Some old and new problems in combinatorics and geometry

Gil Kalai
Hebrew University of Jerusalem

Erdős 100 Conference, Budapest, July 2013

Part 1: Around Borsuk's problem:

Let $f(d)$ be the smallest integer so that every set of diameter one in R^d can be covered by $f(d)$ sets of smaller diameter

Borsuk conjectured that $f(d) \le d+1$. It is known (Kahn and Kalai, 1993) that : $f(d) \ge 1.2^{\sqrt{d}}$, and also that (Schramm, 1989) $f(d) \le (\sqrt{3/2}+o(1))^d$.

A remaining problem

Is $f(d)$ exponential in d?

Best shot (in my opinion) for an example:
(a) Start with binary linear codes of length n (based on AG-codes) with the property that the number of maximal-weight codewords is exponential in n.
(b) Show that the code cannot be covered by less than exponential number of sets which do not realize the maximum distance.
Part (b) can be difficult.

Low dimensional counterexamples and another problem of Larman

The Borsuk conjecture is false for $n = 1305$ and all $n > 2014$ (Kahn, Kalai 1993)

... Gradual improvements...946 (A. Nilli) 561 (Raigorodski) 560 (Weissbach) 323 (Hinrichs), 321 (Pikhurko), 298 (Hinrichs and Richter).

Larman asked: Is the Borsuk conjecture correct for 2-distance sets?

Low dimensional counterexamples and another problem of Larman

The Borsuk conjecture is false for $n = 1305$ and all $n > 2014$ (Kahn, Kalai 1993)

... Gradual improvements...946 (A. Nilli) 561 (Raigorodski) 560 (Weissbach) 323 (Hinrichs), 321 (Pikhurko), 298 (Hinrichs and Richter).

Larman asked: Is the Borsuk conjecture correct for 2-distance sets?

(May 19, 2013) Andriy V. Bondarenko found a 2-distance set with 416 points in 65 dimensions that cannot be partitioned into less than 83 parts of smaller diameter. Remarkable!

Volumes of sets of constant width

Let us denote the Volume of the n-ball of radius $1/2$ by V_n.
Question (Oded Schramm): Is there some $\epsilon > 0$ so that for every $n > 1$ there exist a set K_n of constant width 1 in dimension n whose volume satisfies $VOL(K_n) \leq (1-\epsilon)^n V_n$.

(A negative answer for spherical sets will push the $(3/2)^{n/2}$ upper bound to $(4/3)^{n/2}$.)

Part 2: Around Tverberg's theorem

Tverberg's Theorem states the following: Let $x_1, x_2, \ldots, x_m$ be points in R^d with $m \geq (r-1)(d+1)+1$. Then there is a partition $S_1, S_2, \ldots, S_r$ of $\{1, 2, \ldots, m\}$ such that $\cap_{j=1}^{r} conv(x_i : i \in S_j) \neq \emptyset$. This was a conjecture by Birch who also proved the planar case.

The bound of $(r-1)(d+1)+1$ in the theorem is sharp as easily seen from configuration of points in sufficiently general position. The case $r = 2$ is Radon's theorem.

Reay's conjecture

Reay asked what is the smallest integer $R(d,r)$ such that If $x_1, x_2, \ldots, x_m$ be points in R^d with $m \geq R(d,r)$, then there is a partition $S_1, S_2, \ldots, S_r$ of $\{1, 2, \ldots, m\}$ such that $conv(x_i : i \in S_j) \cap conv(x_i : i \in S_k) \neq \emptyset$, for every $1 \leq j < k \leq r$. Reay conjectured that you cannot improve the value given by Tverberg's theorem, namely that $R(d,r) = (r-1)(d+1)+1$.

Micha A. Perles conjectures that Reay's conjecture is false even for $r = 3$ for large dimensions, but with Moria Sigron he proved the strongest positive results in the direction of Reay's conjecture.

My '74 conjecture

For a set A, denote by $T_r(A)$ those points in R^d which belong to the convex hull of r pairwise disjoint subsets of A. We call these points Tverberg points of order r.

Conjecture: For every $A \subset R^d$,

$$\sum_{r=1}^{|A|} \dim T_r(A) \geq 0.$$

(Note that $\dim \emptyset = -1$.) The conjecture was proved for $d \leq 2$ by Akiva Kadari (unpublished M. Sc thesis in Hebrew).

A question about directed graphs that can be described as the union of two trees

Let G be a directed graph with n vertices and $2n-2$ edges.

Question: When can you divide your set of edges into two trees T_1 and T_2 (so far we disregard the orientation of edges,) so that when you reverse the directions of all edges in T_2 you get a strongly connected digraph.

A question about directed graphs that can be described as the union of two trees

Let G be a directed graph with n vertices and $2n-2$ edges.

Question: When can you divide your set of edges into two trees T_1 and T_2 (so far we disregard the orientation of edges,) so that when you reverse the directions of all edges in T_2 you get a strongly connected digraph.

I conjectured that if G can be written as the union of two trees, the only additional obstruction is that there is a cut consisting only of two edges in reversed directions.

A counterexample by Maria and Paul

... But Maria Chudnovsky and Paul Seymour found a counterexample with five vertices, and some additional necessary conditions.

```
Here is another attempt at a counterexample;
hopefully we are doing the right problem now.
It's a 5-vertex graph, with vertices a1,a2,b1,b2,c,
and directed edges
ai->bj for all i,j b1->c,b2->c c->a1,c->a2
In fact, here is what seems to be another necessary
condition:
There is no induced cycle c1-...-c2k-c1 in G, s.t.
each ci is cubic, the edges of the cycle alternate in
direction, and none of c1,..,c2k are sources or sinks
of G.
```

Another old problem

How many points $T(d; s, t)$ in R^d guarantee that they can be divided into two parts so that every union of s convex sets containing the first part has a non empty intersection with every union of t convex sets containing the second part.

Part 3: Erdős-Ko-Rado meets Catalan

Conjecture: Let C be a collection of triangulations of an n-gon so that every two triangulations in C share a diagonal. Then $|C|$ is at most the number of triangulations of an $(n-1)$-gon.

Part 4: The $F \leq 4E$ conjecture

Theorem (follows from Euler's theorem): Let G be a simple planar graph with V vertices and E edges. $E \leq 3V$.

Now let K be a two-dimensional simplicial (or polyhedral) complex and suppose that K can be embedded in R^4 . Denote by E the number of edges of K and by F the number of 2-faces of K.
Here is a really great conjecture:
Conjecture:

$$F \leq 4E$$

A weaker version which is also widely open and very interesting is: For some absolute constant C,

$$F \leq CE$$

Part 5: The Polynomial Hirsch Conjecture

Conjecture: Let P be a d-dimensional polytope with n facets, then the diameter of the graph of P is at most polynomial in d and n.

Intermission 1: a question from analysis

Problem: Let K be a convex body in R^d. (Say, a ball, say a cube...) For which class $\mathcal{C}$ of functions, every $f \in \mathcal{C}$ which takes K into itself admits a fixed point in K.

Intermission 2: a question from number theory

Problem: Find a (not extremely artificial) set A of integers so that for every n, $|A \cap [n]| \leq n^{0.499}$, where you can prove that A contains infinitely many primes.

Problem: Find a (not extremely artificial) set A of integers so that for every n, $|A \cap [n]| \leq n^{0.499}$ where you can prove that

$$\sum \{\mu(k) : k \leq n, k \in A\} = o(|A \cap [n]|).$$

Intermission 3: a question about computation

Problem: Does a noisy version of Conway's game of life support universal computation?

Part 6: Ramsey's type conjecture for polytopes

Conjecture: For a fixed k, every d-polytope of sufficiently high dimension d contains a k-face which is either a simplex or a (combinatorial) cube.

Part 7: Threshold and Expectation threshold

Consider a random graph G in $G(n,p)$ and the graph property: G contains a copy of a specific graph H. (Note: H depends on n; a motivating example: H is a Hamiltonian cycle.) Let q be the minimal value for which the expected number of copies of H' in G is at least $1/2$ for every subgraph H' of H. Let p be the value for which the probability that G contains a copy of H is $1/2$.

Conjecture: [Kahn - Kalai 2006]

$$p/q = O(\log n).$$

The conjecture can be vastly extended to general Boolean functions, and there are possible connection with harmonic analysis and discrete isoperimetry.

Part 8: Traces

Problem: Let X be a family of subsets of $[n] = \{1, 2, \ldots, n\}$. How large X is needed to be so that the restriction (trace) of X to some set $B \subset [n]$, $|B| = (1/2 + \delta)n$ has at least $3/4 \cdot 2^{|B|}$ elements?

Kahn-Kalai-Linial (1988): $2^n/n^c$ for some constant $c > 0$ is ok.

Kahn Kalai (2013): $2^n/2^{n^\beta}$, for some $\beta < 1$ is not enough.

Bourgain (2013): $2^n/n^C$ is ok, for some $\delta = \delta(C)$.

Part 9: Graph-codes

Let P be a property of graphs. Let $\mathcal{G}$ be a collection of graphs with n vertices so that the symmetric difference of two graphs in $\mathcal{G}$ has property P. How large can $\mathcal{G}$ be.

Part 10: Graphs without induced cycles of length divisible by three.

A conjecture by Roy Meshulam and me (motivated by Helly-type theorems): There is a constant C such that every graph G with no induced cycles of order divisible by 3 is colorable by C colors.

Part 10: Graphs without induced cycles of length divisible by three.

A conjecture by Roy Meshulam and me (motivated by Helly-type theorems): There is a constant C such that every graph G with no induced cycles of order divisible by 3 is colorable by C colors.

Another conjecture by Roy Meshulam and me: For every $b > 0$ there is a constant $C = C(b)$ with the following property: Let G be a graph such that for all its induced subgraphs H

(*) The number of independent sets of odd size minus the number of independent sets of even size is between $-b$ and b.

Then G is colorable by C colors.

Paul Erdős' way with people and with mathematical problems

There is a saying in ancient Hebrew writings:

> *Do not scorn any person and do not discount any thing, for there is no person who has not his hour, and there is no thing that has not its place.*

Paul Erdős had an amazing way of practicing this saying, when it came to people, and when it came to his beloved "things" - mathematical problems. And his way accounts for some of our finest hours.

Thank you very much !