The Odd-Distance Plane Graph

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Abstract

The vertices of the odd-distance graph are the points of the plane $\mathbb{R}^2$. Two points are connected by an edge if their Euclidean distance is an odd integer. We prove that the chromatic number of this graph is at least five. We also prove that the odd-distance graph in $\mathbb{R}^3$ is not countably choosable.

1 Introduction

In 1950 Edward Nelson, a student at the University of Chicago, formulated the *alternative four-color problem*: What is the minimum number of colors for coloring the points of the plane so that points at unit distance apart receive distinct colors. Nelson himself showed that at least four colors are needed. Soon after learning about the problem from Ed Nelson, John Isbell proved that the plane can be colored by seven colors. Fifty seven years later, and numerous efforts by many researchers, these are still the best known bounds. Some authors call it a *disappointment* or a *disaster* while others call it *frustrating.* We would like to call it a *a great opportunity* as evidenced by its high popularity and interesting history. After all, at least five mathematicians were credited as the creators of the unit-distance problem. Without a doubt, its popularity can be traced to its simplicity, its elusive solution and to Paul Erdős who repeatedly publicized it in his presentations and papers. It has produced many variations, numerous papers, dissertations and occasional *headaches*, but no improvements on the original bounds. Its interesting history was traced by A. Soifer [9]. A nice account of the problem and its derivatives can be found in the book Research Problems in Discrete Geometry [2].

In this paper we present yet another variation of the unit-distance graph: the *odd-distance graph.* It is a simple, natural generalization of the unit-distance graph. We wish to color the points of the plane so that points at odd distance apart receive distinct colors. Since four points in the plane cannot have pairwise odd integral distances, this graph does not contain K_4 as a subgraph [5], [8]. It is thus natural to ask whether the plane can be colored by a finite number of colors such that points at odd integral distance receive distinct colors. In 1994 M. Rosenfeld asked Paul Erdős this question at the 25^{th} Southeastern International Conference on Combinatorics, Graph Theory and Computing Florida Atlantic University, Boca Raton, Florida. Erdős [4] presented this problem in his talk. Uncharacteristically, the proceedings did not include his review. The first time this question appeared in print was in 1996 [8] where it was called the *Ruby - Rose problem.* Erdős also asked to determine the maximum number of odd distances among n points in $\mathbb{R}^2$. It is curious to note that an identical question for the unit distance was investigated by Erdős in 1946 [3] but not in the context of graphs (in this context there is some justification to attribute the unit distance problem also to P. Erdős). Since the odd distance graph spanned by n points does not contain a K_4 as a subgraph, this number is bounded by Turán's function $t(n)$, the maximum number of edges in a graph of order n and no K_4. L. Piepmeyer [7] showed that Turán's extremal graph, the complete tri-partite graph $K_{m,m,m}$ can be embedded in the plane (actually on a circle) such that two vertices connected by an edge will be at odd distance apart. Note that this is a faithful embedding as no other vertices are at odd distance apart. As an aside, answering a question of Erdős, Maera, Ota and Tokushige [6] proved that every finite graph admits a faithful representation in $\mathbb{R}^2$ such that two vertices connected by an edge will be at an integral distance apart.

It is interesting to compare the odd-distance graph and the unit-distance graph. Clearly, the unit-distance graph is a subgraph of the odd-distance graph. Its diameter is 2, while the diameter of the unit distance graph is not bounded, any two vertices have countably many common neighbors while in the unit distance graph any two points have at most two common neighbors. As a result of these differences, some problems that are easy for one graph are difficult for the other and vice versa. For instance, the lower bound and upper bound for the chromatic number of the unit-distance graph are relatively easy to obtain, while the same bounds for the odd-distance graph are more difficult. Actually it is not known whether the upper bound is finite. Since any two distinct circles have at most two points in common, it is easy to find many graphs which are not subgraphs of the unit-distance graph. For instance, $K_{2,3}$ is such a graph. On the other hand, the only *forbidden* subgraph of the odd-distance graph known to us is K_4. Finding the maximum number of edges among all subgraphs of order n is unknown for the unit-distance graph, but follows a very predictive upper bound for the odd-distance graph as proved in [7].

In this note we prove that the chromatic number of the odd-distance graph is at least five. We could not find a finite upper bound other than the trivial bound $\aleph_0$. We suspect that this graph cannot be colored by a finite number of colors.

In the next section we use the triangular lattice to construct a graph of order 21 with chromatic number five which, is a sub-graph of the odd-distance graph. We also show that the odd-distance graph is countably colorable, the rational odd-distance graph is 2-colorable and prove that the odd-distance $\mathbb{R}^3$ is not $\aleph_0$-choosable.

Let D be a set of numbers. We shall denote by $G^D(\mathbb{R}^d)$ the graph whose vertices are the points of the Euclidean space $\mathbb{R}^d$ and edges between points whose distance $\in D$. Its chromatic number will be denoted by $\chi^D(\mathbb{R}^d)$

2 The Odd-Distance Graph

2.1 Triangular grid

Let T be the triangular grid in the plane with edges of unit length. Aside from points at odd integral distance along the grid lines, there are other points on the grid at odd distance. In particular, consider the points B and D in Figure 1. The triangle ΔBCD has sides 3 & 5 along the grid, $\angle BCD = 120^0$. Hence $d(B,D)^2 = d(B,C)^2 + d(C,D)^2 - 2 \times d(b,c) \times d(c,d)\cos(120^0) = 5^2 + 3^2 + 3 \times 5 = 49$ or $d(B,D) = 7$.

Lemma 1. *Let f be a proper 4-coloring of T. For any two points X and Y at distance 8, we have $f(X) = f(Y)$.*

Proof. Let A and B be two points at distance 8 such that $f(A) \neq f(B)$, say $f(A) = c_1$ and $f(B) = c_2$. Then, since $d(A,C) = 3$ and $d(C,B) = 5$, C must have a different color than c_1 and c_2, say c_3. Since $d(A,D) = d(C,D) = 3$, and $d(B,D) = 7$, $f(D) = c_4$. Similarly, it follows that $f(E) = c_3$, $f(F) = c_4$, $f(G) = c_3$ and $f(H) = c_4$. Since each of I_1, I_2, I_3 is at distance 3 from one

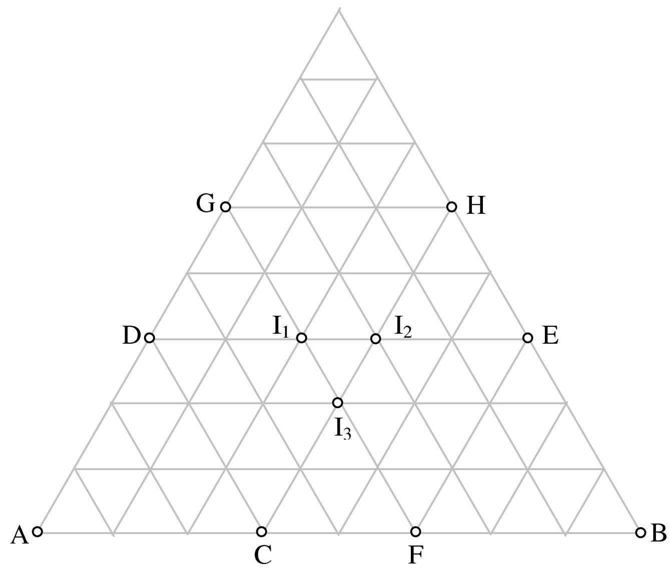

Figure 1: The triengular grid

point colored c_3 and one colored c_4, we have $f(I_1), f(I_2), f(I_3) \in \{c_1, c_2\}$. This is a contradiction since they are vertices of a triangle with side 1. □

Corollary 1. $\chi^{Odd}(\mathbb{R}^2) \geq 5$.

Proof. Take 2 copies of the triangle in Figure 1. Rotate the second copy around the point A so that the image of B in the second copy will be at distance 1 from B. By Lemma 1 this configuration is not 4-colorable. □

The graph thus obtained has 21 vertices and only four odd distances: $D = \{1, 3, 5, 7\}$. Clearly, the plane can be colored in a finite number of colors so that two points with distance $\in D$ have distinct colors. We did not try to find an upper bound for the chromatic number $\chi^D(\mathbb{R}^2)$.

The following theorem shows that the triangular grid by itself can be colored by 4 colors.

Theorem 1. *Let* $T = \left\{ \left(n + \frac{m}{2}, \frac{m}{2}\sqrt{3}\right) : \ n, m \in \mathbb{Z} \right\}$. *Then* $\chi^{Odd}(T) = 4$.

Proof. It is easy to show that $\chi^{Odd}(T) > 3$.

The coloring function f is defined as follows:

$$f\left(n+\frac{m}{2},\frac{m}{2}\sqrt{3}\right)=\begin{cases}1 & \text{if } n \text{ and } m \text{ are both even,}\\ 2 & \text{if } n \text{ is odd and } m \text{ is even,}\\ 3 & \text{if } n \text{ is even and } m \text{ is odd,}\\ 4 & \text{if } n \text{ and } m \text{ are both odd.}\end{cases}$$

Note that all the color classes of this coloring are translations of the color class of the color 1.
Let $\left(n_1+\frac{m_1}{2},\frac{m_1}{2}\sqrt{3}\right)$ and $\left(n_2+\frac{m_2}{2},\frac{m_2}{2}\sqrt{3}\right)$ be two points colored with the color 1. Then

$$\begin{aligned}&\left\|\left(n_1+\tfrac{m_1}{2},\tfrac{m_1}{2}\sqrt{3}\right)-\left(n_2+\tfrac{m_2}{2},\tfrac{m_2}{2}\sqrt{3}\right)\right\|^2\\ &\qquad=(n_1-n_2+\tfrac{m_1-m_2}{2})^2+(\tfrac{m_1-m_2}{2}\sqrt{3})^2\\ &\qquad=(n_1-n_2)^2+(n_1-n_2)(m_1-m_2)+(m_1-m_2)^2\end{aligned}$$

which is an even integer. Hence, f is a proper 4-coloring of T.
Therefore, $\chi^{Odd}(T)=4$. □

2.2 The chromatic number of $G^{Odd}(\mathbb{Q}^2)$

We begin with two simple observations:
1. Let $x,y,z\in\mathbb{Z}$ be a Pythagorean triple, i.e. $x^2+y^2=z^2$. Then the following statements are both true.

1. If $z\equiv 0 \pmod 2$ then $x\equiv y\equiv 0 \pmod 2$.
2. If $z\equiv 1 \pmod 2$ then one of x and y is even and the other is odd.

2. Let $x,y,z\in\mathbb{Z}$ be a Pythagorean triple. Then if 2^k divides z then 2^k divides both x and y.

Lemma 2. *Let $X=(x,y)=(2^k m,2^l n)$ and $U=(u,v)=(2^p r,2^q s)$ for some $k,l,p,q,m,n,r,s\in\mathbb{Z}$, where m,n,r,s are all odd. Then if $d(X,U)$ is an odd integer then $x-u$ and $y-v$ are both integers with different parity modulo 2.*

Proof. Assume $d(X,U) = o$ for some odd integer o. It is enough to show that $x - u \in \mathbb{Z}$, since then $y - v \in \mathbb{Z}$ and the rest of the theorem follow by Theorem **??**.

$$\|(x,y) - (u,v)\|^2 = (2^k m - 2^p r)^2 + (2^l n - 2^q s)^2 = o^2 \tag{1}$$

Case 1.1. Assume $k, p \geq 0$ (similarly for $l, q \geq 0$). Then
$x - u = 2^k m - 2^p r$ is an integer.

Case 1.2. Assume $p < 0 \leq k$ (similarly for $q < 0 \leq l$). Then

$$(2^{k-p} m - r)^2 + (2^{l-p} n - 2^{q-p} s)^2 = (2^{-p} o)^2.$$

Since both $2^{-p} o$ and $2^{k-p} m - r$ are integers, $2^{l-p} n - 2^{q-p} s$ is also an integer. Therefore by Corollary **??**, 2^{-p} divides $2^{k-p} m - r$. But since $k - p \geq -p$, 2^{-p} divides $2^{k-p} m$. Hence 2^{-p} divides r. But this is impossible since r is odd.

Case 1.3. Assume $p \leq k < 0$ and $q \leq l < 0$. Let $a = -k,\ b = -p,\ c = -l$ and $d = -q$. Then $0 < a \leq b$ and $0 < c \leq d$ and (1) becomes

$$\left(\frac{m}{2^a} - \frac{r}{2^b}\right)^2 + \left(\frac{n}{2^c} - \frac{s}{2^d}\right)^2 = o^2.$$

Assume without loss of generality that $b \geq d$. Then

$$(2^{b-a} m - r)^2 + \left(2^{b-c} n - 2^{b-d} s\right)^2 = (2^b o)^2.$$

Since all the terms in the above equality are integers and $2^b | (2^b o)$, by Corollary **??**, $2^b | \left(2^{b-a} m - r\right)$. But since m and r are both odd, this is possible only if $b = a$, and in this case $2^b | m - r$. i.e.

$$x - u = \frac{m - r}{2^b} = \frac{m}{2^a} - \frac{r}{2^b} \in \mathbb{Z}.$$

□

Theorem 2. *Let H be the induced subgraph of $G^{Odd}(\mathbb{R}^2)$ on $\mathbb{Q} \times \mathbb{Q}$. Then H is a bipartite graph, i.e. H does not contain an odd cycle.*

Proof. Assume by contradiction that it contains an odd cycle, say (x_1, y_1), (x_2, y_2), $\cdots$, (x_{2t+1}, y_{2t+1}) for some $t \in \mathbb{N}$. Then $x_i = \frac{2^{k_i} n_i}{m}$ and $y_i = \frac{2^{l_i} p_i}{m}$ for some $k_i, l_i, n_i, p_i \in \mathbb{Z}$ for $1 \leq i \leq 2t + 1$ and $m \in \mathbb{Z}$ and n_i, p_i and m are odd. Then $\left(2^{k_1}n_1, 2^{l_1}p_1\right)$, $\left(2^{k_2}n_2, 2^{l_2}p_2\right)$, $\cdots$, $\left(2^{k_{2t+1}}n_{2t+1}, 2^{l_{2t+1}}p_{2t+1}\right)$ is also an odd cycle. Then by Lemma 2, $\left(2^{k_i}n_i - 2^{k_{i+1}}n_{i+1}\right)$, $\left(2^{l_i}p_i - 2^{l_{i+1}}p_{i+1}\right) \in \mathbb{Z}$ and they have different parity for all $1 \leq i \leq 2t+1$ where indices are calculated $\mod (2t+1)$. Then

$$\sum_{i=1}^{2t+1} \left(2^{k_i}n_i - 2^{k_{i+1}}n_{i+1}\right) + \left(2^{l_i}p_i - 2^{l_{i+1}}p_{i+1}\right) = 0.$$

But this is a contradiction since in the left hand side all the summands are odd and there are $2t+1$ of them.

Therefore, H is a bipartite graph. □

Remark
It is easy to see that the odd-distance graph is $\aleph_0$ colorable. Just use any tiling of the plane by countably many tiles of diameter < 1 (such as the traingular lattice with side < 1) or squares of side $\frac{1}{2}$.

3 Choosability

A graph G is κ -choosable if for every assignment of an arbitrary set of colors of cardinality κ to each vertex of G it is possible to properly color each vertex of G with a color from its assigned set. Since the unit-distance graph has finite regular subgraphs of arbitrarily large degree, it follows from N. Alon's Theorem [1] that the unit-distance graph and hence the odd-distance graph are not finitely choosable. We believe that the odd-distance graph is $\aleph_0$ choosable but we can prove that the odd-distance $\mathbb{R}^3$ is not.

Consider the set of points

$$A = \{(x, y, 0) : x^2 + y^2 = 1\}$$

and

$$B = \{B_n = (0, 0, \sqrt{4n^2 + 4n)} : n \in \mathbb{N}\}.$$

Note that for any $u \in A$ and any B_n, $d(u, B_n) = 2n + 1$, an odd integer.

Since $|A| = \aleph$, there is a one to one correspondence ψ, between A and the set of all infinite subsets of $\mathbb{N}$. Now for each $n \in \mathbb{N}$, let $\tau(B_n) = \{n, n+1, n+2, ...\}$. Let f be an $\aleph_0$-*coloring* of G. It follows from the definition of τ that

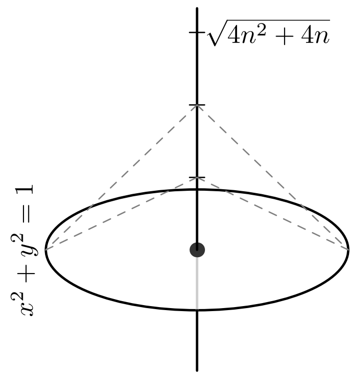

Figure 2: List coloring of R^3

$f(B)$ is infinite so that there are points $u \in A$ such that $\psi(u) \subseteq f(B)$. But since the distance from u to any point in B is an odd integer, $f(u)$ can not be in $f(B)$. This gives the desired contradiction. Therefore, $\chi_l(G) = \aleph$.

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