Neighborly embedded manifolds

G. Kalai* and A. Wigderson†

1 Introduction

An embedding of an n-dimensional (connected) manifold M into R^d is called *k-neighborly* if for every k points on the embedded manifold there is a hyperplane H in R^d which supports the manifold precisely at these points. Namely, H contains these k points and all other points of the embedded manifold are in the same open half space determined by H.

The moment curve $m(x) = (x, x^2, \ldots, x^d) \subset R^d$ is a $[d/2]$-neighborly embedding of R^1 into R^d, see, e.g., [14]. The trigonometric moment curve $(\cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos(kt), \sin(kt)) \subset R^d$, $d = 2k$ is an example of a k-neighborly embedding of S^1 in R^d.

Micha A. Perles [10] posed the following problem:

Problem 1 *What is the smallest dimension $d(k,n)$ of the ambient space in which a k-neighborly n-dimensional manifold exists?*

A k-neighborly embedding of any n manifolds M is also k-neighborly when we restrict to a submanifold of M and, in particular, to a small neighborhood of a point which is homeomorphic to R^n. Therefore for this problem we may assume that the manifold is R^n. Perles' problem is related to the problem of finding k-regular embeddings of manifolds into R^n. A map $f : X \to R^n$ is called *k-regular* if the images of every k distinct points in X are linearly independent. k-regular embeddings were studied in the context of approximation theory since the 50s, see e.g., [5, 7].

A simple dimension count shows that

$$d(k,n) \geq (k+1)n.$$

Indeed, let K be the convex hull of a k-neighborly embedded n-manifold M. We have a 1-1 map from every vector $(\alpha_1, \ldots, \alpha_k, x^1, \ldots, x^k)$, where $x^i \in M$, for every i and every α_i is a positive real number, $\sum \alpha_i = 1$, to the boundary of K which is $(d-1)$-dimensional. This implies that $d - 1 \geq kn + k - 1$.

However there is good evidence that this lower bound is never tight for $k > 1$. Vassiliev [13] considered a slightly stronger definition of k-neighborly C^2 embeddings, where the k-neighborliness is stable under C^2 perturbations, and showed, by an intricate topological argument, that the analogous function $d'(n,k)$ satisfies $d'(k,n) \geq 2kn - bin(n)$, where $bin(n)$ is the number of ones in the binary expansion of n. Related lower bounds for k-regular embeddings can be found in [5, 7].

A straightforward extension of the moment curve gives an upper bound for $d(k,n)$ which is exponential. For this purpose we use the embedding $\Phi_{k,n}$ defined as follows: Let $p_0 = 1, p_1, p_2, \ldots, p_d$ be all monomials of degree $\leq k$ in n variables $x_1, x_2, \ldots, x_n$. Embed R^n into R^d, by assigning to $x = (x_1, x_2, \ldots, x_n)$, the d-vector $(p_1(x), p_2(x), \ldots, p_d(x))$. It is easy to see that this embedding is $[k/2]$-neighborly. Here is the proof: Let $x^1, \ldots, x^m$ be $m = [k/2]$ points in R^n and let

$$P(x) = < x - x^1, x - x^1 > \cdot < x - x^2, x - x^2 > \cdots < x - x^m, x - x^m > .$$

*Institute of Mathematics, The Hebrew University of Jerusalem, and Departments of Computer Science and Mathematics, Yale University, kalai@math.huji.ac.il. Supported by NSF and BSF grants.

†Department of Mathematics, Institute for Advanced Studies, Princeton, avi@ias.edu.

$P(x)$ is a non-negative polynomial of n variables ($x = (x_1, x_2, \dots, x_n)$) of degree $2m$ which vanishes precisely on the points $x^1, \dots, x^m$. Therefore, it defines a hyperplane in R^d which supports the embedded R^n and touches it precisely at the given points. Convex hulls of the images of finite sets S of points in R^n under the map $\Phi(k, n)$ form an interesting class of convex polytopes that we briefly discuss in Section 3. However, $\Phi_{k,n}$ provides only upper bound on $d(k, n)$ which is exponential in k.

The main purpose of this paper is to present a simple construction showing a polynomial upper bound on $d(k, n)$.

Theorem 2 *There is a k-neighborly embedding of R^n into R^d for $d = 2k(k-1)n$.*

Our construction is based on hashing and is related to natural "continuous" analogs considered by Clark, McColm, and Shekhtman [4] of standard hashing problems.

An obvious remaining challenge is to close the gap between the lower and upper bounds for $d(n, k)$. A bound of the form $O(kn)$ or even $2kn$ may be realistic. Further problems and connections are discussed in Section 3. Let us mention here that 2-neighborly embeddings of S^3 into R^{17} occurs naturally as faces of the "mass ball" (the convex hull of the Grassmanian $G(n, m)$ embedded into the mth exterior power of R^n), see [8, 2].

2 Continuous hashing and the construction

Call a function $g : R^n \to (R^s)^r$, $g = (g_1, \dots, g_r)$ where each g_i maps R^n to R^s k-universal with parameters (n, s, r), if any k distinct points in R^n are mapped into k distinct points in R^s by at least one of the g_is, $i \in [r](= \{1, \dots, r\}$. Finding k-universal functions is a standard hashing problem here unusually situated in continuous domain, as opposed to the usual discrete one.

In order to prove Theorem 2 we require a beautiful construction of Clark, McColm, and Shekhtman [4] of k-universal family (consisting of linear functions) of parameters $(n, 1, r)$ for $r = n(k-1)$. In [4] a matching lower bound for smooth functions and a lower bound $r \geq [(n-1)k/2]$ is proved. To make the paper self contained, here is a simple construction for k-universal family of parameters $(n, 1, t)$ where $t = n\binom{k}{2} + 1$.

Set t distinct real numbers $\alpha_1, \alpha_2, \dots, \alpha_t$, and define

$$f_i(z_1, z_2, \dots, z_n) = \sum_{\ell=1}^{n} \alpha_i^{\ell} z_{\ell}.$$

To see that this gives a universal family consider k distinct points in R^n, $z^1, z^2, \dots, z^k$. For $j \in [k]$, define

$$P_j(\alpha) = \sum_{j=1}^{n} z_i^j \alpha^j.$$

Note that $f_i(z^j) = P_j(\alpha_i)$. For every $j_1, j_2 \in [k]$, P_{j_1} and P_{j_2} can agree on at most n points. Therefore, since $t > (n\binom{k}{2})$, there is $i \in [t]$ such that all $P_j(\alpha_i)$, $j \in [k]$, are different.

Remark: We thank Jirka Matousek for bringing [4] to our attention.

Proof of Theorem 2: We will now show that the existence of a k-universal function of parameters (n, s, r), implies that $d(n, k) \leq r \cdot d(s, k)$, since we can compose it with the k-neighborly solution for R^s on each of the r blocks separately. We will use this fact only for the case $s = 1$ that we now explain in more details.

Let g be a k-universal function of parameters $(n, 1, r)$ and consider the embedding that map a point $x = (x_1, x_2, \ldots, x_n)$ to the $(r \times 2k)$ matrix $M(x)$ whose (i,j)-entry is $g_i(x)^j$, $i \in [r]$ and $j \in [2k]$. First note that for every k points $x^1, x^2, \ldots, x^k$, the matrices $M(x^1)$, $M(x^2)$, $\ldots M(x^k)$ are linearly independent. This follows from the fact that there is a row i such that $g_i(x^\ell)$ are all distinct for $\ell \in [k]$ and therefore restricting to the first k columns, the ith rows of the k matrices correspond to the Vondermonde matrix and are linearly independent.

Next, from the neighborliness property of the moment curve it follows that for every k points $x^1, \ldots, x^k$ and every $\ell \in [r]$, there is an affine functional ρ_ℓ on the space of $(2k \times r)$ real matrices with the following properties:

1) ρ_ℓ depends only on the entries of the ℓth row.

2) ρ_ℓ is non negative on matrices whose mth row is of the form $(a, a^2, \ldots, a^{2k})$, and vanishes on such a vector a if and only if $a = (g_i^j(x^p))$ for some $p \in [k]$.

A positive linear combination of $\rho_1, \ldots, \rho_r$ will vanish only for x so that for every ℓ, $M_\ell(x)$ affinely depend on $M_\ell(x^1)$ $, \ldots, M_\ell(x^k)$. This implies that for every ℓ there is j so that $M_\ell(x)$ is equal to $M_\ell(x^j)$ but we showed that this is not the case unless $x = x^j$ for some j.

3 Discussion

We will briefly discuss Perles' question in the wider context of understanding convex hulls and order types for embedded manifolds.

The (affine) order type of a set X of points in R^d is a map from each $d+1$ ordered tuples $(x^0, \ldots, x^d)$ to $\{-1, 0, 1\}$ defined as the sign of the determinant of the $d+1$ by $d+1$ matrix whose i-th row is $(1, x^i)$. There is an extensive study of order types of finite set of points and the more general combinatorial notion of oriented matroids [3, 6]. The questions considered here can be regarded as studying the case that X is a manifold or a more general topological space rather than a finite set.

Perles' question can also be regarded as a question about the convex hull of embedded n-dimensional manifolds in R^d. The theory of convex polytopes ([14]) which studies convex hulls of finite sets of points is rather substantial, but there are only sparse results about convex hulls of embedded manifolds. An important example to keep in mind for this context is the convex hull of rank-one positive semi-definite matrices. (For this and other related examples see [1, 2].) A remarkable class of embeddings of the circle to R^4 was studied by Smilansky [12].

Problem 3 *What can be said about the face structure/order type of a convex embedding of R^1 and S^1 into R^d.*

Embedded manifolds give rise to interesting classes of polytopes which depend on choosing v points on the manifolds. The case of polytopes obtained via the embedding $\Phi_{k,n}$ is of particular interest. For example, the embedding $\Phi_{k,2}$ gives for each configuration C of v points in the plane a convex polytope in $R^{\binom{k+1}{2}-1}$. The image of v planar points under $\Phi_{2,2}$ is a configuration of points in R^5. The order type of this configuration of points record the location of an original point relative to the conic described by five other points.

We can consider also isometric embeddings:

Problem 4 *Are there k-neighborly isometric embeddings of R^r, T^r and S^r (equipped with the standard metric) for every $r > 1$. (Here, T^r is the r-dimensional torus.)*

We conclude with two problems which are further away from Perles' problem but still appear to be in the same general spirit. The next problem is joint with Chris Connell:

Problem 5 *Find the isometric embedding of T^r and S^r (equipped with the standard metric) with maximum volume of the convex hull.*

For $r = 1$ and d even, Schoenberg [11] showed that the maximum volume of the convex hull is attained by the trigonometric moment curve.

Finally, let K be an arbitrary convex body in R^n and let $F_r(P)$ be the space of r-dimensional faces of K.

Conjecture 6 *If all $F_r(K)$ are compact then*

$$\sum_{r=0}^{d-1} (-1)^r \chi(F_r(K)) = 1 + (-1)^{d-1}.$$

The case that K is a convex polytope is the Euler-Poincaré formula and the case that K is a smooth body is just the Euler-characteristic of a $(d-1)$-sphere. An extension of this conjecture to arbitrary convex bodies (giving up the compactness of the set of k-faces,) might be possible under an appropriate definition of Euler characteristic for the individual $F_r(K)$s. Results by Mulmuley [9] may be relevant.

References

[1] A. Barvinok, *A course in convexity.* Graduate Studies in Mathematics, 54. American Mathematical Society, Providence, RI, 2002.

[2] A. Barvinok and G. Blekherman, Convex geometry of orbits Combinatorial and Computational Geometry, MSRI Publications, 52, 2005, 51–77.

[3] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler, *Oriented matroids*, Encyclopedia of Mathematics and its Applications, 46. Cambridge University Press, Cambridge, 1993.

[4] E. W. Clark, G. L. McColm, and B. Shekhtman, An application of spanning trees to k-point separating families of functions, J. Lond. Math. Soc., II. Ser. 58, No.2, 297-310 (1998).

[5] F. R. Cohen and D. Handel, k-regular embeddings of the plane, Proc. AMS 72 (1978), 201-204.

[6] J. Goodman, and R. Pollack, Allowable sequences and order types in discrete and computational geometry, in *New trends in discrete and computational geometry*, 103–134, Algorithms Combin., 10, Springer, Berlin, 1993.

[7] D. Handel, $2k$-regular maps on smoth manifolds, Proc. AMS 124 (1996), 1609-1613.

[8] R. Harvey, and F. Morgan, The faces of the Grassmannian of three-planes in R^7 (calibrated geometries on R^7). Invent. Math. 83 (1986), 191–228.

[9] K. Mulmuley, A generalization of Dehn-Sommerville relations to simple stratified spaces, Discrete Comput. Geom. 9 (1993), 47–55.

[10] Micha A. Perles, Problems presented in Oberwolfach conference on "Convexity", 1982.

[11] J. Schoenberg, An isoperimetric inequality for closed curves convex in even-dimensional Euclidean spaces, Acta Math. 91 (1954), 143-164.

[12] Z. Smilansky, Convex hulls of generalized moment curves, *Israel J. Math.* 52 (1985), 115–128.

[13] V. A. Vassiliev, On r-neighbourly submanifolds in R^M, Topol. Methods Nonlinear Anal. 11 (1998), 273–281.

[14] G. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, 152. Springer-Verlag, New York, 1995.