$ and let $\Gamma_0 \subset \Gamma$ be a $d-$dimensional subspace. For a function h on G and $z \in G$ define a function h_z on $\Gamma_0^\perp$ by $h_z(x) = h(z + x)$.

Claim 3 *Suppose $B \subset \Gamma$ is such that $\overline{B} \subset \Gamma/\Gamma_0 = \widehat{\Gamma_0^\perp}$ is a $\Lambda(p)$ set with a constant C. Then for $E = B + \Gamma_0$ and any function h on G:*

$$\sum_{\chi \in E} |\widehat{h}(\chi)|^2 \leq C^2 N^2 \|h\|_{q,n-d}^2 \quad . \tag{12}$$

Proof:

$$\sum_{\chi \in E} |\widehat{h}(\chi)|^2 = \sum_{\beta \in B} \sum_{\gamma \in \Gamma_0} \sum_{x \in G} h(x)\beta(-x)\gamma(-x) \sum_{y \in G} \overline{h(y)}\beta(y)\gamma(y) =$$

$$\sum_{\beta \in B} \sum_{x \in G} h(x)\beta(-x) \sum_{y \in G} \overline{h(y)}\beta(y) \sum_{\gamma \in \Gamma_0} \gamma(y-x) =$$

$$|\Gamma_0| \sum_{\beta \in B} \sum_{x \in G} h(x)\beta(-x) \sum_{y \in x+\Gamma_0^\perp} \overline{h(y)}\beta(y) =$$

$$|\Gamma_0| \sum_{\beta \in B} \sum_{z \in G \setminus \Gamma_0^\perp} \sum_{w \in \Gamma_0^\perp} h(z+w)\beta(-z-w) \sum_{y \in z+\Gamma_0^\perp} \overline{h(y)}\beta(y) =$$

$$|\Gamma_0| \sum_{\beta \in B} \sum_{z \in G \setminus \Gamma_0^\perp} \sum_{w \in \Gamma_0^\perp} h_z(w)\beta(-w) \sum_{w \in \Gamma_0^\perp} \overline{h_z(w)}\beta(w) =$$

$$|\Gamma_0| \sum_{\beta \in B} \sum_{z \in G \setminus \Gamma_0^\perp} |\widehat{h_z}(\beta)|^2 = |\Gamma_0| \sum_{z \in G \setminus \Gamma_0^\perp} \sum_{\beta \in B} |\widehat{h_z}(\beta)|^2 \leq$$

$$|\Gamma_0| \sum_{z \in G \setminus \Gamma_0^\perp} C^2|\Gamma_0^\perp|^2\|h_z\|_q^2 \leq C^2 N^2 \|h\|_{q,n-d}^2 \quad .$$

A Dual Formulation

Define a norm on functions on $\mathbb{F}_2^n$ by

$$\|f\|_{\overline{p,d}} = \min\{\frac{1}{2^{n-d}} \sum_{L\in\mathcal{L}_d} \lambda_L \ : \ \sum_{L\in\mathcal{L}_d} \lambda_L h_L(x) \geq |f(x)| \ , \ \|h_L\|_p \leq 1\} \ .$$

The norms $\| \ \|_{q,d}$ and $\| \ \|_{\overline{p,d}}$ are dual:

$$|\sum_{x\in\mathbb{F}_2^n} f(x)g(x)| \leq N \ \|f\|_{\overline{p,d}} \ \|g\|_{q,d} \ .$$

We need an estimate of the following form: If $f \in C_E$ where $|E| = n^\gamma$ then

$$\|f\|_{\overline{p,n^\alpha}} \leq C\|f\|_2 \tag{13}$$

where $p \to \infty$ and $\alpha \to 1$.