

Quantum Computers: Noise Propagation and Adversarial Noise Models

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Abstract

In this paper we consider adversarial noise models that will fail quantum error correction and fault-tolerant quantum computation.

We describe known results regarding high-rate noise, sequential computation, and reversible noisy computation. We continue by discussing highly correlated noise and the “boundary,” in terms of correlation of errors, of the “threshold theorem.” Next, we draw a picture of adversarial forms of noise called (collectively) “detrimental noise.”

Detrimental noise is modeled after familiar properties of noise propagation. However, it can have various causes. We start by pointing out the difference between detrimental noise and standard noise models for two qubits and proceed to a discussion of highly entangled states, the rate of noise, and general noisy quantum systems.

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1 Introduction

The feasibility of computationally superior quantum computers is one of the most fascinating scientific problems of our time. The main concern regarding quantum-computer feasibility is that quantum systems are inherently noisy. The theory of quantum error correction and fault-tolerant quantum computation (FTQC) provides strong support for the possibility of building quantum computers. In this paper we will discuss adversarial noise models that may fail quantum computation.

This paper presents a critique of quantum error correction and skepticism on the feasibility of quantum computers. An early critique regarding noise and quantum computation (put forward in the mid-90s by Landauer [31, 32], Unruh [51], and others) was instrumental to the development of FTQC. Some of the ideas in the paper are provocative and speculative and they certainly **do not** express established scientific material. We will also make some deviations from standard notation regarding quantum operations. We will use ordinary function notation for quantum operations (superoperators). So when E is a quantum operation and ρ is a state (described in terms of a density matrix), we will denote by $E(\rho)$ the action of E on ρ .

The paper describes in part my research, and relies on three (closely related) discussion papers [22, 23, 24]. It has also benefited from several weblog discussions. Many colleagues contributed helpful comments, and allow me to single out Greg Kuperberg for his ongoing, patient adversarial partnership, and Daniel Lidar for a careful editing of an earlier version of the paper.

We will now describe the structure of the paper. Section 2 presents an example that can be regarded as a “role model” for adversarial noise. Section 3 describes the basic framework for noisy quantum computers and the “threshold theorem.” In Section 4 we describe several obstructions in terms of the noise to quantum computation. We discuss high-rate noise, reversible noisy computation, and sequential noisy computation. In Section 5 we discuss highly correlated noise and we also come back to the rate of noise. We

try to draw a line between the types of correlations to which the “threshold theorem” applies and those to which it cannot apply.

In Sections 6-8 we propose a hypothetical form of noise that we call “detrimental noise” that can cause quantum error correction and fault-tolerant quantum computing to fail. Detrimental noise is described via some (counter-intuitive) properties. Perhaps the simplest way to think about our proposed detrimental noise picture is to regard the characteristics of noise propagation as the fundamental properties of noise in quantum systems, and to note that these properties can have other causes. We discuss noise propagation in Section 6.

We describe detrimental noise starting with the case of two qubits (Section 7), consider error synchronization for systems with many highly entangled qubits, and discuss general open quantum systems (Section 8).

The conjectures toward the end of the paper can be regarded as proposed properties for noise models for quantum computers (and more general quantum systems) that will cause quantum error correction and FTQC to fail. Alternatively, the conjectures can be regarded as **consequences** of a lack of fault tolerance in quantum systems. As such, they can be relevant to the nature of decoherence of quantum physical systems in nature even if computationally superior quantum computers are possible.

In Section 9 we discuss some concerns regarding our conjectures and, in particular, what their cause can be, and we return to the issue of modeling the rate of noise. In Section 10 we discuss some further conceptual issues regarding noise and quantum computing.

2 Example first!

Consider a quantum memory with n qubits whose intended state is ρ_0 . Suppose that ρ_0 is a tensor product state. The noise affecting the memory can be described by a quantum operation E_0 . Let us suppose that E_0 acts inde-

pendently on different qubits and its action on the k th qubit is as follows: with some small probability p the noise changes the state of the qubit into the completely mixed state τ_k .

This depolarizing noise is a very simple form of noise that can be regarded as basic to the understanding of the standard models of noise as well as of detrimental noise.

In the standard model of noise E_0 describes the noise of the quantum memory regardless of the state ρ stored in the memory. This is quite a natural and indeed expected form of noise.

A detrimental noise will correspond to a scenario that, when the quantum memory is supposed to be in a state ρ and $\rho = U\rho_0$, the noise E will be UE_0U^{-1} . Such noise is the effect of first applying E_0 to ρ_0 and then applying U to the outcome noiselessly.

In reality we cannot perform U instantly and noiselessly and the most we can hope for is that ρ will be the result of a process. Our main conjecture is that for a noisy process intended to lead to $\rho = U\rho_0$ the noise will contain a component of the form $E = UE_0U^{-1}$.

Two remarks are in order: 1) The noise described by the quantum operation E depends on the evolution of the quantum computer leading to ρ . The dependence of E on the prior evolution is linear and there is nothing in this description that violates quantum mechanics linearity. In fact, this noise is a simple and familiar expression of noise propagation. The quantum computer whose intended state is ρ can be subject to a whole **envelope** $D(\rho)$ of possible noise operations depending on the evolution leading to ρ . The relation between $D(\rho)$ and ρ is nonlinear.

2) How can we claim that this example is damaging while noise propagation is successfully dealt with by fault-tolerant methods? We will discuss this question later.

3 Quantum computers, noise, fault tolerance, and the threshold theorem

3.1 Quantum computers

The state of a digital computer having n bits is a string of length n of zeros and ones. As a first step toward quantum computers we can consider (abstractly) stochastic versions of digital computers where the state is a (classical) probability distribution on all such strings. Quantum computers are similar to these (hypothetical) stochastic classical computers and they work on qubits (say, n of them). The state of a single qubit q is described by a unit vector $u = a|0\rangle + b|1\rangle$ in a two-dimensional complex space U_q . (The symbols $|0\rangle$ and $|1\rangle$ can be thought of as representing two elements of a basis in U_q .) We can think of the qubit q as representing ‘0’ with probability $|a|^2$ and ‘1’ with probability $|b|^2$. The state of the entire computer is a unit vector in the 2^n -dimensional tensor product of these vector spaces U_q ’s for the individual qubits. The state of the computer thus represents a probability distribution on the 2^n strings of length n of zeros and ones. The evolution of the quantum computer is via “gates.” Each gate g operates on k qubits, and we can assume $k \leq 2$. Every such gate represents a unitary operator on U_g , namely the (2^k -dimensional) tensor product of the spaces that correspond to these k qubits. At every “cycle time” a large number of gates acting on disjoint sets of qubits operate.

Moving from a qubit q to the probability distribution on ‘0’ and ‘1’ that it represents is called a “measurement” and it can be considered as an additional 1-qubit gate. We will assume that measurement of qubits that amount to a sampling of 0-1 strings according to the distribution that these qubits represent is the final step of the computation.

3.2 Noisy quantum computers

The postulate of noise asserting that quantum systems are inherently noisy is essentially a hypothesis about approximations. The state of a quantum computer can be prescribed only up to a certain error. For FTQC there is an important additional assumption on the noise, namely, on the nature of this approximation. The assumption is that the noise is “local.” This condition asserts that the way in which the state of the computer changes between computer steps is approximately statistically independent for different qubits. We will refer to such changes as “storage errors” or “qubit errors.” In addition, the gates that carry the computation itself are imperfect. We can suppose that every such gate involves a small number of qubits and that the gate’s imperfection can take an arbitrary form, and hence the errors (referred to as “gate errors”) created on the few qubits involved in a gate can be statistically dependent. We will denote as “fresh errors” to the storage errors and gate errors in one computer cycle. Of course, qubit errors and gate errors propagate along the computation. The “overall error” describing the gap between the intended state of the computer and its noisy state takes into account also the cumulated effect of errors from earlier computer cycles.

The basic picture we have of a noisy computer is that at any time during the computation we can approximate the state of each qubit only up to some small error term ϵ . Nevertheless, under the assumptions concerning the errors mentioned above, computation is possible. The noisy physical qubits allow the introduction of logical “protected” qubits that are essentially noiseless.

What does the error rate ϵ refer to? Perhaps the simplest way to think about it is as follows. If we measure a qubit (with respect to every basis of its Hilbert space) the outcome will agree with the same measurement for the intended state, with probability of at least $1 - \epsilon$. More formally, recall that the trace distance $D(\sigma, \rho)$ between two density matrices ρ and σ is equal to the maximum difference in the results of measuring ρ and σ in the same

basis. $D(\sigma, \rho) = 1/2\|\sigma - \rho\|_{tr}$. When the error is represented by a quantum operation E the rate of error for an individual qubit is the maximum over all possible states ρ of the qubit of the trace distance between ρ and $E(\rho)$.

For most of the paper we will consider the same model of quantum computers with more general notions of errors. We will study more general models for the fresh errors. (We will not distinguish between the different components of fresh errors, gate errors and storage errors.) Our models require that the storage errors not be statistically independent (on the contrary, they should be very dependent) or that the gate errors not be restricted to the qubits involved in the gates and be of sufficiently general form. (Note that the errors may also reflect the translation from this ideal notion of noisy quantum computers to a physical realization.)

There are several other models of quantum computers that are equivalent in terms of their computational power to the one described here. This equivalence does not extend automatically to noisy versions and exploring fault tolerance in noisy versions of these models is an important challenge in FTQC.

The discrete models of noisy quantum computers we discuss here are direct analogs to continuous-time models described, for example, via Lindblad's equations. There were some concerns raised by quantum computer skeptics that the crux of the matter is in the translation from continuous-time models to discrete-time models. (Those are referred to respectively as Hamiltonian modeling and phenomenological modeling in Alicki's chapter [11].) Namely, the concerns were that certain Hamiltonian models will lead to non-local fresh errors when translated to the discrete-time description. Such a possibility was first raised and studied by Alicki, Horodecki, Horodecki, and Horodecki [8]. Later, it was suggested that non-local behavior for the fresh errors can result from "slow gates" (see [9]), from "high frequency noise" (in the Hamiltonian model from [50]), and from "non-exponential tail" ([11]).

Recent extensions of the threshold theorem to cases that allow time- and

space-dependence [50, 12, 6, 38] start with continuous-time models.

Damaging models of noise of the kind we describe in this paper can be described also for the continuous-time case. (Under such models, continuity properties needed to move from continuous-time to discrete-time may also pose additional difficulties.)

3.3 The threshold theorem

The existence of fault-tolerant schemes turns the problem of building a quantum computer into a hard but possible-in-principle engineering problem: if we just manage to store our qubits and operate upon them in a level of noise below the fault-tolerance threshold, then we can perform arbitrary long quantum computations. — Kempe, Regev, Unger, and Wolf, 2008 [25].

Let \mathcal{D} be the following envelope of noise operations for the fresh errors: the envelope for storage errors \mathcal{D}_s will consist of quantum operations that have a tensor product structure over the individual qubits. The envelope for gate errors \mathcal{D}_g will consist of quantum operations that have a tensor product structure over all the gates involved in a single computer cycle (more precisely, over the Hilbert spaces representing the qubits in the gates). For a specific gate the noise can be an arbitrary quantum operation on the space representing the qubits involved in the gate. (The threshold theorem concerns a specific universal set of gates \mathcal{G} that is different in different versions of the theorem.)

Theorem 1 (Threshold theorem) [5, 26, 29] *Consider quantum circuits with a universal set of gates \mathcal{G} . A noisy quantum circuit with a set of gates \mathcal{G} and noise envelopes \mathcal{D}_s and \mathcal{D}_g is capable of effectively simulating an arbitrary noiseless quantum circuit, provided that the error rate for every computer cycle is below a certain threshold $\eta > 0$.*

Here is some further information regarding the threshold theorem:

1. At every computer cycle the specific error operation can be chosen from the noise envelope by an adversary. The adversary can make his choice based on the entire intended evolution and his own earlier choices.

Part of the fault-tolerance process is identifying an “error syndrome,” i.e., a set of faulty qubits.¹ Once this is done we can give the adversary even greater power to manipulate the faulty qubits in an arbitrary way.

2. If we are allowing a smaller numerical value for the threshold η we can even assume that the envelope for fresh errors will be fixed for the entire computation and will include at every computer cycle all possible gate errors (not just gates involved in this computer cycle).

3. Recent works [50, 12, 6, 38] show that the threshold theorem prevails if we allow certain space- and time-dependencies for the noise operations. For example, the quantum computer is described by a lattice in space and the fresh-noise envelope allows dependencies among qubits that are “close together.” A certain amount of dependence of the noise on the earlier evolution is also permitted.

4. The value of the threshold in original proofs of the threshold theorem was around $\eta = 10^{-6}$ and it has since been improved by at least one order of magnitude. There are several works showing that under various reasonable assumptions on the noise the value of the threshold can be pushed up further. Statistical properties of the noise, and certain biases, can be used to improve the threshold! (See, e.g., [14].) A breakthrough work by Knill [30] uses error-detection codes rather than error-correction codes and massive post-selection. This allows one to raise the value of η (based on numerical simulations) to 3%. (It also leads to substantially higher provable bounds [12].)

5. The threshold theorem relies on a supply of auxiliary fresh qubits called “ancillas.” Roughly speaking, they are needed to “cool” the system.

¹More precisely, the noise is measured in terms of the tensor product of Pauli operators. The faulty qubits come with a Pauli operator indicating the error.

See Section 4.2 below.

6. One of the basic properties of FTQC is that the overall error for a single physical qubit will be bounded above along the entire computation by a small factor times the rate of the fresh error.

7. A weak version of the threshold theorem was first proved by Shor [48] for the case where the error rate is $O(1/\log^c n)$, where n is the number of qubits and $c > 0$ is some constant. Quantum error correction pioneered by Shor himself [47] and by Steane [49] plays a crucial role in Shor's as well as all later FTQC schemes.

8. Most proofs of the threshold theorem use concatenation codes. A crucial observation that led to an improvement of Shor's result was that it is enough to have codes that deal well with a random set of faulty qubits. A different approach by Kitaev [26, 27] is closely related to "topological quantum computing." In addition to the reliance on quantum error correction the proofs of the threshold theorem also rely on a basic theorem of Kitaev and Solovay ([36], Appendix 3).

9. The overhead in terms of the number of additional qubits needed for fault tolerance is polylogarithmically in the number of qubits in the original circuit.

10. FTQC was extended to other models for quantum computers. (Let me just mention measurement-based models based on cluster states [37].) The case of adiabatic quantum computation is still open.

The threshold theorem was one of the most outstanding developments in the theory of computation towards the end of the last century. It is fair to say also that efforts to extend the scope of the theorem and to reduce the numerical value of the threshold in various situations that can be realistic have demonstrated substantial progress over the last decade. As the opening quotation of this section indicates, the threshold theorem may well be the basis for the construction of operating quantum computers, an achievement that in terms of scientific and technological significance can be compared to

the discovery of X-rays and their applications towards the first half of the 20th century and the construction of digital computers in the second half. If quantum computers cannot be constructed then the detailed understanding of the assumptions in the threshold theorem that fail will be a first-rate achievement and may also lead to important developments in the theory of computation and in physics.

4 Noisy obstructions

4.1 High-rate noise

There are several papers showing that if the error rate is large then FTQC fails. Both in positive and negative results, the threshold η is not a universal constant but depends on the specific assumptions on the noisy quantum computer. We will restrict our description to the case where the computer involves only 1- and 2-qubit gates and to depolarizing noise.

The first negative result of this kind was proved by Aharonov and Ben-Or [4]. They proved that a quantum computer in which every qubit is subject to depolarizing noise with probability 97% in every computer cycle can be simulated by a classical computer.

There are two basic strategies for negative results of this kind.

- Strategy 1: After a logarithmic depth computation we will not be able to distinguish the noisy output from a random output.

Two recent papers in this direction are by Razborov [45], who showed that FTQC fails when the amount of depolarizing noise exceeds 50%. Kempe, Regev, Unger, and Wolf [25] managed to reduce this bound to 35.7%.²

²For circuits with unitary k -qubit gates, Razborov shows an upper bound of $1 - \Theta(1/k)$ and Kempe et al. improve it to $1 - \Theta(1/\sqrt{k})$. (Razborov's model is somewhat more general.)

Razborov's proof (which follows some ideas from [7] mentioned below) is based on tracking the trace distance of the intended state to the noisy state of the computer. New ingredients from [25] express the effect of depolarizing noise in terms of multi-Pauli operators and consider the *Frobenius distance* instead of the trace distance.

- Strategy 2: Efficiently simulate the noisy quantum computation by a classical computer.

The most recent paper in this direction is by Buhrman, Cleve, Laurent, Linden, Schrijver, and Unger [17]. They showed that a quantum circuit cannot be made fault-tolerant against a depolarizing noise level of 45.3%. Their model allows perfect gates from the Clifford group and additional noisy one-qubit gates. (For this particular model they show that a lower level of noise allows universal quantum computing!) While the computational complexity conclusion of this strategy is stronger, typically it applies to more restricted models (in terms of the set of gates).

4.2 Sequential computation and reversible computation

We will now describe two additional early negative results regarding fault tolerance.

The first result by Aharonov and Ben-Or [4] asserts that sequential noisy quantum computers can be simulated by classical computers. This result shows that the computational power of decohered quantum computers depends strongly on the amount of parallelism in the computation. A computation on the model of noisy quantum circuits with the additional assumption that at every round only a single gate is applied can be simulated classically.

The second result is by Aharonov, Ben-Or, Impagliazzo, and Nisan [7], who proved that the computational power of noisy **reversible** quantum computers reduces to log-depth quantum computation. The proof follows the

physics intuition that without a cooling mechanism the increase in entropy will eventually make computation impossible. A similar result is proved for classical computation.³

5 Highly correlated noise

Objection: *Coding does not protect against highly correlated errors.*

Response: *Correlated errors can be suppressed with suitable machine architecture.* —John Preskill, Quantum Computing: Pro and Con, 1996 [40].

5.1 The error syndrome and error synchronization

The concern regarding highly correlated noise has been raised in several papers, yet there have been only a few systematic attempts to study what kind of correlated errors will cause the threshold theorem to fail.⁴

Error synchronization refers to a situation where, while the expected number of qubit errors is small, there is a substantial probability of errors affecting a large fraction of qubits.

A simple way to describe error synchronization is via the expansion of the quantum operation E in terms of multi-Pauli operators. A quantum operation E can be expressed as a linear combination

$$E = \sum v^I P_I,$$

³One of the interesting aspects of this paper is a beautiful extension to the quantum case of an entropy inequality by Shearer [20].

⁴Of course, everyone has always known that the threshold theorem will fail for some noise models, e.g., it's hard to protect your quantum computer (or digital computer for that matter) from a meteor strike. But such models were considered as uninteresting and unrealistic.

where I is a multi-index i_1, i_2, \dots, i_n , where $i_k \in \{0, 1, 2, 3\}$ for every k , v^I is a vector, and P^I is the quantum operation that corresponds to the tensor product of Pauli operators whose action on the individual qubits is described by the multi-index I . The amount of error on the k th qubit is described by $\sum\{\|v^I\|_2^2 : i_k \neq 0\}$. For a multi-index I define $|I| := |\{k : i_k \neq 0\}|$. Let

$$f(s) := \sum\{\|v^I\|_2^2 : |I| = s\}.$$

We regard $\sum_{s=1}^n f(s)s$ as the **expected amount of qubit errors**.

Measuring the noise in terms of tensor product of Pauli operators is an important ingredient of several fault-tolerance schemes. Such a measurement leads to a word w of length n in the letters $\{I, X, Y, Z\}$, called the **error syndrome**. We will define the **coarse error syndrome** as the binary word of length n obtained from w by replacing I with '0' and the other letters by '1'. Given a noise operation E , the distribution \mathcal{E} of the error syndrome is an important feature of the noise. Given E we will denote by \mathcal{D} the probability distribution of coarse error syndrome. $f(s)$ is simply the probability of a word drawn according \mathcal{D} having s '1's.

Suppose that the expected amount of qubit errors is αn where n is the number of qubits.

All noise models studied in the original papers of the “threshold theorem,” as well as some extensions that allow time- and space-dependencies (e.g., [50, 12, 6]), have the property that $f(s)$ decays exponentially (with n) for $s = (\alpha + \epsilon)n$, where $\epsilon > 0$ is any fixed real number. (This is particularly simple when we consider storage error, which is statistically independent over different qubits.)

In contrast, we say that E leads to **error synchronization** if $f(\geq s)$ is substantial for some $s \gg \alpha n$. We say that E leads to a *very strong* error synchronization if $f(\geq s)$ is substantial for $s = 3/4 - \delta$ where $\delta = o(1)$ as n tends to infinity. By “substantial” we mean larger than some absolute constant times α/s , or, in other words, the multi-Pauli terms for $|I| \geq s$ contributes a constant fraction of the expected amount of qubit errors.

5.2 Examples and models

Proposition 2 *Conditioning on the expected number αn of qubit errors, a random unitary operator acting on all the qubits of the computer yields a very strong error synchronization.*

The proposition extends to the case where we allow additional qubits representing the environment.

The proof of Proposition 2 is based on a standard “concentration of measure” argument (see, e.g., [33]). (We will give only a rough sketch.) When we consider a typical expression of the form $\sum a_I P^I$ where $\sum a_I^2 = 1$ and $\sum \{a_I^2 |I|\} = \alpha n$, it will have a large support on $a_{\mathbf{0}}$ and the other coefficients will be supported on a_I where I itself is typical, i.e., I (the error syndrome) behaves like a random string of length n with entries I,X,Y,Z. Hence $|I|$ is around $(3/4)n$.

How relevant is Proposition 2? It is well known that random unitary operations on the entire 2^n -dimensional vector space describing the state of the computer are not “realistic” (in other words, not “physical” or not “local”). The best formal explanation why random unitary operators are “not physical” is actually computational and relies on the following lemma.

Lemma 3 *For large n , it is impossible to express or even to approximate a random unitary operator using a polynomial-size quantum circuit with gates of bounded fanning (namely, gates that operate on a bounded number of qubits).*

An interesting problem (posed in [24]) is to what extent we can describe the basic statistical properties of a random unitary operation U , conditioned on the value of $a(U)$, as the outcome of simple polynomial-size quantum circuits. As it turns out, there are various other reasons arising from quantum algorithms to seek computationally feasible unitary operators that resemble the behavior of random unitary operators.

Klesse and Frank [28] described a physical system in which qubits (spins) are coupled to a bath of massless bosons and they reached (after certain simplifications) a noise model with error synchronization.⁵

5.3 The boundary of the threshold theorem

For a quantum operation E describing the noise for a quantum computer with n qubits we denote by $\alpha(E)$ the expected number of qubit errors in terms of the multi-Pauli expansion as described above.

Proposition 4 *For the known noise models (e.g., [50, 12, 6]) that allow FTQC via the threshold theorem:*

- 1) *The fresh noise E expanded in terms of multi-Pauli operations decays exponentially above $\alpha(E)$.*
- 2) *The overall (cumulated) noise E' expanded in terms of multi-Pauli operations decays exponentially above $\alpha(E')$.*

There is an even simpler property of fresh and cumulated noise for noise models for which the threshold theorem holds.

Proposition 5 *For the known noise models (e.g., [50, 12, 6]) that allow FTQC via the threshold theorem:*

- 3) *The fresh noise (at every computer cycle) for almost every pair of qubits in the computer is almost statistically independent for the two qubits in the pair.*
- 4) *The overall noise for almost every pair of qubits in the computer is almost statistically independent for the two qubits in the pair.*

Here when we talk about “almost every pair” we refer to $(1 - o(1))\binom{n}{2}$ of the pairs when n is large.

⁵On the other hand, Shabani [46] argues that in certain cases correlated errors can lead to better performance of quantum codes.

The error syndrome will provide a simple way to express correlation between the noise acting on two qubits. For two qubits i and j , denote by $cor_{ij}(E)$ the correlation between the events that the qubit 'i' is faulty and the event that the qubit 'j' is faulty. In other words, $cor_{ij}(E)$ is the correlation between the events that w_i is not I , and w_j is not I when w is a word drawn according to the distribution of error syndromes described by E . Proposition 5 implies, in particular, that for models allowing the threshold theorem, $cor_{ij}(E)$ and $cor_{ij}(E')$ are close to 0 for most pairs i, j of qubits. (Another simple way to formulate approximate independence is in terms of the trace distance between the noise operation restricted to two qubits from a tensor product operation on these two qubits.) We will further discuss two-qubit behavior in the next section.

Note that properties 1 and 3 refer to the noise model, which is one of the assumptions for the threshold theorem, while properties 2 and 4 are consequences of the threshold theorem and, in particular, of suppressing error propagation. For the very basic noise models where the storage errors are statistically independent property 3 follows from the fact that the number of pairs of interacting qubits at each computer cycle is at most linear in n . Property 3 continues to hold for models that allow decay of correlations between qubit errors that depend on the (geometric) distance between them. Property 1 is a simple consequence of the independence (or locality) assumptions on the noise for noise models that allow the threshold theorem.

5.4 The rate of highly correlated noise

Highly correlated errors are bad for quantum error correction, but a potentially even more damaging property we face for highly correlated noise is that the notion of "rate of noise for individual qubits" becomes sharply different from the rate of noise as measured by trace distance for the entire Hilbert space describing the state of the computer.

Consider two extreme scenarios. In the first scenario, for a time interval

of length t there is a depolarizing storage noise that hits every qubit with probability pt . In the second scenario the noise is highly correlated: all qubits are hit with probability pt and with probability $(1 - pt)$ nothing happens. In terms of the expected number of qubit errors both these noises represent the same rate. The probability of every qubit being corrupted at a time interval of length t is pt . However, in terms of trace distance (and here we must assume that t is very small), the rate of the correlated noise is n^{-1} times that of the uncorrelated noise. What should be the correct assumption for the rate of noise when we move away from the statistical independence assumption?

Consider now our first example where for an intended state $\rho = U\rho_0$ the noise is described by UE_0U^{-1} . Since conjugation by a unitary operator preserves trace distance, the rate of noise in terms of trace distance will not depend on U . However, the rate of noise in terms of the expected number of qubit errors can be much greater. If the unitary operation U that describes the computation is “complicated enough” that UE_0U^{-1} is highly synchronized, we can even expect a situation where the number of qubit errors increases linearly with the number of qubits.⁶

6 Noise propagation

6.1 Noise propagation as a role model

The basic insight of fault-tolerant quantum computing is that if the incremental errors are standard and sufficiently small then we can make sure that the accumulated errors are too.⁷

⁶In Section 7 we will propose (and define formally) “highly entangled” states as those states that are necessarily “complicated enough.”

⁷By “standard” we refer to the assumptions that qubits errors are independent and that gate errors are confined to the Hilbert space describing the qubits of the gates and are independent for different gates. As we already mentioned these assumptions can be

The main issue is therefore to understand and describe the fresh (or infinitesimal) noise operations. The adversarial models we consider here should be regarded as models for fresh noise. But the behavior of accumulative errors in quantum circuits that allow error propagation is sort of a “role model” for our models of fresh noise.

The common picture of FTQC asserts:

- Fault tolerance will work if we are able to reduce the fresh gate/qubit errors to below a certain threshold. In this case error propagation will be suppressed.

What we propose is:

- Fault tolerance will not work because the overall error will behave like accumulated errors for standard error propagation (for circuits that allow error propagation), although **not necessarily** because of error propagation.

Therefore, for an appropriate modeling of noisy quantum computers the fresh errors should behave like accumulated errors for standard error propagation (for circuits that allow error propagation).

(As a result, in the end we will not be able to avoid error propagation.)⁸

Suppose that in your quantum computer at some period along the computation, you have two qubits (say, two photons) that are entangled. (This entanglement was created along the computation and we expect further changes in the joint state of these two qubits.) The entanglement between the two qubits is the result of a chain of gates acting on the computer’s qubits and if

widened and we can regard as “standard” those operations satisfying properties 1 and 3 in Propositions 4 and 5.

⁸On the face of it, this alternative description looks less natural than the common one. The main reason to examine it is in view of the extraordinary consequences of the common description.

error propagation cannot be suppressed we can expect that the accumulated errors for these two qubits will be correlated. But there are other reasons for correlation between the errors. The device may lead to such a correlation in order to make future interaction between the qubits possible. Even if the device does not induce such a correlation but pairs of qubits are postselected according to the interaction, such a postselection may induce correlation between the errors.

The conjectures of this paper amount to saying that noise propagation is the fundamental property of noisy quantum systems and that we need to identify the basic mathematical properties of noise propagation and use them in modeling noisy quantum computers or noisy quantum systems.

6.2 Forcing noise propagation

A way to force noise propagation into the model is as follows. Let K is a positive continuous function on $[0,1]$. We write $\bar{K}(t) = \int_0^t K(s)ds$ and assume $\bar{K}(1) = 1$. Start with an ideal quantum evolution $\rho_t : 0 \leq t \leq 1$ and suppose that $U_{s,t}$ denotes the unitary operator describing the transformation from time s to time t , ($s < t$). Now consider a noisy version with E_t be a noise operation describing the infinitesimal noise at time t . Now replace E_t by

$$E'_t = (1/\bar{K}(t)) \cdot \int_0^t K(t-s)U_{s,t}E_sU_{s,t}^{-1}ds. \quad (1)$$

Relation (1) represents some sort of smoothing of the noise operator in time. If E_t represents standard (local) noise operations for noisy quantum computers then E'_t will be similar, to some extent, to our example from Section 2. (See below for a discrete-time version of equation (1).)

Main Conjecture: Relation (1) properly models natural noisy quantum systems, and will not allow quantum fault tolerance.

For the rest of the paper we will restrict somewhat the class of noise operators and we will suppose that E_t and hence E'_t are described by POVM-measurements (see [36], Chapter 2).

Definition: Detrimental noise refers to noise (described by a POVM-measurement) that can be described by equation (1).

What could be a motivation for our main conjecture? we will mention four reasons:

1) For modeling systems we encounter in nature there is no noticeable difference between relation (1) and the standard description of noisy quantum evolution.

2) Regardless of the feasibility of quantum computers, noise propagation appears to be the rule for open quantum system in nature. Therefore, relation (1) should allow modeling information leaks for quantum systems in nature.

3) If FTQC is not possible by whatever fundamental principle, the conclusion is that noise propagation cannot be avoided. If noise propagation is a consequence of any hypothetical fundamental principle that would cause FTQC to fail, we may as well consider noise propagation as such a fundamental principle.

4) It is expected that the main conjecture will have interesting mathematical consequences leading to a coherent picture.

Let me elaborate on the first point. For modeling systems encountered in nature the standard noise models suffice in the following sense: probing the noise in short time intervals is difficult and the outcomes may be ambiguous.⁹ For longer time periods, standard noise models are sufficient to describe noisy systems that allow noise propagation because moving the incremental noise in time will have a similar effect to introducing non-standard noise of the kinds we propose.

⁹Knowing the intended state and the noisy state is not enough to determine the noise operation uniquely. In addition, we also lose information by measuring the noisy state.

More formally, we can try to approximate the evolution described by (1) by defining

$$E_t'' = 1/(1 - \bar{K}(t)) \cdot \int_t^1 E_s K(s - t) ds. \quad (2)$$

For noisy quantum computers if E_t represents a standard noise operators for every t then so does E_t'' . (But not E_t' .) Moreover, for systems that do not involve fault tolerance a noisy system with the standard noise E'' will give a good approximation to the system described by the non-standard noise E' .

We can replace relation (1) by a discrete time description. When we consider a quantum computer that runs T computer cycles, we start with standard storage noise E_t for the t -step. Then we consider instead the noise operator

$$E_t' = 1/(\sum_{s=1}^t K(s/T)) \cdot \sum_{s=1}^t K((s - t)/T) U_{s,t} E_t U_{s,t}^{-1}, \quad (3)$$

where again $U_{s,t}$ is the intended unitary operation between step s and step t .

Remark: Since we do not witness quantum error correction in nature, understanding the behavior of noisy quantum systems where noise propagation is “forced” can be of interest not just in the context of quantum-computer skepticism. Another possibility to force noise propagation is to consider the properties of random quantum circuits leading to a given state ρ . It will also be interesting to examine whether the model of noisy adiabatic computers (see [19]) satisfies our main conjecture.

7 Detrimental noise

7.1 Two conjectures

We can fight entanglement with entanglement. — John Preskill, Reliable Quantum Computers, 1998 [41].

In this subsection we present qualitative statements of two conjectures concerning decoherence for quantum computers which, if (or when) true, are damaging for quantum error correction and fault tolerance.

The first conjecture concerns entangled pairs of qubits.

Conjecture A: A noisy quantum computer is subject to error with the property that information leaks for two substantially entangled qubits have a substantial positive correlation.

We emphasize that Conjecture A refers to part of the overall error affecting a noisy quantum computer. Other forms of errors and, in particular, errors consistent with current noise models may also be present.

Recall that error synchronization refers to a situation where, although the error rate is small, there is nevertheless a substantial probability that errors will affect a large fraction of qubits.

Conjecture B: In any noisy quantum computer in a highly entangled state there will be a strong effect of error synchronization.

We should informally explain already at this point why these conjectures, if true, are damaging. We start with Conjecture B. The states of quantum computers that apply error-correcting codes needed for FTQC are highly entangled (by any formal definition of “high entanglement”). Conjecture B will imply that at every computer cycle there will be a small but substantial probability that the number of faulty qubits will be much larger than the threshold.¹⁰ This is in contrast to standard assumptions that the probability of the number of faulty qubits being much larger than the threshold decreases exponentially with the number of qubits. Having a small but substantial probability of a large number of qubits to be faulty is enough to fail the quantum error correction codes.

¹⁰Here we continue to assume that the probability of a qubit being faulty is small for every computer cycle.

We move now to Conjecture A. Let us first make the assumption that individual qubits can be measured without inducing errors on other qubits. This is a standard assumption regarding noisy quantum computers. When we start from highly entangled states needed for FTQC and measure (and look at the results for) all but two qubits, we will reach pairs of qubits (whose intended state is pure) with almost statistically independent noise, in contrast to Conjecture A. Under this assumption it is also possible to deduce Conjecture B from Conjecture A. Without making such assumptions on measurement, Conjecture A as stated above is not damaging, and we will need to extend Conjecture A to disjoint blocks of qubits.

7.2 Two qubits and two blocks of qubits

In this subsection we will describe a mathematical formulation of Conjectures A and B.

The first step in this formal definition is to restrict our attention to noise described by POVM-measurements. This is a large class of quantum operations describing information leaks from the quantum computer to the environment.

Our setting is as follows. Let ρ be the intended (“ideal”) state of the computer and consider two qubits a and b . Consider a POVM-measurement E representing the noise. We describe correlation between the qubit errors via the expansion in tensor products of Pauli operators, or, in other words, by the error syndrome.

Associated to E (see Section 5.1) is a distribution $\mathcal{E}(E)$ of error syndromes, i.e., words of length n in the alphabet $\{I, X, Y, Z\}$. A coarser distribution $\mathcal{D}(E)$ of binary strings of length n is obtained by replacing the letter I with '0' and all other letters by '1'.

As a measure of correlation $cor_{i,j}(E)$ between information leaks for the i th and j th qubit we will simply take the correlation between the events $x_i = 1$ and $x_j = 1$ according to $\mathcal{D}(E)$. When we have two disjoint sets of

qubits X and Y we will denote by $cor_{X,Y}(E)$ the correlation between the distributions \mathcal{D}_X and \mathcal{D}_Y , namely, the correlation between the distributions of coarse error syndromes on these two sets of qubits.

We also define $r_i(E)$ as the probability that $x_i = 1$ according to the distribution \mathcal{D} . We let $r_X(E)$ be the average of $r_i(E)$ for $i \in X$. (To start with, assume that $r_i(X)$ is small for every i .)

Here and below, $S(*)$ is the (von Neumann) entropy function; see, e.g., [36], Ch. 11. For a set Z of qubits and a state ρ we denote by $\rho|_Z$ the density matrix obtained after tracing out the qubits not in Z . If Z contains only the i th qubit, we write ρ_i instead of $\rho|_Z$.

Suppose that ρ is the intended state of the computer, consider two disjoint sets of qubits X and Y , let $Z = X \cup Y$, and suppose that the joint state $\rho|_Z$ is pure (for example, this is the case when Z is the set of all qubits). The entropy function $S(\rho|_X)$ is a standard measure of entanglement between the state ρ on X and on Y . (Recall that in this case $S(X) = S(Y)$.) In particular, ρ is a tensor product state iff the restriction of ρ to X is pure, hence $S(\rho|_X) = 0$.

Here is the statement of Conjecture A for two qubits whose intended state is pure and an extension to two blocks of qubits.

Conjecture A: (mathematical formulation)

(1)(For two qubits in intended joint pure state.) Suppose that the intended state ρ restricted to $Z = \{i, j\}$ is pure.

$$cor_{i,j}(E) \geq K(r_i(E), r_j(E)) \cdot S(\rho_i). \quad (4)$$

(2) (For two disjoint blocks of qubits.) Let X and Y be two disjoint sets of qubits whose intended joint state ρ is pure:

$$cor_{X,Y}(E) \geq K(r_X(E), r_Y(E))(\min |X|, |Y|)^{-1} S(\rho|_X). \quad (5)$$

Here, $K(x, y)$ is a function of x and y so that $K(x, y)/\min(x, y)^2 \gg 1$ when x and y are positive and small. (Note that Conjecture A(1) does not claim anything when the two qubits are noiseless.) If $r_i(E) = r_j(E) = \alpha$ for a small real number α , then the conjecture asserts that $cor_{i,j}(E) \gg \alpha^2$, and, as we will see later, this is what is needed to derive error synchronization.

The main mathematical challenge is to show that Conjecture A is satisfied when we force noise propagation, for example, via relation (1).

Main mathematical conjecture: The assertions of Conjecture A are satisfied for noisy quantum computers where the noise is described by equation (1).

It will be interesting to check whether the assertion of Conjectures A and B holds for noisy adiabatic computers and also for our very first example from Section 2.

We mention a second mathematical conjecture related to Section 5.4.

Second mathematical conjecture: For noisy quantum computers described by relation (1), the rate of fresh noise in terms of the expected number of faulty qubits scales up linearly with the number of qubits if the intended state is highly entangled.

Remarks

1) As an alternative measure of entanglement we can simply take the trace distance between the state induced on the two qubits (or, more generally, two disjoint sets of qubits) and a separable state. Formally, let $SEP(A, B)$ denote the set of mixed separable states on $A \cup B$, namely, states that are mixtures of tensor product pure states $\tau = \tau_A \otimes \tau_B$. Define $Ent(\rho : A, B) = \max\{\|\rho_{A,B} - \psi\| : \psi \in SEP(A, B)\}$.

2) We will use only Conjecture A for the cases where the intended joint state is pure. The conjecture itself extends to the case where the intended joint state is not pure. If we use the trace distance from a separable state

as the measure for entanglement then the conjecture carries over without change. If we want to use an entropy-based measure we can use the minimum of the relative entropy $S(\rho|_{X \cup Y} \|\psi)$ over all $\psi \in SEP(A, B)$.

7.3 Why are the conjectures damaging?

We already described why error synchronization fails current methods for fault tolerance. We need to describe formally Conjecture B and explain why Conjecture A implies Conjecture B.

Proposition 6 *Let $\eta < 1/20$ and $s > 4\eta$. Suppose that \mathcal{D} is a distribution of 0-1 strings of length n such that $p_i(\mathcal{D}) \geq \eta$ and $c_{ij}(\mathcal{D}) \geq s$. Then*

$$\mathbf{Prob}\left(\sum_{i=1}^n x_i > sn/2\right) > s\eta/4. \quad (6)$$

The proof of this proposition is indicated in [22] and we expect that a similar argument will also yield:

Proposition 7 *Let \mathcal{D} be a probability distribution on 0-1 strings of length n . Suppose that for a random partition of the bits into two parts X and Y , the expected value of the correlation satisfies:*

$$\mathbf{E}(\text{cor}\mathcal{D}(X, Y)) \geq s.$$

Then

$$\mathbf{Prob}\left(\sum_{i=1}^n x_i > sn/2\right) > s\eta/4. \quad (7)$$

We can now state formally also Conjecture B. The notion of “highly entangled state” in Conjecture B can be taken as a state for which when we partition the qubits into two parts at random the expected amount of entanglement between the two parts is large. This is indeed the case for

states used for error correction. With this definition, Proposition 7 asserts that Conjecture A(2) (for disjoint blocks of qubits) implies Conjecture B.

Remark: The following critique of the possibility of any systematic damaging relation between the state of the quantum computer and the noise was raised by several people. Having a classical computer control a quantum computer makes it possible to run a variant of any quantum computer program where at the initial state we apply random Pauli operators on every qubit and modify the action of the gates accordingly. In this way the state of the quantum computer will always be the same mixed state for the entire computation. A detailed proof of such a result along with an interesting interpretation and discussion was offered by Dorit Aharonov [3]. (Her work extends and relies on earlier works by Preskill, Shor and Ben-Or.)

A response to this critique is based on the following point made by Aharonov in the same paper. Consider the qubits of the mixed-state quantum computer, together with the qubits (which are simply random bits) of the computer that controls its state, as a single larger pure-state quantum computer. We assume that the quantum qubits are noisy but the control classical bits are noiseless. Then (with very high probability) there will be a strong entanglement when we partition all the qubits into two parts. Conjecture A will imply that a large correlation between information leak for the two parts. Now we can apply a variant of Proposition 7 to deduce strong error synchronization for the noisy quantum qubits and hence the failure of FTQC.

We conclude this subsection with a description of another avenue ([23, 22]), which goes from the two-qubit case of Conjecture A (extended in another direction) to Conjecture B. This goes through a notion of “emergent entanglement.” The emergent entanglement of two qubits is the maximum expected amount of entanglement between two qubits when the other qubits are measured (separably) and we look at the outcome of the measurements. (This is a less drastic notion than the definition in [22, 23], which appears to

be too strong.)

In standard noise models for quantum computers, measuring and looking at the results for all but two qubits of the computer will not affect the errors on these two qubits. We can define a highly entangled state as a state where the expected emergent entanglement among pairs is large. This is the case for states used in quantum error correction. A strong form of Conjecture A is obtained if we take emergent entanglement as the measure of entanglement. Using Proposition 6, this strong form of Conjecture A for pairs of qubits implies Conjecture B.

7.4 Censorship

The conjectures regarding noisy quantum computers and error synchronization are rather counterintuitive. The possibility that when the state of the quantum computer is highly entangled then for the period of time when the probability of every qubit being corrupted is very small there will still be a substantial probability of a large fraction of faulty qubits seems strange. One comment is that the argument will be to some extent counterfactual and that these properties of noise will imply severe restrictions on feasible states of noisy quantum computers. The counterintuitive forms of noise will occur for infeasible states. (Yet the conjectures on the nature of noise can be tested on feasible states.)

Computational complexity poses severe restrictions on the feasible states of (noiseless) quantum computers. For example, as we already mentioned, a state that is approximately the outcome of a random unitary operator on the entire 2^n -dimensional Hilbert space is computationally out of reach when the number of qubits is large.

Adversarial forms of noise may lead to further restrictions on feasible states for noisy quantum computers. Here is a specific conjecture in this direction (partially responding to a challenge posed by Aaronson in [1].) We assume that the “ideal” state of the quantum computer (before the noise is

applied) is a pure state. (Some adjustment to our conjecture will be required when the ideal state itself is a mixed state.)

Let ρ be a pure state on a set $A = \{a_1, a_2, \dots, a_n\}$ of n qubits. Define

$$ENT(\rho; A) = -S(\rho) + \max S(\rho^*),$$

where ρ^* is a mixed state with the same marginals on proper sets of qubits as ρ , i.e., $\rho^*|_B = \rho|_B$ for every proper subset B of A .

Next, define

$$\widetilde{ENT}(\rho) = \sum_B \{ENT(\rho; B) : B \subset A\}.$$

In this language a way to formulate the censorship conjecture is:

Conjecture C: There is a polynomial P (perhaps even a quadratic polynomial) such that for any quantum computer on n qubits, which describes a pure state ρ ,

$$\widetilde{ENT}(\rho) \leq P(n). \tag{8}$$

The parameter \widetilde{ENT} can serve as an alternative measure for the notion of “highly entangled states” from Conjecture B. States (admitting some symmetry in order to ensure that the entanglement is not confined to a small subset of qubits) where \widetilde{ENT} is quadratic (perhaps even super-linear) in the number of qubits can already be regarded as “highly entangled.”

7.5 Testing it

Objection: *In the near term, experiments with quantum computers will be mere demonstrations. They will not teach us anything.*

Response: *... We will learn about correlated decoherence.*— John Preskill, Quantum Computing: Pro and Con, 1996 [40].

The conjectures regarding pairs of qubits or error synchronization can be examined on rather small quantum computers.

For example, under the standard assumptions on noise, a circuit able to correct two errors will be able to create pairs of entangled qubits with almost independent errors even if gates used in the circuit each have a small but otherwise arbitrary form of errors. (This will require a small overhead on the rate of error.)

Creating pairs of entangled qubits (say, EPR pairs) with almost uncorrelated errors, which runs counter to our conjectures, can be tested on a rather small quantum computer with 10-20 qubits.

Here we propose to test properties of the overall (cumulative) noise. It is probably harder to probe the “fresh noise” directly (and the outcomes will be less conclusive), but probing “fresh noise” will enable one to test these ideas for systems operating already with a small number of qubits. Some detrimental noise behavior may be witnessed in the realization of quantum error correction for a single error.

An important experimental quantum error correction challenge is the ability to approximate in small quantum computers every possible pure state on a few qubits (three, four, five). Achieving this will go a long way toward refuting the conjectures on detrimental noise.

One point to notice is that the conjectures we consider in this paper are not equivalent to the familiar concerns about scalability of quantum computers. Our conjectures may come into play, as anticipated in Preskill’s quotation starting this section, already with small quantum computers.

Two warnings are in order:

- 1) Empirical support for the conjectures from one device does not apply to other devices. The mechanism leading to the conjectured behavior is not universal but may depend on the device.

- 2) We need a low error rate to start with. In order to identify the effects of non-standard noise we still need to suppress standard noise of a higher

rate.

Here is an example. Most current implementations of ion trap computers creating entanglement between two qubits require physically moving them together. This suggests that for these ion trap computers fresh errors will be correlated for **every** pair of qubits and that using them to create entangled pairs of qubits with uncorrelated errors will not be possible. Of course, this suggestion should be tested experimentally. (While it seems rather clear that for these ion trap computers, fresh errors for every pair of qubits are going to be correlated, the stronger claim of positive correlation for information leaks is not clear.)

In principle, for some other implementation of ion trap computers it may be possible to induce entanglement between pairs of qubits without affecting any other qubits.

8 Detrimental noise for general quantum systems

Right now the only way I can see engineering worlds with classical but not quantum computation is to engineer a world in which “phase”-type decoherence is massive or crazily correlated but “amplitude”-type decoherence is not.—Dave Bacon, The Quantum Pontiff, 2006.

8.1 Our first example revisited

Consider our first example of a quantum computer where when the quantum memory is in a state ρ and $\rho = U\rho_0$, the noise E will be UE_0U^{-1} . When we try to describe the relation between the state of the computer and the noise, this example describes, for every state ρ , an envelope of noise $D_\rho = \{UE_0U^{-1} : U\rho_0 = \rho\}$. This is a huge class of quantum operations most of which are irrelevant (being computationally infeasible.) An important

property of this noise is:

$$\mathcal{D}_{U\rho} = U\mathcal{D}_\rho U^{-1}. \quad (9)$$

Relation (9) amounts to saying that there is a component of quantum noise that is invariant under unitary operations and thus does not depend on the device that carries these operations.

Remark: Note that contrary to Bacon’s assertion quoted at the beginning of this section, our conjectures on the nature of noise do not treat amplitude errors and phase errors differently. Rather, the conjectures and especially relation (9) do precisely the opposite in asserting that some ingredient of noise is inherently invariant under symmetries of the Hilbert space describing the states of the computer. Such a symmetry for decoherence may account for the symmetry-breaking leading to the classical behavior of large quantum systems.

8.2 Noisy quantum systems

When we talk about general noisy quantum systems and not about controlled systems with a clear “intended” evolution there is no obvious meaning to the notion of “errors.” There are two related issues to consider:

1. Information leaks from the system to its environment.
2. Errors in any **description** of the evolution of a noisy quantum system.

As before, we restrict our attention to noise described by POVM-measurements.

We can now ask: what are the laws of decoherence for general noisy quantum systems that follow the properties of noise propagation?

As with the case of standard models of noise, we would like to describe an envelope of noise, i.e., a large set of quantum operations, so that when we model noisy quantum operations or more general processes the incremental (or infinitesimal) noise should be taken from this envelope. Conjectures A

and B and our first example propose some systematic connection between the noise and the state. However, in these conjectures both the assumption in terms of entanglement and the conclusion in terms of correlation rely on the tensor product structure of \mathcal{H} .

Here is a (rather tentative) proposal on how to formalize this connection for general systems:

Definition: A D-noise of a quantum system at a state ρ is a quantum operation E that commutes with some non-identity unitary quantum operation that stabilizes ρ .

This definition describes a (huge) class \mathcal{D}_ρ of quantum operations that respect the relation $\mathcal{D}_{U\rho} = U\mathcal{D}_\rho U^{-1}$.

Conjecture D: D-noise cannot be avoided in every noisy quantum process.

On its own our suggested definition of D-noise is extremely inclusive, and so is any (nonempty) envelope of noise operations that satisfies relation (9). For example, a D-noise on a state of the form $\rho \otimes \rho$ can be standard even if ρ is highly entangled. However, there are two additional conditions we have to keep in mind:

1. The hypothesis that the overall noise contains a large D-component applies to every subsystem of our original system. (An appropriate “hereditary” version of Conjecture D may suffice to imply Conjectures A and B for noisy quantum computers. This has yet to be explored.)
2. The operation describing the noise should be “local” or more formally (see Section 5.2) “computationally feasible” in terms of local operations describing the system.

Trying to express the noise envelope in terms of the entire evolution (not just the temporal state) or in terms of a set of “gates” that describe the evolution may lead to sharper descriptions. We will not pursue these directions here. Another interesting issue is extending the censorship conjecture (Conjecture C) to general quantum systems. This conjecture and the whole notion of entanglement rely on a tensor product structure that we do not have in the general case. It is true in rather general cases that a tensor product structure emerges (not necessarily the “natural” tensor product structure). It is not known how general this phenomenon is. And we can ask whether Conjecture C would extend to arbitrary noisy quantum systems for some emerging tensor power structure.

9 Linearity, causality, memory, and rate

9.1 Some concerns

9.1.1 Linearity

Our conjectures for noisy quantum computers and for noisy quantum systems amount to nonlinear relation between the noise envelope and the state of the computer (system). Such nonlinear relations do not violate linearity of quantum mechanics. For example, if we consider the noise in our main relation (1) or in our opening example as a function of the entire earlier evolution then it is completely linear. Nonlinearity is caused by ignoring the earlier evolution and considering the relation between the noise and the state for all possible evolutions leading to this state.

9.1.2 Memory

Do our conjectures and relation (1) mean that the environment necessarily “memorizes” the past evolution, or, at least, a very crude property of the past evolution encoded by the noise envelope? In order to relate to this question

we note that while the models of noisy quantum evolutions and noisy quantum computers are sufficiently rich to model any noisy quantum evolutions that we can imagine or create, these models can give wrong or incomplete intuitions regarding issues like memory and causality. The distinction between the quantum computer that performs the intended evolution and the environment that induces the noise is a property of the mathematical model and not a description of the physical reality. The mathematical dependence of the noise on the past evolution can represent the effect of the past evolution on the environment, but it can also represent various other things, such as consequences of the feasibility of the past evolution on the physical device performing it, and postselection.

9.1.3 Causality

When a gun shows up in the first act, it will go off in the third.

Chekhov's gun principle.

Consider an intended pure-state evolution ρ_t , $0 \leq t \leq 1$ of a quantum computer, and a noisy realization σ_t , $0 \leq t \leq 1$. Assuming that σ is close¹¹ to ρ for the **entire** time interval may create a systematic relation of the infinitesimal noise at an intermediate time t on the entire intended evolution of ρ .

It is a **consequence** of FTQC that dependence of the errors on the past evolution and on the future (intended) evolution becomes negligible.

Of course, we need **also** to be able to describe the noise as the outcome of a local, computationally feasible process which depends only on the past. (Indeed relation (1) offers such a description.)

Perhaps the following (completely classical) example can shed some light on the causality issues we discuss. Suppose that an airport-averse professor is planning a trip as follows: Leave Davis at 8:00; arrive at San Francisco

¹¹In some sense, e.g., in terms of the expected number of qubit errors.

airport at 9:30; take the 10:00 flight to Chicago; present a lecture the next morning at UC at 11:00. Assuming that this plan is realized up to small errors we can deduce that it is much more likely that the professor arrived at the airport earlier rather than later. The errors compared to the original plan may thus depend on the entire planned evolution (assuming its success).

Of course, it also necessary that the errors compared to the planned time-estimate to reach the airport can be described as consequences of events occurring in California before arriving at the airport, e.g., the number of people taking the highway being less than average.¹²

9.1.4 Faraway photons

Suppose we have two faraway photons at a given entangled state at time T . Consider their state at time $T + t$. Is there any reason to believe that the changes will not be independent? We can expect detrimental noise at the time the entanglement is created but we cannot expect it at a later time. Is this a counterexample to our conjecture regarding pairs of qubits?

We relate to this concern in the next subsection.

9.2 Modeling the rate of noise for noisy quantum evolutions

The physical systems in which qubits may be implemented are typically tiny and fragile (electrons, photons, and the like). This raises the following paradox: On the one hand we want to isolate these systems from their environment as much as possible, in order to avoid the noise caused by unwanted interaction with the environment — so called “decoherence.” But on the other hand we need to manipulate these qubits very precisely in order

¹²This can also be influenced by a future event, e.g., a major sports game shown on TV that evening at 10.

to carry out computational operations. A certain level of noise and errors from the environment is therefore unavoidable in any implementation. — Kempe, Regev, Unger, and Wolf, 2008 [25].

The quote from Kempe et al. points to some genuine difficulty in modeling noisy quantum systems. We can exhibit extremely stable entangled quantum states, and yet we believe that quantum systems are inherently noisy. We can also have isolated qubits that do not interact at all that are subject to uncorrelated noise, and yet we propose in this paper that for the appropriate model of noisy quantum computers the noise should be highly correlated. The noise (its rate and its form) depends on the fact that we need to manipulate the qubits, but what is the formal description of such a dependence?

When we model the fresh (or infinitesimal) noise for the evolution of a noisy quantum computer or even a general noisy system, what should be a lower bound on the rate of noise? This is an interesting issue even when it comes to a single noisy qubit.

Recall that the usual assumption regarding the rate of noise is that for every qubit the probability of it being faulty is a small constant for every computer cycle. We propose the following refinement of this assumption.

Conjecture E: A noisy quantum computer is subject to (detrimental) noise with the following property: the rate of noise at time t (in terms of trace distance) is bounded from below by a measure of noncommutativity between the operators describing the evolution prior to time t and those describing it after time t .

The lower bound according to Conjecture E for the rate of noise when the process starts or ends is zero. The rate of noise can also vanish for classical systems where all the operations commute. Conjecture E can be regarded as a proposed refinement on the assumptions regarding the rate of noise even for a single qubit.

10 Discussion

10.1 Classical noisy systems

When it rains it pours. English proverb.¹³

Our definitions of detrimental noise and our various conjectures as stated here do not have any implications for classical noisy systems. Still, some of our conjectures were originally formulated also for “natural” noisy classical correlated systems; see [23]. (As a matter of fact, the behavior of classical noisy systems was one of the motivations for Conjectures A and B.)

For example, we can expect error synchronization for attempts to describe (or prescribe) noisy highly correlated stochastic systems such as the weather or the stock market.

Understanding noise and the study of de-noising methods span wide areas. For example, in machine learning we can see the example where text and speech represent respectively the intended (ideal) and noisy signals. Certain statistical methods of de-noising are based on assumptions that run counter to our conjectures. However, our conjectures are in agreement with insights asserting that such statistical de-noising methods will leave a substantial amount of noise uncorrected. (Moreover, “natural” examples of noisy highly correlated classical systems exhibit a moderate degree of dependence, much less than the sort of dependence required for quantum error correction and various basic quantum algorithms.)

Because of the heuristic (or subjective) nature of the notion of noise in classical systems (and of the notion of probability itself), such a formulation, while of interest, leads to several difficulties. Moreover, we can exhibit counterexamples to classical analogs of Conjectures A and B based on the ability to have noiseless classical memory and computation. Therefore, the analogy with a classical noisy system does not make the conjectures of this

¹³Similar proverbs asserting that troubles come together exist in other languages.

paper more compelling (or less compelling) but rather gives a wider context in which to discuss them.

Let me mention a question that is often raised in discussions on quantum fault tolerance and deserves better understanding.

How is it possible that quantum fault-tolerant computation fails while classical fault tolerant computation succeeds?

One conceptual difference between quantum and classical error correction (mentioned in [23]) is that clean bits can be extracted from noisy signals that do not erase all information. (For example, when you have a stream of bits and every bit is replaced by a random bit with a probability of 0.9999.) However, in the quantum case, there is a whole range of noise that does not enable extracting clean qubits from a stream of noisy qubits. (Extracting clean **bits** is still possible.) Another interesting conceptual difference related to error correction of correlated noise (with a single error) is described by Ban-Aroya, Landau and Ta-Shma [15]. The paper of Alicki and Horodecki [10] can also be regarded as a proposed explanation.

10.2 Theoretical and empirical physics

The development of the theory of quantum error correction may in the long run have broader and deeper implications than the development of quantum complexity theory — John Preskill, Quantum Computing: Pro and Con 1998, [40].

Implementing quantum error correction requires complicated and specific constructions of quantum processes that we do not encounter in nature. There are, however, interesting suggestions regarding usefulness of quantum error correction in the study of black holes [21], quantum gravity, and other areas (see [43]). On the other hand, there has been little effort within the QM framework to understand what could be the implications of failure of FTQC

for physics. Some proponents of quantum computers regard the feasibility of computationally superior quantum computers as a logical consequence of quantum mechanics. Some skeptics regard it as a not particularly interesting far-fetched idea.

An obvious concern regarding adversarial noise models (and other skeptical claims about quantum computers) is whether they are consistent with well-established phenomena from physics and current empirical evidence. For example, are such noise models consistent with superconductivity? Since detrimental noise appears to express familiar properties of noise propagation it seems reasonable that detrimental noise is consistent with the physics that we see around us, but this deserves much closer examination.

On the other hand, detrimental noise is in conflict with hypothetical physics constructions. The construction of stable non-Abelian anyons [27, 35] might be inconsistent with the conjectures regarding detrimental noise since (at least according to some models describing them) such non-Abelian anyons demonstrate quantum error correction based on highly entangled systems.

A potential implication of a deeper understanding of quantum error correction (and their limitations) may lead to a better understanding of the “quantum measurement problem” [34]¹⁴ as well as to:

Conjecture F: Stable non-Abelian anyons do not exist in nature and cannot be created.

Third mathematical conjecture: (i) Show that the model of noisy quantum evolutions with forced noise propagation (relation (1)) and the conjectures on the relation between the noise envelope and the state do not support non-Abelian anyons.

¹⁴Leggett’s view in [34] is that regarding the phenomenon of decoherence as an explanation of the “measurement paradox” is a “gross logical fallacy.” Perhaps, contrary to his view, the crux of the matter resides in a deeper understanding of the phenomenon of decoherence itself.

(ii) Show that this model and these conjectures do support Abelian anyons as well as even more basic quantum mechanics phenomena.

10.3 Classical simulation of noisy quantum systems

Here is an interesting question:

Does a (hypothetical) failure of computationally superior quantum computers necessarily mean that classical computers are capable, in principle, of simulating efficiently the behavior of the quantum processes we witness in nature?

Of course, we can ask if in view of the complex nature of fault tolerance based on quantum error correction classical computers are capable, in principle, of simulating natural quantum processes, anyway (even if quantum computers are feasible). Candidates for processes that may occur in nature and possibly hard to simulate classically are distributions represented by bounded-depth quantum circuits (even random such circuits). Understanding the computational complexity of such distributions is a question of great importance.

10.4 Engineering, science, and time

One of the interesting aspects of quantum error correction (and of quantum information in general) is the mixture of theory and practice, science and engineering, and various areas of mathematics, physics, and computer science (and more). It is often the case that the borders between engineering issues and abstract theoretical and conceptual matters are rather blurred. We will mention one example.

In his paper [40] Preskill (see also [43]) proposes small quantum computers with quantum error correction capability as a way to engineer more

accurate clocks than those available at present. Far-fetching (and flipping) Preskill’s suggestion we can ask: Does a failure of FTQC (in principle) have any conceptual bearing on the notion of time itself?

10.5 Computational complexity issues

The foundations of noisy quantum computational complexity were laid by Bernstein and Vazirani in [16]. The problem of describing complexity classes of quantum computers subject to various models of noise was proposed by Peter Shor in the nineties. (Although we naturally expect computational power between BQP and BPP it is possible, in principle, that certain noise models will allow efficient algorithms even for problems not in BQP.) Scott Aaronson [1] asked for the computational complexity consequences of various hypothetical restrictions on feasible (physical) states for quantum computers. In particular, he posed the interesting “Sure/Shor challenge”: to describe such restrictions that do not allow for polynomial-time factoring of integers and at the same time do not violate what can already be demonstrated empirically.

The threshold theorem and some of its recent versions give a fairly good description of the wide models of noise that allow universal quantum computing when the noise rate is sufficiently small. We mentioned several results ([7, 45, 25]) showing that for the standard noise models when the computation is reversible or when the noise rate is high, the computational power reduces to BPP (for some results) or BPP^{BQNC} (the power of classical computers together with log-depth quantum circuits). (This is sufficient for polynomial-time factoring! Cleve and Watrous [18] gave a polynomial algorithm for factoring that requires, beyond classical computation, only log-depth quantum computation.)

How bad can the effect of correlated errors be? I tend to think that for an arbitrary form of noise, if the expected numbers of qubit errors in a computer cycle is sufficiently small then problems in BPP^{BQNC} and, in

particular, polynomial-time factoring can prevail. A rough argument in this direction would go as follows. First replace a given log-depth circuit by a larger one capable of correcting standard errors; then run the computation a polynomial or quasi-polynomial (depending on the precise overhead in the fault-tolerant circuit) number of times to account for highly synchronized errors.

On the other hand, it may be possible (but not easy) to prove that highly correlated errors of the kind under consideration do not allow fault tolerance based on quantum error correction, and perhaps also that they suffice to reduce the computational power to BPP^{BQNC} .

The most interesting direction, in my opinion, would be to show that with the full power of detrimental errors, e.g., as defined in equation (1), including the conjectured effect on the expected number of qubit errors in one computer cycle (Section 5.4), the computational power of noisy quantum computers reduces to BPP.

10.6 An analogy: the $NP \neq P$ problem

In this section we draw a quick analogy between the $NP \neq P$ problem and a skeptical point of view regarding quantum computers. (An analogy in the opposite direction, namely, a parallel discussion of reasons to believe $NP \neq P$ and the feasibility of quantum computers, is offered by Aaronson [2].)

A point of view that can be found implicitly or implied even today, and certainly could have been found much more in the middle of the twentieth century, asserts that:

*Every finite problem can be solved, in principle, by a digital computer.*¹⁵

¹⁵Of course, the opposite point of view was also present since the early days of computers. But it was not until the seventies that it was fully realized that the limitation of computers is an important scientific question.

The major scientific change that happened in the last fifty years is twofold. Attempts to find algorithms for certain “hard” problems, or special-purpose computing devices to deal with them, have failed. In addition, the conjecture that some problems are computationally infeasible was stated in concrete mathematical terms and has led to an elegant, coherent, and rich mathematical theory of computational complexity.

This has led to the modern point of view where the common wisdom is:

Large NP-complete problems cannot be solved by any realistic computational device.

Of course, when an algorithm or a physical device whose purpose is to solve NP-complete problems is offered it is not always easy to explain why it is going to fail, and there is no “universal” reason or mechanism for such a failure. Often, this leads to rather interesting research.

We come to the issue at hand, starting with a common belief or assumption that every quantum state that we can imagine can, in principle, actually be created.

It is clear that computational complexity does restrict the type of states that can be created, and this agrees with earlier physics insights. So a realistic version is: Every (computationally feasible) state that we can imagine can, in principle, be created.

The research of the past fifteen years has led to a more detailed insight centered around quantum error correction.

Every (computationally feasible) state that we can imagine can, in principle, be created via quantum error correction.

The alternative possibility proposed here and in various other papers is:

Highly entangled states cannot be created; full-fledged quantum error correction is not feasible, nor are computationally superior quantum computers.

In order to support such a position we will need strong experimental evidence, most importantly, strong evidence that attempts to build quantum computers face some solid obstacles already for handful of qubits. In addition, we need a coherent and elegant mathematical explanation for a principle that can imply that quantum error correction and quantum computers fail. In words, the principle we propose in this paper is a familiar one: “noise propagation.”¹⁶ It comes with the important asterisk that properties of noise propagation can have other causes. A main point of this paper is that there is more to be explored and understood in the mathematical study of noise propagation.

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¹⁶In a few more words: “the process for creating entanglement necessarily leads to noise synchronization.”

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