Algebraic Shifting

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Abstract

Algebraic shifting is a correspondence which associates to a simplicial complex K another simplicial complex $\Delta(K)$ of a special type. In fact, there are two main variants based on symmetric algebra and exterior algebra, respectively. The construction is algebraic and is closely related to "Gröbner bases" and specifically to "generic initial ideals" in commutative algebra.

Algebraic shifting preserves various combinatorial and topological properties of K while others disappear. For example, $\Delta(K)$ has the same Betti numbers as K while the ring structure on cohomology is destroyed as $\Delta(K)$ is always a wedge of spheres. One of the important challenges is to deepen the relation between algebraic shifting and the basic notions and constructions of algebraic topology. Some important progress in this direction was achieved by Duval.

Algebraic shifting also preserves the property that K is Cohen-Macaulay. At the forefront of our knowledge in this direction is a far-reaching extension of this fact achieved by Bayer, Charalambous and Popescu (symmetric shifting) and Aramova and Herzog (exterior shifting). In a different context extensions to Buchsbaum complexes have been made by Schenzel and by Novik (available only for symmetric shifting). These results apply to triangulations of manifolds and have interesting combinatorial consequences. Among the challenges which remain are: To understand algebraic shifting of simplicial spheres and simplicial manifolds, to find relations between shifting and embeddability and to identify intersection homology groups via algebraic shifting.

We will also describe the relation of algebraic shifting to framework rigidity, the connection with the original notion of "combinatorial shifting" which goes back to Erdös, Ko and Rado and some possible applications to extremal combinatorics.

Contents

1 Introduction and Background

1.1 Introduction

Algebraic shifting is a correspondence which associates to a simplicial complex K another simplicial complex $\Delta(K)$ of a special type. It was introduced in Kalai [55, 58] (see also Björner and Kalai [16, 17]). There are two main variants of algebraic shifting. The original one was based on exterior algebra while a variation based on symmetric algebra was considered in [61]. The constructions are algebraic and closely related to "Gröbner bases" and specifically to "generic initial ideals" in commutative algebra [38]. $\Delta(K)$ belongs to a special class of simplicial complexes called "shifted complexes" (closely related to "Borel-fixed ideals").

We associate to a simplicial complex K the exterior face algebra $\bigwedge(K)$, and with it the exterior shifting of K denoted by $\Delta^{ext}(K)$. The symmetric shifting of K denoted by $\Delta^{symm}(K)$ is based on the Stanley-Reisner ring $R(K)$. When the type of shifting is clear from the context or when we are discussing properties that apply to both versions, we omit the superscripts ext and $symm$.

Algebraic shifting preserves various combinatorial and topological properties of K while others disappear. Thus, for example $\Delta(K)$ has the same Betti numbers as those of K while the ring cohomology of $\Delta(K)$ is always trivial as $\Delta(K)$ is always a wedge of spheres. If K has the Cohen-Macaulay property then so does $\Delta(K)$ and if every two r-faces of K have nonempty intersection then the same is true for $\Delta^{ext}(K)$.

The main application of algebraic shifting is the study of face numbers of various classes of simplicial complexes. However, in this paper I primarily discuss algebraic shifting for its own sake. I will also present various open problems.

The basic problem is as follows:

Problem 1. Find interesting relations between topological and combinatorial properties of the complex K, commutative-algebraic properties of the algebras $R(K)$ and $\bigwedge(K)$ and combinatorial properties of the shifted complexes $\Delta^{symm}(K)$ and $\Delta^{ext}(K)$. Extract combinatorial consequences.

In the first part of the paper (Sections 2-3) we will discuss some basic properties of algebraic shifting and will concentrate on its relation to simplicial homology. We will also briefly mention the connection between framework rigidity and (symmetric) shifting of graphs. The second part (Sections

4-5) describes connections to finer homological properties of simplicial complexes and their links. The Cohen-Macaulay property will play a central role. The third part (Sections 6-7) is devoted to combinatorial properties and applications and to some extensions and variations.

1.2 Comments on the early literature

In Eisenbud's book [38] the reader will find a historical description and references concerning initial ideals, Gröbner basis and generic initial ideals and, in particular, references to works of Hartshorne (1966), Grauert (1972) and Galligo (1994). Let me mention in particular the seminal works by Bayer and Stillman from 1987 (see, for example, [11]). Green [48] is a recent influential paper concerning generic initial ideals.

Face rings and the application of commutative algebra to combinatorics were pioneered by Stanley [71] in 1975. For a discussion of connections between commutative algebra and combinatorics see Stanley [76], Hibi [52] and Burns and Herzog [24]. I should also mention the important early papers by Hochester [53], in 1972, in which combinatorics was applied in commutative algebra, and by Reisner [70].

Applications of exterior and polynomial algebras in extremal combinatorics which were introduced by Lovàsz [65] in 1977, are also closely related to the mathematics of this paper.

Of course, algebraic shifting is related to the classical notion of "combinatorial shifting" due to Erdös, Ko and Rado [41] and later in full generality by Kleitman (see the survey article by Frankl [44]). The combinatorics of Kruskal-Katona and Macaulay's theorems are also relevant (see [40]).

1.3 Simplicial complexes

An (abstract) *simplicial complex* K is a collection of finite sets with the property that $S \in K$ and $R \subset S$ implies that $R \in K$.

Let K be a finite simplicial complex. A set $S \in K$ where $|S| = k+1$ is called a k-face of K. 0-faces of K are called *vertices*. Denote by $f_k(K)$ the number of k-faces in K. The vector $f(K) = (f_{-1}(K), f_0(K), f_1(K), \dots)$ is called the *f-vector* of K.

Simplicial complexes are basic combinatorial objects and also arise in geometry and topology. A *geometric realization* $\overline{K}$ of a simplicial complex K

is a collection of Euclidean simplices such that for every r-face of K there is an associated r-dimensional simplex $\overline{S}$, so that for every $S, T \in K$

$$\overline{S} \cap \overline{T} = \overline{S \cap T}.$$

Given a geometric realization $\overline{K}$ of K define $|K| = \cup\{\overline{S} : S \in K\}$. As a topological space $|K|$ does not depend (up to a homeomorphism) on the specific geometric realization $\overline{K}$. A simplicial complex K is called a *triangulation* of a topological space X if $|K|$ is homeomorphic to X. Various topological invariants of X were defined and studied via triangulations of X.

Let K be a simplicial complex and S be a face of K. The link of S in K, denoted by $lk(S,K)$ is defined by

$$lk(S,K) = \{T \backslash S : T \in K, \quad S \subset T\}.$$

The link of the empty set is K itself. Links of non-empty faces of K will be called proper links.

For an excellent description of simplicial complexes and simplicial homology the reader is referred to Munkers' book [68].

1.4 Shifted complexes

A collection A of k-sets of positive integers (or any ordered set) is *shifted* if whenever $S \in A$ and R is obtained from S by replacing an element with a smaller element, then R belongs to A. For example: If $\{2, 5, 11\} \in A$ then $\{1, 5, 9\}$ must also be in A. We will write $S <_p T$ if $S \neq T$ and S can be obtained from T by successively replacing elements with smaller elements. In other words, if $S = \{s_1, s_2, \ldots, s_k\}_<$ (the subscript $_<$ indicates that $s_1 < s_2 < \cdots < s_k$) and $T = \{t_1, t_t, \ldots, t_k\}_<$ then $S \leq_p T$ if $s_i \leq t_i$ for every i, $1 \leq i \leq k$.

A simplicial complex K whose vertices are positive integers is *shifted* if the set of r-faces of K is shifted for every r. Algebraic shifting assigns a shifted simplicial complex $\Delta(K)$ to every simplicial complex K. $\Delta(K)$ has the same f-vector as K.

2 The definition of algebraic shifting, basic properties and basic problems

2.1 Exterior shifting

From herein $\mathbf{k}$ will be a fixed field of coefficients. The lexicographic ordering on finite k-subsets of $\mathbf{N}$ is defined by $S <_L T$ if and only if $min(S\Delta T) \in S$. ($S\Delta T$ is the symmetric difference between S and T.) In other words, if $S = \{s_1, s_2, \dots, s_k\}_<$ and $T = \{t_1, t_t, \dots, t_k\}_<$ then $S <_L T$ if for some j, $1 \leq j \leq k$, we have: $s_i = t_i$ for $i < j$ and $s_j < t_j$. Thus,

$$\{1,2\} <_L \{1,3\} <_L \{1,4\} <_L \cdots <_L \{2,3\} <_L \{2,4\} <_L \cdots <_L \{3,4\} <_L \cdots$$

Let K be a collection of k-subsets of $[n] = \{1, 2, ..., n\}$. In this section we will define the algebraic shifting operation $K \to \Delta(K)$ and discuss some of its basic properties. For a general set system K denote by K_k the collection of k-sets in K, and define $\Delta(K) = \cup\Delta(K_i)$.

Let $X = (x_{ij})_{1\leq i\leq n, 1\leq j\leq n}$ be an n by n matrix. Let $X^{\wedge k}$ be the k-th compound matrix of X, i.e, the $\binom{n}{k}$ by $\binom{n}{k}$ matrix of k by k minors of X. Assume that the rows and columns of $X^{\wedge k}$ are ordered lexicographically.

Given a collection K of k-subsets of $[n]$ with $|K| = m$, let $M(K)$ be the m by $\binom{n}{k}$ submatrix of $X^{\wedge k}$ whose rows correspond to the k-sets in K. Now choose a basis of columns for the column-space of $M(K)$ in the greedy way: by simply taking those columns which are not spanned by previous columns in the lexicographic ordering. Define $\Delta_X(K)$ as the family of sets which are the indices of the chosen columns. $\Delta(K) = \Delta_X(K)$ for a generic matrix X, i.e., when $(x_{ij})_{1\leq i,j\leq n}$ are n^2 variables.

Remark: We can replace the lexicographic order $<_L$ in this definition with any "term order", namely a linear extension of the partial order $<_p$. We will rarely consider other term orders.

We now describe two other ways to define algebraic shifting. Let E be an n-dimensional vector space over $\mathbf{k}$ with a standard basis $\mathbf{e}= (e_1, e_2, ..., e_n)$. Let $\bigwedge^k E$ be the k-th exterior product over E and let $\bigwedge E$ be the entire exterior algebra over E. Let $\mathbf{f} = (f_1, f_2, \cdots f_n)$ be a basis of E, given by $f_i = \sum x_{ij}e_j$. Let $(f_S : S \subset [n])$ be the corresponding basis of $\bigwedge E$.

An equivalent way to define shifting is as follows: For a subspace I of $\bigwedge^k(E)$ consider the quotient space $A = \bigwedge^k(E)/I$. For $m \in \bigwedge^k(E)$ let $\widetilde{m}$ be its image in A. Define

$$\Delta_X(I) = \{S : \widetilde{f}_S \notin span\{\widetilde{f}_R : R <_L S\}\}. \tag{2.1}$$

Starting with a family K consider $I = span\{e_S : S \notin K\}$ and then $\Delta_X(I)$ coincides with $\Delta_X(K)$ as defined above.

For a simplicial complex K denote

$$I(K) = span\{e_S : S \notin K\}$$

and

$$\bigwedge(K) = \bigwedge E/I(K) \tag{2.2}$$

as above. $\bigwedge(K)$ is called the *exterior algebra* of K. It is a graded quotients algebra of $\bigwedge E$ which is the exterior analog of the Stanley-Reisner ring.

In order to obtain $\Delta(K)$ one must choose X to be a generic matrix. For this replace $\mathbf{k}$ with the field of rational functions with n^2 variables $\mathbf{k}(x_{ij}; 1 \leq i, j \leq n)$ and let $f_i = \sum x_{ij}e_j$.

Finally, let M be a subspace of $\bigwedge^k(K)$. For each $m \in M$ express $m = \sum_S \alpha_S f_S$, let $i(m) = min\{S : \alpha_S \neq 0\}$ and define $\Delta_X(M) = \{i(m) : m \in M\}$. For a collection K of k-subsets of n let $M(K) = span\{e_S : S \in K\}$ and then $\Delta_X(K)$ as defined in the previous paragraphs coincides with $\Delta_X(M(K))$. (Recall that a Gröbner basis for M is a set of elements m_j in M such that $i(m_j)$ gives every set in $\Delta(M)$ precisely once.)

The equivalence of the definitions can easily be shown. Note that $f_S = \sum X_{ST}e_T$, where X_{ST} is the k by k minor of X with rows and columns corresponding to the elements of S and T respectively. Thus, the second definition is a direct translation of the first to the language of exterior algebra. To see the equivalence of the first and third definitions note that a column i in a matrix M is linearly dependent on the previous columns if and only if there is no linear combination of the rows in M whose first nonzero element is in the i-th place.

2.2 Symmetric shifting

Let K be a simplicial complex and let $R(K)$ be its Stanley-Reisner ring (face ring).

$$R(K) = R[x_1, x_2, \ldots, x_n]/I, \tag{2.3}$$

where I is the ideal spanned by monomials $x_{i_1} \cdot x_{i_2} \cdots x_{i_r}$ where $\{i_1, i_2, \ldots, i_r\} \notin K$. Here R will be the field of coefficients $\mathbf{k}$.

Consider now $y_1, y_2 \ldots, y_n$ which are n generic linear combinations of $x_1, x_2 \ldots x_n$. All monomials in the y_i's span the ring $R(K)$ and we will now construct the basis $GIN(K)$ of monomials in the new variables in a greedy way w.r.t. the lexicographic order. Thus a monomial m belongs to $GIN(K)$ if and only if its image $\widetilde{m}$ in $R(K)$ is not a linear combination of (images of) monomials which are lexicographically smaller.

Recall that The lexicographic order $m_1 <_L m_2$ is defined as follows: If the variable with the smallest index which appears with a different exponent in the two monomials appears with a larger exponent in m_1. Thus,

$$y_1^2 <_L y_1y_2 <_L y_1y_3 <_L \cdots <_L y_1y_n <_L y_2^2 <_L \ldots.$$

$GIN(K)$ is the direct analog to the shifting operation described for the exterior algebra. However the combinatorial information in $GIN(K)$ is redundant as a result of the following property: If m is a monomial in $GIN(K)$ of degree $i \leq d$ then the monomials $y_1m, y_2m, \ldots y_im$ belong to $GIN(K)$. The symmetric shifting of K denoted by $\Delta^{symm}(K)$ is a simplicial complex obtained from $GIN(K)$ as follows:

For every monomial m of degree r in $GIN(K)$ which does not involve the variables $y_1, \ldots y_{r-1}$, write $m = y_{i_1} \cdot y_{i_2} \cdots y_{i_r}$, where $i_1 \leq i_2 \leq \cdots \leq i_r$, and associate the set $S(m) = \{i_1 - r + 1, i_2 - r + 2 \ldots, i_r\}$ to the monomial m.

$$\Delta^{symm}(K) = \cup\{S(m) : m \in GIN(K)\}. \tag{2.4}$$

Note that $GIN(K)$ and $\Delta^{symm}(K)$ carry precisely the same combinatorial information. $GIN(K)$ is essentially equivalent to the notion of generic initial ideal (except that it is a collection of monomials rather than an ideal). In some cases, (e.g. when K is Cohen-Macaulay or a manifold) it is easier to explain the combinatorial properties of $\Delta^{symm}(K)$ in terms of $GIN(K)$. However, the translation is straightforward.

2.3 Exterior shifting and symmetric shifting

Problem 2. 1. What is the relation between the exterior face algebra and the Stanley-Reisner ring?

2. What is the relation between exterior shifting and symmetric shifting?

Most of the general theorems we can prove for one of these operations are either true or conjectured to be true for the other operation. (Even in cases where we can prove the results for both types of shifting the proofs may be very different.) The recent paper by Eisenbud, Popescu and Yuzvinsky [39] is relevant to these questions.

A case in which the outcomes of the two operations differ is $K_{3,3}$ - the complete bipartite graph with two color classes of size three. The symmetric shifting $\Delta^{symm}(K_{3,3})$ contains all edges lexicographically smaller than $\{2,6\}$ while $\Delta^{ext}(K_{3,3})$ contains all edges lexicographically smaller than $\{2,5\}$ and, in addition, the edge $\{3,4\}$.

There is a further explanation of this example: The presence of $\{2,3\}$ in $\Delta(G)$ is equivalent to the property that the graph contains a cycle or, in algebraic terms, to the existence of a linear combination m of the edges whose boundary vanishes. The boundary operation in question is differs for exterior and symmetric shifting. In the case of exterior shifting, the boundary of the edge $\{i,j\}$ (that corresponds to $e_i \wedge e_j$) is $\alpha_j e_i - \alpha_i e_j$. In the case of symmetric shifting, the boundary of the edge $\{i,j\}$ is $(\alpha_i - \alpha_j)e_i + (\alpha_j - \alpha_i)e_j$. In both cases $(\alpha_1, \alpha_2, \ldots \alpha_n)$ is a generic vector of coefficients.

In the case of a single boundary operation these two boundaries are equivalent to the usual boundary.

The presence of $\{3,4\}$ is equivalent to the existence of a linear combination of edges which simultaneously vanishes for two independent generic boundary operations. This differs in the exterior and symmetric case. (In the symmetric case this is equivalent to the existence of a non-zero stress for a generic embedding of the graph in $\mathbb{R}^3$.)

Let K_1 and K_2 be simplicial complexes on the vertex set $[n]$. We say that K_2 is lexicographically smaller than or equal to K_1, denoted by $K_2 \leq_L K_1$, if for every $r > 0$ the lexicographically first r-face in the symmetric difference between K_1 and K_2 belongs to K_2. Note that for $K = K_{3,3}$, $\Delta^{symm}(K)$ is lexicographically smaller than $\Delta^{ext}(K)$ as the first edge (w.r.t. the lexicographic order) in their symmetric difference is $\{2,6\} \in \Delta^{symm}(K)$.

Problem 3. Is it always the case that $\Delta^{symm}(K) \leq_L \Delta^{ext}(K)$?

We will now mention three properties of exterior shifting which may also apply for symmetric shifting.

Problem 4. Is $\Delta^{symm}(K; \mathbf{k})$ shifted when the characteristic p of the field $\mathbf{k}$ of coefficients is not zero?

For exterior shifting, if K itself is shifted then $\Delta^{ext}_{<}(K)$ is shifted with respect to every term order.

Problem 5. Let K be a shifted complex. Is $\Delta^{symm}_{<}(K)$ shifted with respect to every term order?

We will show (in Section 6) that if $A \subset \binom{[n]}{k}$ is *intersecting*, namely every two sets in A have nonempty intersection, then $\Delta^{ext}(A)$ is intersecting as well.

Problem 6. Let K be an intersecting family of k-sets. Is $\Delta^{symm}(K)$ also intersecting?

Goresky and MacPherson proposed (in a private communication) that Koszul duality may shed light on the relation between exterior and symmetric face rings and the associated shifting operations.

2.4 An example

Consider the boundary complex K of an octahedron. If we number the vertices of the octahedron by $\{1, 2, \cdots, 6\}$ the 2-faces are given by 123, 126, 135, 234, 156, 246, 345 and 456. (Here, 123 stands for $\{1, 2, 3\}$.)

K is pure and therefore its 1-skeleton consists of all the edges included in these triangles. These are all the possible $\binom{6}{2}$ edges except for 14, 25 and 36.

What is $\Delta(K)$? We will reveal the identity of $\Delta(K)$ step-by-step and we will use several properties of algebraic shifting (written with a special `font`) which we are going to discuss in various places of this paper.

- The algebraic shifting of K also consists of 8 triangles. The first two triangles in the partial order are 123 and 124 which must be included. Next we have 125 and 134 which are not comparable. 125 is smaller (w.r.t. $<_p$) than all the other triangles but four 123, 124, 134 and 234. Since we must have 8 triangles altogether 125 must be included. (`Note that so far we have only used the facts that` $\Delta(K)$ `is shifted and has the same number of triangles as` K.)
- The triangle 134 is smaller than all the other triangles except those of the form 12x. The first six of these include 127 which cannot be in $\Delta(K)$ since $\Delta(K)$ is a simplicial complex. So $134 \in \Delta(K)$ as well. A similar argument applies to 135. If $135 \notin \Delta(K)$ then $\Delta(K)$ must be included in the simplicial complex spanned by the triangles of the form

12x together with 134 and 234. Again there are not enough triangles of this type. `(Here we used that` $\Delta(K)$ `is a simplicial complex.)`

- We will see below (Section 3) that $\Delta(K)$ `has the same Betti numbers as` K. and that $\beta_2(\Delta(K))$ `is the number of triangles in` $\Delta(K)$ `which do not contain '1'.` Since $\beta_2(K) = 1$ and 234 is the first triangle not containing 1 it must be included in $\Delta(K)$, and all other triangles in $\Delta(K)$ must contain 1. From this we can conclude that the shifted complex also contains 126. We would not have enough triangles if this were not the case. (Alternatively, we can use the fact that `since` K `is Cohen Macaulay,` $\Delta(K)$ `is pure` (see Section 4) and therefore includes a triangle containing '6' since 126 is the first such triangle it must be included in the shifted complex.)

- It is left to decide whether 136 or 145 is included in $\Delta(K)$.

  We will see below that `since` K `is Cohen Macaulay` $\Delta(K)$ `is as well and that this means that` $\Delta(K)$ `is a pure simplicial complex.`

  It is therefore left to decide which of the edges 45 or 36 belongs to $\Delta(K)$. The answer is 36. It follows that the triangle 136 is included in $\Delta(K)$. What we need is the following fact: `If` G `is a planar graph then` $\Delta(G)$ `does not contain 45 (equivalently,` $\Delta(G)$ `does not contain a complete graph on 5 vertices.)`

  We do not have good conceptual explanation for this last property neither can we prove higher dimension analogs (see Section 5.2). It is equivalent to the fact that when you shift a maximal planar graph G, 36 is included in the shifted graph. For symmetric shifting it is equivalent to the fact that maximal planar graphs are generically rigid when embedded in space (see Section 2.7). (This follows, from the Cauchy-Dehn-Alexandrov rigidity theorem for polyhedra.)

To sum, $\Delta(K)$ (for both exterior and symmetric shifting) consist of the pure simplicial complex whose maximal faces are: 123, 124, 125, 126, 134, 135, 136 and 234.

2.5 Basic properties of algebraic shifting

We will now list some basic properties of the operation $K \to \Delta(K)$.

Theorem 2.1. *Let A be a family of r-subsets of $[n]$.*

1. $|A| = |\Delta(A)|$.

2. $\Delta(A)$ *is shifted (see [16, 60]).*

3. If A' is combinatorially isomorphic to A then $\Delta(A) = \Delta(A')$ (see [16, 60]).

4. If A is a shifted family then $\Delta(A) = A$ [60].

5. For every nonsingular matrix X, $\Delta_X(\Delta(A)) = \Delta(A)$ (this follows from 4). It is possible that $\Delta(\Delta_X(A)) \neq \Delta(A)$.

6. $\Delta(A)$ *depends only on the characteristics of* $\mathbf{k}$.

The following Theorem relates shifting to several operations on families of sets.

For a family A of k-sets the *shadow* of A, $\partial(A)$, is a family of $(k-1)$-sets defined by

$$\partial(A) = \{R : |R| = k-1, R \subset S \in A\}.$$

Define a *cone* over A as a family of $(k+1)$-sets of the form $\{S \cup \{w\} : S \in A\}$, where $w \notin S$ for any $S \in A$.

Theorem 2.2. *Let A be a family of k-subsets of $[n]$.*

6. $\partial(\Delta(A)) \subset \Delta(\partial(A))$ *[55].*

7. If $L \subset A$ then $\Delta(L) \subset \Delta(A)$.

8. $\Delta(Cone(A)) = Cone(\Delta(A))$.

Properties 1,4,6,7 and 8 hold for $\Delta_X(K)$ for every non-singular matrix X. Properties 2 and 3 rely on the genericity of X.

For certain applications it is enough to assume that X is in a *general position* which means that all minors of the form $X_{[r],R}$ are not singular for every r and R, with $|R| = r$.

Property 5 is intriguing, and it leads to the following question:

Problem 7. 1. What can be said about a family of shifted complexes which comprise the set of $\Delta_X(K)$ for some fixed simplicial complex K (when X varies)?

2. For which simplicial complexes K is it the case that if $\Delta_X(K)$ is shifted then $\Delta_X(K) = \Delta(K)$?

2.6 How to shift?

Problem 8. Is there a deterministic polynomial algorithm or at least a Las Vegas polynomial algorithm for determining $\Delta(K)$?

A randomized algorithm is called Monte Carlo when it depends on some internal randomization and produces the right answer with a probability larger then $1 - \epsilon$ but may give a wrong answer otherwise. (By repeating the algorithm the probability of failure can be reduced to whatever level is desired.) A Las Vegas algorithm is a superior type of randomized algorithm since it never produces a wrong answer. It produces the right answer with probability $> 1 - \epsilon$ but may fail to give any answer with probability $< \epsilon$.

There is a simple Monte Carlo algorithm for finding $\Delta(K)$: Choose the entries of the matrix M at random from a large field with the correct characteristics. If a non-shifted complex is obtained then this is not the correct answer. But it is possible (with low probability) that you will obtain an incorrect shifted complex.

2.7 Shifting of graphs and framework rigidity

Consider a graph $G =< V, E >$ with n vertices and e edges. An embedding of G into $\mathbb{R}^d$ is a map $\phi : V \to \mathbb{R}^d$. Assume that $V = [n]$ and put $x_i = \phi(i)$.

An embedding ϕ is rigid if any perturbation of the embedded vertices which preserves all distances between adjacent vertices is induced by a rigid motion of $\mathbb{R}^d$. The embedding is infinitesimally rigid if every assignment of velocity vectors $v_i \in \mathbb{R}^d$ to the vertices, which satisfies $< v_i - v_j, x_i - x_j >= 0$ whenever i, j are adjacent vertices, must satisfy the same relation for every two vertices. The graph G is generically d-rigid if it is rigid for almost all embeddings in $\mathbb{R}^d$ or equivalently if it is infinitesimally rigid for all embeddings into $\mathbb{R}^d$.

G is generically d-rigid if and only if $\{d, n\} \in \Delta^{symm}(G)$. No deterministic (or even Las Vegas) polynomial algorithm for determining if a graph is generically d-rigid is known for $d > 2$. $\{d, n\} \in \Delta^{ext}(G)$ is a related but different property of graphs called "hyperconnectivity", [56].

It is worth mentioning that shifted graphs are called threshold graphs and they have been studied extensively [66].

2.8 Finer and coarser invariants

Start with a simplicial complex K and its exterior face algebra $\bigwedge K$. (The following applies to the symmetric case as well.) For a generic n by n matrix we can consider the following invariants of K.

1. The ring $\bigwedge(K)$.

2. The symmetric matroid $M(K)$ determined by $\wedge(K)$.
3. The ranks of shifted families of sets in the matroid $M(K)$.
4. $\Delta(K)$.
5. The f-vector of K (or equivalently the Hilbert series of $\bigwedge(K)$).

We will elaborate on the new items 2 and 3.

The symmetric matroid $M(K)$ is a matroid defined on subsets of $[n]$ so that the rank of a collection of subsets $S_1, S_2, \dots S_u$ is the dimension of the vector space spanned by $\widetilde{f}_{S_1}, \widetilde{f}_{S_2}, \dots \widetilde{f}_{S_u}$ in $\bigwedge(K)$. This matroid is invariant under permutations of $[n]$. It seems to yield very fine yet quite intractable information on K.

If we restrict our attention to the ranks of shifted families of sets in this matroid we loose some information but are still able to determine the outcome of shifting with respect to any term order. This seems to be more tractable and yet to carry much information on K and its face algebra.

2.9 Shifting subspaces and decompositions of $G(\bigwedge^k V, m)$

The definition of algebraic shifting can be applied to an arbitrary subspace of $\bigwedge^k V$. Given an m-dimensional subspace M of $\bigwedge^k(V)$, consider the family of subsets $\Delta_X(M)$. There are two cases of particular interest: in one X is the identity matrix while in the other X is a generic matrix. For a vector space W let $G(V, m)$ denote the space of m-dimensional subspaces of W.

Define two decompositions $\mathcal{D}$ and $\mathcal{E}$ of $G(\bigwedge^k V, m)$ as follows: For a family $\mathcal{F}$ of $k-subsets$ of $[n]$ such that $|\mathcal{F}| = m$, let $U_{\mathcal{F}}$ be the set of m-dimensional subspaces M of $\bigwedge^k V$ that satisfy $\Delta_I(M) = \mathcal{F}$. Let $\mathcal{D}$ be the decomposition of $G(\bigwedge^k V, m)$ into the sets $U_{\mathcal{F}}$. The parts of $\mathcal{D}$ are indexed by k-uniform hypergraphs with m-edges. For $k = 1$, $\mathcal{D}$ is simply the standard Schubert decomposition of $G(V, m)$. More generally, $\mathcal{D}$ is the Schubert decomposition of $G(\bigwedge^k V, m)$ with respect to the standard basis $\{e_S : S \in \binom{[n]}{k}\}$ ordered by the lexicographic ordering on $\binom{[n]}{k}$.

Similarly, for a shifted family $\mathcal{F} \subset \binom{[n]}{k}, |\mathcal{F}| = m$, let $W_{\mathcal{F}}$ be the set of m-dimensional subspaces M of $\bigwedge^k V$ such that $\Delta_X(M) = \mathcal{F}$. $\mathcal{E}$ is the decomposition of $G(\bigwedge^k V, m)$ into the sets $W_{\mathcal{F}}$. In this case, the parts of $\mathcal{E}$ are indexed by shifted k-uniform hypergraphs with m edges.

Note that $GL(V)$ acts on each of the parts of the decomposition $\mathcal{E}$. For $k = 1$, $\mathcal{E}$ consists of only one part, i.e. the entire $G(V, m)$. (It corresponds to $[m]$ the only shifted family of singeltones of size m.) If a generic matrix X can be found with entries in the field k itself (e.g. for the fields of real or complex numbers), then $\mathcal{E}$ can be regarded as the decomposition of $G(\bigwedge^k V, m)$ given by the orbits of the cells in $\mathcal{D}$ under the action of $GL(V)$.

2.10 How many decomposable elements are there?

Suppose now that V is a vector space over a field with q elements and U is an m-dimensional subspace of $\bigwedge^k V$. Let $f(U)$ be the number of decomposable elements in U or, in other words, the number of k-dimensional subspaces W of V such that $f_W \in U$. (Here, f_W is the exterior product of vectors in a basis of W.)

Problem 9. 1. Show that $f(U)$ does not decrease under shifting.

2. Show that given m, $f(U)$ is a maximum when U is spanned by an initial set of m basis vectors with respect to the reverse lexicographic order.

3 Algebraic shifting and homology

3.1 Simplicial homology and cohomology

Let K be a simplicial complex and let $H_k(K)$ and $H^k(K)$ be respectively the k-th (reduced) homology group and the k-th (reduced) cohomology group of K with coefficients in the field $\mathbf{k}$. $H_k(K)$ and $H^k(K)$ are $\mathbf{k}$-vector spaces of the same dimension. This dimension is called the k-th Betti number of K and is denoted by $\beta_k(K)$. The cohomology of K has the following simple expression in terms of the exterior face algebra $\bigwedge(K)$. Let $f = e_1 + e_2 + \cdots + e_n$. Define

$$Z^k(K) = \{x \in \bigwedge^k(K) : f \wedge x = 0\}, \tag{3.1}$$

$$B^k(K) = f \wedge \bigwedge^{k-1}(K) \tag{3.2}$$

and

$$H^k(K) = Z^k(K)/B^k(K). \tag{3.3}$$

In other words, $H^k(K)$ is the k-th cohomology of the chain complex $C^*(K) = (\bigwedge(K), \delta)$, where the coboundary δ is given by $\delta(m) = f \wedge m$.

3.2 The homotopy type of shifted complexes

Shifted simplicial complexes are homotopically quite simple. They are always homotopically equivalent to a wedge of spheres (possibly of different dimensions). It follows that the homology has no torsion and that the cohomology ring is trivial. These properties hold for a larger class of simplicial complexes that we will now define.

A simplicial complex L is a *near cone* (with apex '1') if for every $S \in L$ and $i > 1, i \in S$, $(S \backslash \{i\} \cup \{1\}) \in L$. Near cones are homotopically equivalent to wedge of spheres. For a simplicial complex L on the vertex set $[n]$, define

$$b_{i-1}(L) =: |\{S \in L : |S| = i, S \cup \{1\} \notin L\}|. \tag{3.4}$$

Lemma 3.1 ([16]). *Let L be a near cone. Then:*

(1) $b_i(L) = \beta_i(L)$.

(2) L is homotopically equivalent to the wedge of spheres: $b_i(L)$ i-dimensional spheres, $i \geq 0$.

3.3 Shifting preserves the Betti number

A fundamental property of algebraic shifting [16] is:

Theorem 3.2.

$$\beta_k(K) = \beta_k(\Delta(K)), \quad \text{for every k} \geq 0. \tag{3.5}$$

The proof of Theorem 3.2 given in Björner and Kalai [16] consists of a combinatorial way to read $\beta_k(K)$ from $\Delta(K)$:

$$\beta_i(K) = b_i(\Delta(K)). \tag{3.6}$$

The assertion of Theorem 3.2 can also be proven for symmetric shifting.

The relation $\beta_i(K) = b_i(\Delta(K))$ holds whenever all entries of f_1 are nonzero. $\Delta(K)$ is a near cone if f_1 is generic.

3.4 Weighted coboundaries

We will mention just one component of the proof of Theorem 3.2. Let $f = a_1e_1 + \cdots a_ne_n$ be an arbitrary vector in E and consider the chain complex $C^*_f(K) = (\bigwedge(K), \delta_f)$ where the coboundary δ_f is given by $\delta_f(m) = f \wedge m$.

If all the a_i's are non-zero then the dimensions of the cohomology groups of $C^*(K)$ and $C^*_f(K)$ are the same. This can be seen by the chain map $D : C^*(K) \to C^*_f(K)$, defined by the relation $D(\widetilde{e}_S) = \prod\{a_i : i \in S\}\widetilde{e}_S$. The proof of theorem 3.2 uses weighted cohomology according to the first basis element f_1 of the generic basis used in the definition of algebraic shifting.

The fact that symmetric shifting preserves the Betti numbers of a complex K is related to the following way for expressing them: Let $R_i(K)$ the part of the Stanley-Reisner ring spanned by monomials of degree i. Consider $\widehat{R}_i(K) = R_i(K)/span < y_1, \ldots, y_{i-1} >$ and define $\delta : \widehat{R}_i(K) \to \widehat{R}_{i+1}(K)$ by $\delta(m) = y_{i+1}m$. Note that $\delta(\delta(m)) = 0$. This is an unusual coboundary operation which expresses the usual Betti numbers.

3.5 Some problems

3.5.1 Non-generic shifting

Let K be a triangulation of a topological space X. Suppose that K has n vertices and consider $\Delta_M(K)$ where M is an n by n matrix. When M is the identity matrix, then $\Delta_M(K) = K$; when M is generic, $\Delta_M(K)$ is a wedge of spheres with the same Betti numbers as K and in addition the links of all faces are a wedge of spheres. What happens in between these two extreme cases? What can be said about the homotopy type and topological properties of the "intermediate" simplicial complexes $\Delta_M(K)$?

Problem 10.

- (1) What can be said about the complexes $\Delta_M(K)$ where M varies over all n by n matrices.
- (2) What can be said about the topological spaces of the form $|\Delta_M(K)|$ where K varies over all triangulations of a space X and M varies over all matrices?
- (3) What can be said about the homotopy type of topological spaces of the form $\Delta_M(K)$ where K varies over all triangulations of a given homotopic type and M varies over all matrices?
- (4) Given a group G, what can be said about the groups $\pi(\Delta_M(K))$ where K varies over all simplicial complexes with $\pi(K) = G$ and M varies over all matrices?

Note that the four parts of this problem correspond to some (pre-)order relations defined on simplicial complexes, (triangulable) topological spaces, homotopy classes of topological spaces and finitely generated groups. For example, for two topological spaces X and Y we will say that $X \leq Y$ if for some triangulation of X and some matrix M, $\Delta_M(X)$ is homeomorphic to Y.

The smallest elements in these orders are respectively: shifted simplicial complexes, sequentially Cohen-Macaulay spaces (see Section 4.5), wedges of spheres and free groups. What can be said about these order relations?

3.5.2 Relative homology

Art Duval [31] proved that exterior algebraic shifting "increases relative homology", that is

$$\beta_i(\Delta(K), \Delta(L)) \geq \beta_i(K, L). \tag{3.7}$$

where K and L are simplicial complexes and β is the reduced relative Betti number. Recently, Tim Römer found a simple proof and various extensions for this result.

Problem 11 *(Björner).* For which K, L, and i is (3.7) an equality?

Duval [32] also studied algebraic shifting and spectral sequences.

3.5.3 Products

For two simplicial complexes K and L, let $K * L$ denote their join. Assume (possibly after renaming the vertices) that $V(K)$ and $V(L)$ are disjoint and define

$$K * L = \{S \cup R : S \in K \quad and \quad R \in L\}.$$

Problem 12. Show that $\Delta(K * L) = \Delta(\Delta(K) * \Delta(L))$.

It is true that $\Delta_X(K * L) = \Delta(K) * \Delta(L)$. For a matrix X which has zeros for entries (u, v) where $u \in V(K)$ and $v \in V(L)$ and generic entries otherwise. Hence $\Delta(\Delta_X(K * L)) = \Delta(\Delta(K) * \Delta(L))$.

3.5.4 Mayer-Vietoris

Problem 13. Given two complexes K and L find the possible relations between $\Delta(K), \Delta(L)$, $\Delta(K \cup L)$ and $\Delta(K \cap L)$.

If C and D are simplicial complexes on disjoint sets of vertices, then it seems true and possibly not difficult to demonstrate that $\Delta(C \cup D) = \Delta(C) \sqcup \Delta(D)$.

The (unusual) operation $K \sqcup L$ is defined inductively as follows: If K and L are 0-dimensional, then $K \sqcup L$ is their disjoint union and

$$K \sqcup L = 1 * (lk(1, K) \sqcup lk(1, L)) \cup ast(1, K) \sqcup ast(1, L).$$

Here, for a face S of a simplicial complex K, the anti-star $ast(S, K) =: \{R \in K : \ R \cap S = \emptyset\}$.

3.5.5 Further relations to algebraic topology

The Dolt-Thom construction of a topological space X is a new topological space $Y = DT(X)$ which satisfies $\pi_i(Y) = H_i(X)$ for $i = 1, 2, \dots$. If K is a simplicial complex describing X, $DT(X)$ can be described as the union $\cup_M \Delta_M(X)$ over all matrices.

Problem 14. Find connections between algebraic shifting and other constructions in algebraic topology such as complexes of differential forms defined on a simplicial complex, minimal models (due to Sullivan and others), the Dolt-Thom construction, etc. Can algebraic shifting be defined for singular homology or de-Rahm homology rather than simplicial homology?

Problem 15. Is there a useful notion of algebraic shifting of maps between simplicial complexes? Is algebraic shifting a functor of any kind?

Problem 16. Can algebraic shifting be axiomatized?

3.5.6 Duality and complementation

Let K be a simplicial complex and let K^{dual} be its Alexander dual (also known as the blocker of K). Thus,

$$K^{dual} = \{S \subset V(K) : (V(K) \backslash S) \notin K\}.$$

It is not hard to show that

$$\Delta(K^{dual}) = (\Delta(K))^{dual}.$$

For a family $A \subset \binom{[n]}{k}$ let $\overline{A}$ be the complement of the family, namely $\overline{A} = \binom{[n]}{k} \backslash A$. $\Delta(A)$ does not in general determine $\Delta(\overline{A})$.

Problem 17. Given $\Delta(A)$, what can be said about $\Delta(\overline{A})$?

Note that $\Delta(K)$ determines the algebraic shifting of A with respect to the reverse lexicographic order. More generally, let $<$ be a total ordering of $\binom{[n]}{k}$ which extends the partial order and let $<'$ be the reverse order to $<$ obtained by 1) reversing the order relation and 2) reversing the role of i and $n-i$. Then by shifting the complement of A with respect to $<'$ and replacing i by $n-i$ the complement of shifting A with respect to $<$ is obtained.

3.5.7 Barycentric and other subdivisions

The next problem is related to the discussion in Sections 4 and 5.

Problem 18. 1. Let K be a subdivision of another complex L. What is the relation between $\Delta(K)$ and $\Delta(L)$?

In particular, it would be useful to find a shifting theoretic interpretation for Stanley's local h-vector theory [75].

2. What can be said about $\Delta(b(K))$ where $b(K)$ is the barycentric subdivision of K?

4 Around Cohen-Macaulay

4.1 Shifting preserves the Cohen-Macaulay property

We have seen that algebraic shifting seems to "forget" all the homotopical information concerning a space K except the Betti numbers themselves. However, other important topological properties are preserved under shifting.

Problem 19. Understand how various topological properties of a simplicial complex are manifested in terms of the shifted complex. In particular, which topological properties are preserved under shifting?

One such property is the Cohen-Macaulay property which originated in commutative algebra. A pure d-dimensional simplicial complex K is Cohen-Macaulay if for every face S of K (including the empty face), $H_i(lk(S,K)) = 0$ when $i < \dim lk(S,K)$.

Theorem 4.1. *If K is Cohen-Macaulay then so is $\Delta(K)$.*

The Cohen Macaulay property has a simple description in terms of $GIN(K)$. Let K be a $(d-1)$-dimensional simplicial complex and consider the set B of monomials m in the variables $y_{d+1}, y_{d+2} \dots$ in $GIN(K)$. B is a finite shifted ordered ideal of monomials in the variables $y_{d+1}, y_{d+2}, \ldots$, and $GIN(K)$ is determined from B by the rule:

$$GIN(K) = \{m \cdot m' : \; m \in B, \; m' \text{ is an arbitrary monomial in } y_1, \ldots y_d\} \tag{4.1}$$

Relation 4.1 characterizes Cohen-Macaulay simplicial complexes, it follows easily from the ring-theoretic definition for Cohen-Macaulayness [76].

4.2 The theorems of Bayer, Charalambous & Popescu and Aramova & Herzog

One can ask which data on the Betti numbers of complexes and their links (or induced subcomplexes) is preserved under shifting. (Note that unlike Cohen-Macaulayness, which is a topological property these conditions are not usually topologically invariant.) The recent results by Bayer, Charalambous & Popescu and by Aramova & Herzog go a long way in this direction.

Theorem 4.2. *Let K be a simplicial complex. Assume that*

$$\beta_k(lk(T, K)) = 0,$$

whenever $|T| = t < j$ and $i \leq k \leq i + (j - |T|)$. Then also

$$\beta_k(lk(T, \Delta(K)) = 0,$$

whenever $|T| = t < j$ and $i \leq k \leq i + (j - |T|)$, and the quantity

$$\sum_{|S|=j} \beta_i(lk(S, K))$$

is preserved under shifting.

This theorem follows from a theorem of Bayer, Charalambous and Popescu (BCP) for symmetric shifting. To derive it from BCP's theorem one has to rely also on a theorem by Aramova, Herzog and Hibi [7] which asserts that the generic initial ideal and the symmetric shifted ideal have the same graded Betti numbers. Theorem 4.2 was proved by Aramova and Herzog (AH) for exterior shifting. (Aramova and Herzog also presented another proof for the symmetric case.)

Remark: The original formulations and (the only known) proofs are ring-theoretic and will not be discussed in this paper. The relations between certain ring-theoretic properties between the classical face rings and their exterior analogs is one of the interesting aspects of these results. To move from the original formulation to the one presented here one must rely on Alexander duality and Hochester's theorem [76], p. 60.

A complex K is *d-Leray* if the Betti numbers of K and all its links vanish at and above dimension d. Theorem 4.2 implies many of the earlier applications of shifting:

- Shifting preserves Betti numbers.
- Shifting preserves the Cohen-Macaulay property.
- Shifting preserve the d-Leray property.
- The property that K and all links of vertices of K are acyclic is preserved under shifting. (It follows that $\Delta(K)$ is a double cone.)

Remark: It is not hard to see (although it has been overlooked for a long time) that the class of d-Leray complexes (for some d) with complete $(d-1)$-dimensional skeletons is precisely the Alexander dual of the class of Cohen-Macaulay complexes. This observation implies that the fact that shifting preserves the Leray property easily follows from the fact that shifting preserves the Cohen-Macaulay property. Moreover, it shows that the characterization of face numbers of d-Leray complexes follows from the corresponding characterization for Cohen-Macaulay complexes.

4.3 Iterated cohomology groups

Iterated cohomology groups are defined by successively applying r generic weighted coboundary operators. There are several variations and the reader is referred to Duval and Rose [27] which introduces one such variation and describes applications for non-pure shellability.

Let K be a simplicial complex with n vertices and let $\bigwedge(K)$ be its face algebra as defined in the previous section. Let $\mathbf{f}=(f_1, f_2, \cdots, f_n)$ be a fixed basis, with coefficient matrix X, in general position in E. Define $f_{[r]} = f_1 \wedge f_2 \wedge, \cdots, \wedge f_r$ and $F_r = span\{f_1, f_2, \cdots, f_r\}$. Define the r-th *iterated* cohomology group of K, $H^k[r](K)$, as follows:

$$H^k[r](K) = Z^k[r](K)/B^k[r](K), \tag{4.2}$$

where

$$Z^k[r] = \{m \in \bigwedge^k(K) : \; f_1 \wedge f_2 \wedge \cdots \wedge f_r(m) = 0\}, \tag{4.3}$$

and

$$B^k[r] = span\{F_r \wedge \bigwedge^{k-1}(K)\}. \tag{4.4}$$

Problem 20. Find relations between iterated homology groups and more standard notions of algebraic topology and/or commutative algebra and, in particular, local cohomology.

Problem 21. Can the theorems of Bayer, Charalambous & Popescu and Aramova & Herzog be further extended by replacing Betti numbers with the dimensions of certain iterated homology groups?

Theorem 4.3.

$$dim H^k[r](K) = |\{S \in \Delta(K) : S \cap [r] = \emptyset, S \cup [r] \notin \Delta(K)\}|. \tag{4.5}$$

Proof: Define $\mathcal{A}_k^r = \{S \subset [n] : |S| = k+1, [r] \cap S \neq \emptyset\}$. First note that $B^k[r](K) = \widetilde{F}_r \wedge \bigwedge_{k-1}(K) = span\{\widetilde{f}_S : S \in \mathcal{A}_k^r\}$. Since $\mathcal{A}_k^r$ is initial w.r.t. the lexicographic ordering $<_L$, it follows that $\{\widetilde{f}_S : S \in \Delta(K) \cap \mathcal{A}_k^r\}$ is a basis of $B^k[r](K) = span\{\widetilde{f}_S : S \in \mathcal{A}_k^r\}$.

Now, let $S_1, ..., S_u$ be the sets in $\Delta_k(K) \backslash \mathcal{A}_k^r$ ordered lexicographically and let

$$U_t = span(B^k[r](K) \cup \{f_{S_1} \cdots f_{S_t}\}).$$

Let $I_t = \widetilde{f}_{[r]} \wedge U_t$. Thus, $I_0 = \{0\}$ and $I_{t+1} = span(I_t \cup \widetilde{f}_{[r]} \wedge \widetilde{f}_{S_{t+1}})$. It follows that $I_{t+1} = I_t$ iff $[r] \cup S_{t+1} \notin \bigwedge_{k+r}(K)$. Therefore,

$$dim Z^k[r](K) - dim B^k[r](K) = |\{S \in \Delta_k(K) : S \cap [r] = \emptyset, S \cup [r] \notin \Delta(K)\}|.$$

Corollary 4.4. *If $\Delta(K)$ is shifted then*

$$H_k[r](K) = H_k[r](\Delta(K)). \tag{4.6}$$

Proof: This follows from the fact that $\Delta(D) = D$ for a shifted family D.

4.4 Collapsing and Shelling

Let K be a $(d-1)$-dimensional simplicial complex. A face S in K is free if it is included in a unique maximal face M. If $|S| = k$ and $|M| = m$ we say that S is of type (k, m). A (k, l)-collapse step is the deletion from K of a free face of type (k, l) and all faces containing it. K is collapsible if it can be reduced to the void complex by a sequence of $(i, i-1)$-collapse steps. A (k, d)-collapse step is called a *shelling* step of type k. K is shellable if it can be reduced to the void complex by successive applications of shelling steps. Collapsible complexes are acyclic while shellable complexes are Cohen-Macaulay. It was proved by Bruggeser and Mani that the boundary complex of every simplicial polytope is shellable.

Theorem 4.5. *Let K, K' be simplicial complexes such that K' is obtained from K by a collapse step of type (i, d) for some i. Then the inclusion map induces an isomorphism between $H^a[b](K)$ and $H^a[b](K')$ for every a, b, $a + b < d$.*

(The proof is similar to the proof of Theorem 5.4 in [55].)

Theorem 4.6. *If K' is obtained from K by a collapse step of type (k, r) then $\Delta(K')$ is obtained from $\Delta(K)$ by a collapse step of the same type.*

Corollary 4.7. *Let K be a shellable $(d-1)$-dimensional simplicial complex. Then $H^{k-1}[d-k](K) = 0$ for every $k \geq 0$.*

The effect of collapsing and shelling on iterated homology groups was crucial for some of my earlier proofs which used algebraic shifting. To show that Cohen-Macaulay is preserved under exterior shifting I needed to show vanishing of the same iterated cohomology groups that appear in the Corollary 4.7 but in a different way: first, show that they are preserved under subdivisions, then prove a nerve theorem using a Mayer-Vietoris long exact sequence and finally use the nerve theorem on good covers of the subdivided complex.

4.5 Sequential Cohen-Macaulayness

Björner and Wachs [19, 20] defined shellability for non-pure complexes and Stanley [76], p. 87 described the commutative algebra content of this notion and defined the notion of sequentially Cohen-Macaulay rings and complexes.

All shifted simplicial complexes are (non-pure) shellable and hence sequentially Cohen-Macaulay. Duval and Rose [27] showed that certain combinatorial invariants of (non-pure) shellable simplicial complexes are preserved under shifting and have simple interpretation in terms of certain iterated homology groups. Duval [29] studied algebraic shifting for sequentially Cohen-Macaulay complexes and showed how certain homological invariants of their face-rings are preserved under shifting.

4.6 Combinatorial decomposition

While collapsible simplicial complexes are acyclic, the converse is far from being true. Is there a natural combinatorial property such as collapsibility that all acyclic complexes satisfy? Note that collapsing K yields a matching between k-faces (the free faces) and $k+1$-faces (the maximal faces). It is not difficult to show that such a mapping exists for arbitrary acyclic (or even $\mathbb{Q}$-acyclic) complex. Following is a very general conjecture in this spirit.

Let D be a shifted simplicial complex. An elementary collapse step $D \to D' = D \backslash [F, G]$ is *shifting preserving* if D' is shifted. A shifting preserving collapse of D is a collapsing of D to the void complex via shifting preserving elementary collapse steps.

Conjecture 22. 1. Let K be a simplicial complex such that $\Delta(K) = D$. Let $D = [F_1, G_1] \cup [F_2, G_2] \cup \cdots \cup [F_t, G_t]$ be the representation of D as a union of intervals given by a shifting preserving collapse of D. Then there is a decomposition of K into disjoint intervals of the form $K = [A_1, B_1] \cup [A_2, B_2] \cup \cdots \cup [A_t, B_t]$ such that $dimA_i = dimF_i$ and $dimB_i = dimG_i$.

2. Moreover, it is possible to find such a decomposition such that $\cup[A_i, B_i)$ is a simplicial complex and, more generally, such that the union of $[A_i, B_i] \setminus Top_j([A_i, B_i])$ is a simplicial complex for every j. Here for an interval I of faces, $Top_j(I)$ is the sets in the highest j-levels.

This conjecture extends earlier theorems by the author, by Stanley and by Duval, (see [28]) and various earlier conjectures including a decomposition conjecture for Cohen-Macaulay complexes formulated by Garsia and Stanley [76], p.85. Duval and Zhang [34] used iterated homology groups to find a very general decomposition theorem which is not as strong as the conjectured one.

Another notion of decomposability is given by combinatorial Morse theory [42, 43].

5 Beyond the Cohen-Macaulay property

5.1 Polytopes, spheres and Gorenstein* complexes

The only shifted complex which is a triangulated sphere (or even a manifold without boundary) is the boundary of a simplex. While being a triangulated sphere is not preserved under shifting, can we still say something about shifting of triangulated spheres? We will consider the more general class of Gorenstein* complexes.

A pure d-dimensional simplicial complex K is Gorenstein* if for every face S of K (including the empty face), $H_i(lk(S,K)) = 0$ when $i < \dim lk(S,K)$ and $\dim H_i(lk(S,K)) = 1$ when $i = \dim lk(S,K)$. Being a Gorenstein* complex manifests a profound duality relation for the face ring ([76]).

Conjecture 23. Let K be a $(d-1)$-dimensional Gorenstein* complex. Let H_k be the set of monomials in $GIN(K)$ in the variables $y_{d+1}, y_{d+2}, \ldots$. Then the map

$$m \to y_{d+1}^{d-2k} \cdot m \tag{5.1}$$

is a bijection between H_k and H_{d-k}.

Conjecture 23 is known for simplicial polytopes where it follows from the hard-Lefschetz theorem for toric varieties. It implies that the characterization of f-vectors of simplicial polytopes (the g-theorem) applies to arbitrary Gorenstein* complexes. But it will give more than a complete description of f-vectors (or Hilbert series) of Gorenstein* complexes. In addition, it will probably yield a complete description of their generic initial ideals.

For a $(d-1)$-dimensional Gorenstein* simplicial complex K let $U(K)$ be the set of all monomials in $GIN(K)$ which involves only the variables $y_{d+2}, y_{d+3}, \ldots$. Assuming Conjecture 23, $U = U(K)$ is a shifted ideal of monomials of degree at most $[d/2]$ in the variables $y_{d+2}, y_{d+3}, \ldots$ and the shifted complex (or $GIN(K)$) of K is determined by $U(K)$. (Since K is Cohen-Macaulay the set B of monomials in $y_{d+1}, y_{d+2}, \ldots$ in $GIN(K)$ determines $GIN(K)$ (relation (4.1)). Conjecture 23 implies that U determines B.)

On the other hand for every shifted ideal of monomials U of degree at most $[d/2]$ in the variables $y_{d+2}, y_{d+3}, \ldots$, a simplicial $(d-1)$-sphere $S(U)$ (called a *squeezed* sphere) was constructed in [59].

Problem 24. Show that $U(S(U)) = U$.

The fact that squeezed $(d-1)$-dimensional spheres are the boundary of d-balls with the same $[d/2]$-skeleton may help to settle this problem.

Regarding face numbers we believe that there is no difference between simplicial spheres and simplicial polytopes. However, this cannot be the case for the shifted complex simply because there are far too many choices for U.

Problem 25. If K is the boundary complex of a simplicial d-polytope, then what more can be said about $\Delta(K)$ (or equivalently about $U(K)$)? Is it true that representability as a simplicial polytope is preserved under squeezing? Namely if K is polytopal, is $S(U(K))$ polytopal as well?

Note that although shifting does not preserve the Gorenstein* property, we can derive (assuming that our conjectured picture is true) another shifting-like operation which associates to every Gorenstein* complex K a new such complex $K' = S(U(K))$. This new complex is always a simplicial sphere and $GIN(K) = GIN(K')$. It would be interesting to understand this operation on the ring-theoretic level.

5.2 Shifting and Embeddability

We will describe now a shifted simplicial complex which will play a crucial role in this section. Let $\Delta(d,n)$ be the pure $(d-1)$-dimensional complex whose set of vertices is $[n] = \{1, 2, \ldots, n\}$ and whose maximal faces are sets $S \subset [n], |S| = d$ which satisfy $k \notin S \Rightarrow [k+1, d-k+2] \subset S$. Let $\Delta(d)$ be the union of $\Delta(d,n)$ for all n. Note that $\Delta(d,n)$ is the restriction of $\Delta(d)$ to the first n vertices.

Define $GIN(d)$ to be the inverse image of $\Delta(d)$ under the operation that transform GIN to Δ^{symm}. $GIN(d)$ is the shifted order ideal of monomials in $y_1, y_2, \ldots$ which does not contain the monomials $y_{d+1}^{d-2i+1} y_{d+2}^{i}, \quad i = 0, 1, \ldots$.

Theorem 5.1. *[61] If K is the boundary complex of a simplicial polytope then $\Delta^{symm}(K) \subset \Delta(d)$ or equivalently $GIN(K) \subset GIN(d)$.*

Let $C(d,n)$ be the boundary complex of the *cyclic* d-dimensional polytope with n vertices.

Proposition 5.2.

$$\Delta(d,n) = \Delta^{symm}(C(d,n)) \tag{5.2}$$

This result probably holds also for exterior shifting. The relation

$$\Delta(K) \subset \Delta(d) \tag{5.3}$$

is referred to as the *shifting-theoretic upper bound relation* since it immediately implies (although it is much stronger) the upper bound theorem for face numbers of simplicial spheres and related stronger combinatorial results. See [61, 63].

Problem 26. Understand the scope of the shifting-theoretic upper bound relation.

A proof of the shifting-theoretic upper bound relation for simplicial spheres or for Gorenstein* complexes will lead to a complete description of their shifted complexes (and hence their f-vectors) because of the following simple combinatorial result.

Proposition 5.3. *The assertion of the shifting-theoretic upper bound theorem for a Gorenstein* complex K is equivalent to the assertion of Conjecture 23 for K.*

The next conjecture links the shifting-theoretic upper bound relation to embeddability.

Conjecture 27. Let K be a simplicial complex with n vertices such that $|K|$ can be embedded in S^{d-1}. Then $\Delta(K) \subset \Delta(d)$. Equivalently, $\Delta(K)$ does not contain any of the sets $T_{d-1} \dots T_{[d/2]}$ where

$$T_{d-j} = \{k+2, k+3, \dots, d-k, d-k+2, \dots, d+2\}. \tag{5.4}$$

Equivalently, $GIN(K)$ does not contain any of the monomials

$$y_{d+1}^{d-2i+1} y_{d+2}^{i}, \quad i = 0, 1, \dots \ . \tag{5.5}$$

Conjecture 27 was stated independently by Sarkaria and by me. Sarkaria proposed in order to prove it, to relate the Van-Kempen obstructions to embeddability [79] for a complex to those for the shifted complex.

Conjecture 28. Let X be a $2r$-dimensional simplicial complex. If σ_r^{2r+2}, the r-dimensional skeleton of a $(2r+2)$-dimensional simplex, cannot be embedded in X then it cannot be embedded in $\Delta(K)$ for every triangulation K of X.

One difficulty that we face in trying to relate shifting with embeddability is that there is a difference between the role of the graphs K_5 and $K_{3,3}$. For a graph G not to contain a topological K_5 is preserved under shifting but this is not the case for $K_{3,3}$. It appears that we need a coarser obstruction for embeddability (for graphs into $\mathbb{R}^2$ and generally for r-dimensional complexes into $\mathbb{R}^{2r}$) which will be non-trivial for K_5 and trivial for $K_{3,3}$.

Conjecture 29. Let K be a pure $(d-1)$-dimensional simplicial complex. Assume that for every face $S \in K$ (including the empty face) with $\dim lk(S, K) = 2r$,

$$\sigma_r^{2r+2} \text{ cannot be embedded into } lk(S, K). \tag{5.6}$$

Then $\Delta(K) \subset \Delta(d)$.

We will come back to related problems in Section 5.6 below. Note that relation (5.6) for $r = 0$ asserts that every face of K of dimension $d-2$ is included in at most two maximal faces.

5.3 Can we trace intersection homology?

Intersection homology [46, 47] is defined for stratified pseudomanifolds (we will only consider triangulated pseudomanifolds) based on numerical sequence $(p_2, p_3 \dots)$ called perversity.

Problem 30. Find an interpretation of intersection cohomology of a pseudomanifold K in terms of the face rings of K and in terms of algebraic shifting.

There are several reasons to think that such a description is feasible: The behavior of intersection homology under forming of a cone and under suspension is simple (see [22]). The various intersection homology groups reduce to the usual homology for manifolds and, as we shall see, in this case the Betti numbers manifest themselves in many different ways in the shifted complex. Finally, there is a certain similarity between iterated homology groups of the type discussed above and intersection homology: If instead of restricting the cycles and boundaries according to the manner in which they intersect the low-dimensional strata, we consider similar algebraic-restrictions (namely, we replace the geometric strata by some generic subspaces of the exterior face algebra) then we obtain similar objects to the iterated homology groups.

5.4 Buchsbaum's complexes and manifolds

A simplicial complex K is Buchsbaum if for every vertex v, $link(v, K)$ is Cohen-Macaulay. (For the commutative algebra definition of Buchsbaum rings see [76, 24]. There are, in fact, several classes of rings which for Stanley-Reisner rings coincide with Stanley-Reisner rings arising from Buchsbaum complexes.) In particular, all triangulations of manifolds are Buchsbaum.

Buchsbaum complexes form a natural extension of Cohen-Macaulay complexes and while the property of being Buchsbaum is not preserved under shifting much can be said about shifting of Buchsbaum complexes. These results are available currently for symmetric shifting only (and only for characteristic 0).

The following important properties of $GIN(K)$ for a Buchsbaum complex K were proved by Novik [69].

Theorem 5.4 (Novik). *Let K be a Buchsbaum complex.*

- *For $i \leq d$, if m is a monomial in $\{y_{i+1}, \ldots, y_n\}$ and $my_i^2 \notin GIN(K)$ then also $my_i \notin GIN(K)$. Let B_i be the set of these monomials m.*
- *if $m \in B_i$ then m is a monomial in $\{y_{d+1}, \ldots, y_n\}$*
- *The number of monomials in B_i of degree r, $r < i$ is $\binom{r-1}{i-1}\beta_{r-1}(K)$*

Remark: Unlike the case of Cohen-Macaulay complexes, Theorem 5.4 does not characterize Buchsbaum complexes. It would be interesting to understand the ring theoretic properties which correspond to the properties of $GIN(K)$ described in Theorem 5.4 and especially the following:

Problem 31. 1. Consider face rings of $(d-1)$-dimensional simplicial complexes (or more general quotient rings of the ring of polynomials with n variables). In which cases do we have the property that for a monomial m which does not involve $y_1, \ldots y_i$, $my_i^2 \notin GIN(K)$ implies $my_i \notin GIN(K)$?

2. What can be said about rings or complexes where, for every $i \leq d$ and m as above, $my_i^{t+1} \notin GIN(K)$ implies $my_i^t \notin GIN(K)$? (Note that for $t = 0$ this is equivalent to the Cohen-Macaulay property and for $t = 1$ this follows from the Buchsbaum property.)

Problem 32. (Stated vaguely.) Is there a way to identify every appearance of β_i in B_i?

Isabella Novik made some progress in this direction.

Problem 33. 1. Prove the assertion derived from Theorem 5.4 for symmetric shifting of Buchsbaum complexes for exterior shifting.

2. Is being a Buchsbaum complex can be described via the exterior face algebra?

5.5 Conjectures concerning simplicial manifolds

We start with the following problem:

Problem 34. Given a triangulation K of a manifold with boundary what are the relations between $\Delta(K)$ and $\Delta(\partial K)$?

There are some far-reaching conjectures concerning the algebraic shifting of triangulations of manifolds with and without boundary. These can be found in [69], Conjectures 7.1 and 7.5(i). (The reader is referred to [69] for more details.)

One of the motivating problem is the following:

Problem 35. Understand Poincaré duality for manifolds in terms of face-rings and algebraic shifting.

In response, Conjecture 37 below, proposes a beautiful connection between Poincaré duality and the Dehn-Somerville relations via a far-reaching extension of the toric hard-Lefschetz theorem.

Let me now describe these conjectures in some details. The first conjecture sharpen Theorem 5.4.

Conjecture 36. Let K be a $(d-1)$-dimensional triangulated manifold. Consider the set A_k of monomials m of degree k in $GIN(K)$ in the variables $y_{d+2}, y_{d+3}, \dots y_n$ such that $my_{d+1} \notin GIN(K)$. Then

$$|A_r| = \binom{d}{k} \beta_{k-1}(K). \tag{5.7}$$

Let H_r denote the monomials of degree r in $GIN(K)$ in the variables $y_{d+1}, y_{d+2}, \dots, y_n$ which are not included in A_r. Now consider the case that K is a manifold without boundary. It follows from Theorem 5.4 (which relies on the fact that K is Buchsbaum) combined with combinatorial relations on the Hilbert polynomial derived from the fact that K is a manifold (the Dehn-Sommerville relations) together with Poincarè-duality that $|H_r| = |H_{d-r}|$. The following is a far reaching extension of Conjecture 23:

Conjecture 37. 1. [Shifting-theoretic Poincarè duality] Let K be a $(d-1)$-dimensional simplicial manifold without boundary. For $k < d/2$ the map

$$m \to m y_{d+1}^{d-2k} \tag{5.8}$$

is a bijection between H_k and H_{d-k}.

2. For an arbitrary $(d-1)$-dimensional simplicial manifold K the set of monomials $m \cdot y_{d+1}^{d-2k}$ for $m \in H_k$ contains H_{d-k}.

In particular, when $d-1=2r$, Conjectures 36, 37 imply that the number of monomials in $GIN(K)$ of degree $r+1$ in the variables $y_{d+2}, y_{d+3}, \ldots$ is precisely $\binom{2r+1}{r} \cdot \beta_r(K)$.

It follows from Conjecture 37 that if K is a simplicial $(d-1)$-dimensional manifold then the monomials $y_{d+1}^{d-2i+1} y_{d+2}^{i}$, $i = 0, 1, \ldots$ do not belong to $GIN(K)$ with one exception only: the monomial y_{d+2}^{r+1} when $d-1=2r$. For an even-dimensional simplicial manifold K, if the middle Betti number of K vanishes then $y_{d+2}^{r+1} \notin GIN(K)$ and K satisfies the shifting-theoretic upper bound relation.

5.6 Pseudomanifolds

We will now consider larger classes of pseudomanifolds. A Witt space is one in which for every link of a proper face, the middle perversity intersection homology vanishes. Towards the connection proposed in Section 5.3 we make the following conjecture:

Conjecture 38. Let K be a triangulation of a Witt space of dimension $2r$.

1.Let A_{r+1} be the set of monomials m in $GIN(K)$ of degree $r+1$ in the variables $y_{d+2}, y_{d+3}, \ldots$. Then

$$|A_{r+1}| = \binom{2r+1}{r} \cdot \dim IH_r(K). \tag{5.9}$$

2. K satisfies the shifting-theoretic Poincaré duality.

3. If $\dim IH_r(K) = 0$ then K satisfies the shifting-theoretic upper bound relation.

However, it appears that these relations go beyond Witt spaces.

Problem 39. (1) Let K be a $(d-1)$-dimensional pseudomanifold with the property that for every proper $2r$-dimensional link K',

$$y_{2r+3}^{r+1} \notin GIN(K'). \tag{5.10}$$

Then the shifting-theoretic Poincarè duality is satisfied!

(2) If, in addition, K is odd-dimensional or if condition (5.10) holds also for K itself then K satisfies the shifting-theoretic upper bound relation, namely, $GIN(K) \subset GIN(d)$.

The class of pseudomanifolds which satisfy condition (5.10) for all proper links appears to be an interesting extension of the class of manifolds. By our conjectures this class contains the class of triangulations of Witt spaces and the class of pseudomanifolds which satisfy relation (5.6) for all proper links of faces.

6 Applications and Connections with Combinatorics

6.1 f-vectors

The main application of algebraic shifting is in the study of f-vectors of classes of simplicial complexes. For a survey (from 1989) the reader is referred to [17]. For a more recent survey of the 'state of the art' concerning f-vectors see [14]. Most of the results described in this paper were accompanied by applications to f-vectors in the original papers. I will not discuss these applications in this paper.

6.2 Combinatorial Shifting

In their seminal paper [41] Erdös, Ko and Rado described an operation on finite set systems which is now called *shifting*. (In this paper we will use the name "combinatorial shifting" (or: CS) to distinguish this operation from algebraic shifting.) The reader is referred to Frankl's survey article [44].

For a family A of k-subsets of $[n]$ and two integers i, j, $1 \leq i < j \leq n$ define a family $C_{ij}(A) = \{C_{ij}(S) : S \in A\}$ as follows: $C_{ij}(S) = S$ if either $i \in S$ or $j \notin S$. If $i \notin S$ and $j \in S$ consider $R = S \cup \{i\} \backslash j$. If $R \in A$ then $C_{ij}(S) = S$ but if $R \notin A$ then $C_{ij}(S) = R$.

Every family A can be transformed into a shifted family $\Delta_{\mathbf{C}}(A)$ by successive applications of the operations $A \to C_{ij}(A)$. $\Delta_{\mathbf{C}}(A)$ depends, of course, not only on A but also on the order in which the operations $C_{ij}(A)$ were applied.

Problem 40. What are the relations between algebraic and combinatorial shifting?

Proposition 6.1. *Let A be a family of r-subsets of $[n]$. Let $B = C_{ij}(A)$ then for a generic n by n matrix X, the set of columns in the matrix $M_X(A)$ indexed by sets in B is linearly independent.*

However, it is not always possible to realize $\Delta(A)$ by applying combinatorial shifting. For example, starting with the 10 triangles of the triangulation of the projective space with 6 vertices, (mod 0) algebraic shifting yields $\Delta(A) = \{S \in [6] : 1 \in S\}$. However, $\{2, 3, 4\} \in \Delta_{\mathbf{c}}(A)$ for every combinatorial shifting of A.

6.3 Applicability of shifting

There are (roughly) four types of behaviors in the application of shifting to the study of a combinatorial (or topological) property:

- The property is preserved under shifting and the situation for shifted families (complexes) is simple.

Examples are:
1. What are the possible f-vectors of simplicial complexes?
2. What is the maximal size of an intersecting family of subsets in $\binom{[n]}{k}$?
3. What are the possible f-vectors of simplicial complexes with prescribed Betti numbers?

For the first two examples combinatorial as well as algebraic shifting can be used (at present only exterior shifting works for the second example). The third example requires algebraic shifting.

- The property is preserved under shifting but the situation for shifted families (complexes) is complicated.

For example: What is the maximum size of a family of k-subsets of $[n]$ such that there are no t sets in the family which are pairwise disjoint? It is not difficult to show that this property is preserved under shifting (either algebraic or combinatorial). The situation for shifted families is still an open question.

- Showing that the property is preserved under shifting is hard but the situation for shifted families (complexes) is simple.

Following is an example of such a property: Chvatal conjectured that when $kr \geq (k-1)n$ every family $A \subset \binom{[n]}{k}$ with more than $\binom{n-1}{k-1}$ must contain r sets whose intersection is empty while the intersection of each $r-1$ of the sets is not empty. (For $r = 2$ this is the theorem of Erdös, Ko and

Rado.) The hard part seems to show that shifting preserves the property. (For $r = 2$ this was the motivation for combinatorial shifting.)

(Embeddability questions that were considered above also fall into this category.)

- The property is not preserved under shifting although shifting may still be useful.

Two such examples from combinatorics, the Turàn problem and the Erdos-Rado sunflower conjecture will be discussed below. (We considered several examples earlier, such as Buchsbaum and Gorenstein complexes.)

6.4 Intersecting families

Intersecting families are of great interest in extremal combinatorics. In this section we will show that if K is an intersecting uniform set system then $\Delta(K)$ is as well.

Let $K \subset \binom{[n]}{k}$ and $L \subset \binom{[n]}{l}$. Define $K \wedge L = \{S \cup T : S \in K, T \in L, S \cap T = \emptyset\}$.

Theorem 6.2. $\Delta(K) \wedge \Delta(L) \subset \Delta(K \wedge L)$.

Proof: Let $m_1 \in M(K)$, $m_2 \in M(L)$ with $i(m_1) = S$, $i(m_2) = T$ and $S \cap T = \emptyset$. Note that $M(K \wedge L) = M(K) \wedge M(L)$. Now, $m_1 = f_S + \sum\{\alpha_R f_R : R <_L S\}$, $m_2 = f_T + \sum\{\beta_R f_R : R <_L T\}$ and therefore $m_1 \wedge m_2 = f_{S\cup T} + \sum\{\alpha_R \beta_{R'} f_{R\cup R'} : R <_L S, R' <_L T, R \cap R' = \emptyset\}$. Thus, $i(m_1 \wedge m_2) = S \cup T$.

Corollary 6.3. *If K is intersecting then $\Delta(K)$ is as well.*

Remarks: 1. If K has the property that among every t members of K there are two with intersection of cardinality of at least m, then the same property holds for $\Delta(K)$. A similar proof applies.

2. Note that the proof did not rely on the term order being used. In this respect shifting preserves intersecting families in a very strong sense.

3. The maximal number of sets in an intersecting family of subsets of size k from $[n]$ is $\binom{n-1}{k-1}$ when $n \geq 2k$. This is also the maximal number of k-sets which do not support a $(k-1)$-dimensional cycle. Is there any connection? We do not know of any for general hypergraphs. For H not to support a

$(k-1)$-dimensional cycle is equivalent to the property that all sets in $\Delta(H)$ contains '1'. This is false in general for intersecting families.

It appears, however, that completely balanced (that is k-colorable) intersecting k-uniform hypergraphs indeed do not support k-dimensional homology and that this follows from an extension of algebraic shifting to the completely balanced case.

6.5 Extremal combinatorics: the sunflower conjecture

I had high hopes for applications of algebraic shifting in extremal combinatorics. So far, there is no real evidence to justify them.

A collection of sets is called a *Delta System* or a *sunflower* if every element that is contained in at leasy two of them is contained in all of them.

Problem 41 *(Erdös and Rado Delta System Conjecture).* There exists a constant C_r depending on r such that every collection F of k-sets without a Delta System of size r has at most C_r^k members.

We will only consider the important case $r = 3$. Consider the simplicial complex K spanned by family F of k-sets without a Delta System of size three. Recall that K is completely balanced if we can color its vertices with k colors such that the vertices of every maximal face represent all the colors. It is easy to see that K contains a completely balanced subcomplex which contains at least $(1/e)^k$ k-sets from F. Therefore, there is no loss of generality for the Delta System conjecture to assume that K is completely balanced. (This fact was pointed out to me by Jeff Kahn.)

We would have liked to be able to prove the following chain of implications:

1 For every face $S \in K$, $lk(S,K)$ does not contain three disjoint sets.

2 For every face $S \in K$, $\Delta^{ext}(lk(S,K))$ does not contain three disjoint sets

3 For every face $S \in K$, all maximal sets in $\Delta^{ext}(lk(S,K))$ contain either '1' or '2'.

4 The monomials $\widetilde{f_S}$, where S ranges over all k-subsets of $[2k]$, is a spanning set for $\bigwedge^d(K)$.

The first property is a reformulation of the fact that the family contains no Delta System of size three. The implication of 2 from 1 follows from the results of the previous section. While 2 does not imply 3 in general, we expect this implication to hold if the complex K is completely balanced. To prove it algebraic shifting for completely balanced complexes should be developed.

The move from 3 to 4 has a similar flavor to that of the general theorems of Aramova and Herzog. Their methods may apply. However, 3 should still be enforced by additional homological properties implied by the Delta System condition. Property 4 would imply that $C_3 \leq 4e$ (< 12).

6.6 The Turàn problem

The Tùran problem can be formulated as follows: What is the smallest number of square free monomials of degree r which span (i.e. the ideal generated by them contains) all square free monomials of degree t? Or in the ususal formulation: what is the minimum size of an r-uniform hypergraph without an independent set of t vertices?

The situation for $r = 2$ is completely explained by a theorem of Turàn and the situation for $r > 2$ is almost entirely not understood. We will mainly be interested in the case $r = 3, t = 4$.

A shifting theoretic approach to the (4,3) case of Turàn's problem was proposed in [57]. Since then, refinements of the conjectures have been formulated. A computer was used to test the conjectures for hypergraphs with a few vertices and for some examples of Kostochka and no counter examples were found. But overall, there has been no real progress in this direction.

6.7 The clique complex of a graph

There is a very interesting "shifting"-type question related to Turàn's theorem for graphs. Consider a graph G with n vertices and the complex of its complete subgraphs $K(G)$. (In this case, the ideals used in the definition of the various face rings for $K(G)$ are quadratic, i.e., they are generated by degree two polynomials.) Let c be the size of the maximal complete subgraph of G.

Consider n generic degree-one elements $f_1, \dots f_2$ and divide them into c parts $A_1, A_2, \dots, A_c$ whose sizes are as equal as possible. Consider the set U of all monomials $\widetilde{f_S}$ in $\bigwedge(K(G))$ such that $|S \cap A_j| \leq 1$ for every j.

Conjecture 42. U span $\bigwedge(K(G))$.

This conjecture is a shifting-theoretic extension of Turàn's theorem which asserts that the number of edges in a graph with no clique of size $c+1$ is attained by a complete c-partite graph where the sizes of the parts are as equal as possible. This conjecture implies a far-reaching conjecture by Eckhoff [36] and myself on face numbers of clique complexes and has applications in the study of f-vectors of nerves of boxes [36, 37]. It also seems related to a conjecture by Charney and Davis [25] on clique complexes that are spheres. See also [76], p. 103.

6.8 Are there more drastic forms of algebraic shifting?

A drastic shifting operation is one which maps every simplicial complex to an even more restricted class of complexes than the shifted complexes while still preserving some useful combinatorial properties. Exterior shifting is never drastic as it fixes all shifted complexes. As already mentioned it is not known whether symmetric shifting, with respect to the reverse lexicographic order, fixes all shifted complexes. In combinatorial applications, shifting is often not the end of the road and some drastic algebraic shifting operations may be helpful.

The Kruskal-Katona theorem asserts that every simplicial complex has the same f-vector as a compressed simplicial complex, namely a simplicial complex whose r faces are initial with respect to the reverse lexicographic order.

Problem 43. Find a drastic form of algebraic shifting which proves the Kruskal-Katona theorem.

There are many simple proofs of the Kruskal-Katona theorem but an algebraic proof may have further applications. The same problem may be asked in the symmetric case for Macaulay's theorem.

The recent remarkable proofs by Alswede and Khchaterian [2, 3] of Frankl's conjecture regarding the Erdös-Ko-Rado problem can be regarded as the application of a drastic form of combinatorial shifting. I feel that the combinatorial content of these proofs can be useful in further understanding the structure of shifted complexes (or equivalently of generic initial ideals). It would be of interest to find an algebraic proof which might be relevant to the following related, and yet unsolved, problem.

Problem 44 *(Erdös).* What is the maximal size of a family of k-subsets of $[n]$ which do not contain a matching of size r?

(A matching is a family of pairwise disjoint sets.)

6.9 Eigenvalues of laplacians, expansion of the dual graphs and shifting

The basic idea behind my paper [61] was the following: Use algebraic shifting to deduce expansion properties of the dual graph of simplicial polytopes and spheres and deduce upper bounds on the diameters of such graphs. At present, this idea only works for neighborly polytopes. It is possible that expansion properties on the dual graph of $\Delta(K)$ imply expansion properties on the dual graph of K (this is related to shifting Meyer-Vietoris).

In a recent paper, Duval and Reiner [33] studied the eigenvalues of laplacians of shifted complexes. The class of shifted simplicial complexes is one of only a handful of classes of complexes with integral Laplacian spectra.

Let s be the Laplacian eigenvalues of a shifted family of k-sets. Consider also d, the generalized degree sequence (number of k-sets each vertex is a member of), and let ^{T} means to take the partition conjugate (transpose the Ferrers diagram of the partition). Duval and Reiner showed that $s = d^T$. (For graphs this result was proved in the sixties by Kelmans and by others.)

Duval and Reiner further conjecture that for an *arbitrary* k-family, s is majorized by d^T. A natural question that arises is the following:

Problem 45 *(Duval and Reiner).* Find the effect of shifting on eigenvalues of laplacians.

In a different direction note that if G is a tree with n vertices then $\Delta(G)$ is always the same: star whose edges contain the vertex '1'. Labelled trees can, of course, be enumerated and some weighted extensions to d-dimensional complexes K on n vertices (so that $\Delta(K)$ is the pure simplicial complex whose d-faces are all sets containing '1') are also known [54]. Is it possible to find an appropriate weighted enumeration for the class of simplicial complexes K with a prescribed $\Delta(K)$?

7 Extensions

In this section we will consider several extensions of algebraic shifting. The first three subsections deal with areas in which algebraic shifting has not kept up with advances concerning f-vectors of simplicial complexes.

7.1 Symmetry

Problem 46. Find an appropriate notion of shifting for simplicial complexes with a group action.

The study of face rings has significant consequences for face numbers of Cohen-Macaulay complexes with symmetry (see [1] and [76]. p.119). However, we are not aware of any useful notion of algebraic shifting in this context.

Problem 47. Characterize f-vectors of simplicial complexes with a free Z_p action. More generally characterize f vectors of such complexes with prescribed Betti numbers. Alternatively, consider general Z_p actions and characterize the pair of f-vectors obtained by the complex and by the faces fixed by the action.

In this context it is worth mentioning an old-standing problem in algebraic topology which asserts that if we have a free action of Z_p^n on a manifold M then the sum of Betti numbers for that manifold must be at least 2^n (the sum of Betti numbers of an n-dimensional torus). Some related results were proven using commutative algebra considerations.

7.2 Balanced and completely balanced complexes

An additional area in which algebraic shifting has not kept with f-vector theory is balanced and completely balanced complexes [72]. It may be feasible to close the gap.

Problem 48. Extend algebraic shifting to balanced and completely balanced families. Characterize pairs of face numbers and Betti numbers for such complexes.

7.3 Shifting more general complexes

In several cases the combinatorial consequences of algebraic shifting have much greater generality than for simplicial complexes. Bjorner and Kalai [18] have shown that the characterization of face numbers of simplicial complexes with prescribed Betti numbers applies to polyhedral complexes and in fact even in much greater generality. Much of the picture concerning the upper and lower bound theorems and the g-theorem also extends to large classes of complexes. The case of cubical complexes is of particular interest.

Problem 49. 1. Extend the definitions of face algebras and algebraic shifting to polyhedral (and more general) complexes.

2. Find analogs to the definitions of face algebras and algebraic shifting for cubical complexes.

7.4 Other combinatorial objects

Together with Hélène Barcelo we considered the possibility of applying something similar to algebraic shifting to other combinatorial objects. The general framework is as follows: We have a class $\mathcal{P}$ of combinatorial objects defined on an underlying set (which is usually taken to be the set $[n] = \{1, 2, \ldots, n\}$). Thus, $\mathcal{P}$ can be the set of subsets of $[n]$, or the set of permutations on $[n]$, or the set of labelled trees with vertex set $[n]$, etc. We need an algebra which is generated *as an algebra* by n variables $x_1, x_2 \ldots, x_n$ and as a vector space has a basis (depending on the variables $x_1, x_2, \ldots, x_n$) in one-to-one correspondence with the elements of $\mathcal{P}$. Algebraic shifting is based on studying Grobner basis w.r.t. a new set of variables $y_1, y_2, \ldots y_n$ obtained from the x_i's by a generic linear transformation.

Thus we can use free Lie algebras (or rather their "square-free" part), or alternatively the Orlik-Solomon algebra, to "shift" families of permutations. And we can try to use the cohomology ring of the variety of flags fixed by a unipotent matrix of Jordan decomposition given by $\mathbf{P} \vdash n$ (see Barcelo [10]) to try to shift families of standard tableaux of a given shape.

It turned out that extending even the simplest properties of shifting can be quite difficult.

Problem 50. Algebraically shift families of permutations, tableaux, trees and partitions and find combinatorial applications.

8 Concluding Remarks

The relations between commutative algebra, algebraic topology and combinatorics have been at the heart of combinatorial commutative algebra since the first works by Richard Stanley which established this field of research. Algebraic shifting appears to be a useful tool for relating topological properties of simplicial complexes to commutative algebraic properties of their (various) face algebras and for extracting combinatorial consequences.

Since the basic connection with simplicial homology was first observed and applied [16], Art Duval has found further connections with algebraic topology [31, 32] and further combinatorial applications [28, 34]. Relating algebraic shifting and, more generally, the face algebras of simplicial complexes with advanced topics from algebraic topology is one of the main challenges.

The recent results by Dave Bayer, Hara Charalambous & Sorin Popescu and by Annette Aramova & Jürgen Herzog appears to give the ultimate extension of the fact that Cohen-Macaulayness is preserved under shifting (which is equivalent to Reisner's theorem in the symmetric case). These results have reached the limit of what was anticipated in my survey with Anders Björner concerning f-vectors and homology [17]. There are further significant developments concerning shifting (generic initial ideals) and commutative algebra mainly due to Aramova, Herzog, Takayuki Hibi and others [4, 5, 6, 7, 8, 9, 51] which have not been properly presented here.

In another direction, the work by Peter Schenzel, followed by the results of Isabella Novik [69] yielded substantial knowledge on shifting (in the symmetric case) for Buchsbaum's complexes including simplicial manifolds. There is a very beautiful emerging picture of algebraic shifting (and face algebras) of simplicial manifolds without boundary. For simplicial spheres and Gorenstein* complexes this picture already includes a far-reaching extension of (a generic version of) the Hard Lefschetz Theorem for toric varieties associated with simplicial polytopes. And further, it appears to extend to simplicial manifolds without boundary giving a deep shifting-theoretic interpretation of Poincaré duality and beyond i.e., to spaces in which certain obstructions for embeddability vanish locally and globally and to Witt spaces. Proving that this picture is the correct one is the most important open problem in this area and perhaps can be considered to be one of the main open problems in algebraic combinatorics. Progress in this direction will reveal profound connections between commutative algebra and topology.

Face algebras and algebraic shifting appears to be also tailor-made for certain problems in extremal combinatorics.

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