Exterior algebra

0.1 Symmetric tensors

Let T^2_0 be the linear space of tensors of type $(2,0)$ on a vector space V of dimension n. The dimension of T^2_0 is n^2. Let a basis of T^2_0 be $e_a\otimes\varepsilon_b$.

Consider a subset of basis elements

$$S(e_a\otimes\varepsilon_b)=e_a\otimes\varepsilon_b+e_b\otimes\varepsilon_a$$

Consider a linear space

$$S(T^2_0)=Span(S(e_a\otimes\varepsilon_b)\,.$$

This is a subset of T^2_0 of *symmetric tensors* .

Proposition 0.1: *1. Every symmetric tensor is represented as*

$$T=\frac{1}{2!}T^{ab}S(e_a\otimes\varepsilon_b)\,,$$

where $T^{ab}=T^{ba}$.

2. The dimension of $S(T^2_0)$ *is* $n(n+1)/2$.

3. The space $S(T^2_0)$ *does not depend on the basis used for its definition, i.e., if a tensor is symmetric in some basis it is symmetric in every basis.*

A symmetric tensor of type $(p,0)$ defined analogously by the basis

$$S(e_{a_1}\otimes\cdots\otimes e_{a_p})$$

where S is the operator of even permutations. Every symmetric tensor is represented as

$$T=\frac{1}{p!}T^{a_1\cdots a_p}S(e_{a_1}\otimes\cdots\otimes e_{a_p})\,.$$

A symmetric tensor of type $(0,p)$ defined by the basis

$$S(\vartheta^{a_1}\otimes\cdots\otimes\vartheta^{a_p})$$

and represented as

$$T=\frac{1}{p!}T_{a_1\cdots a_p}S(\vartheta^{a_1}\otimes\cdots\otimes\vartheta^{a_p})\,.$$

0.2 Antisymmetric tensors

Let T_0^2 be the linear space of tensors of type $(2,0)$ on a vector space V of dimension n. The dimension of T_0^2 is n^2. Let a basis of T_0^2 be $e_a \otimes \varepsilon_b$.

Consider a subset of basis elements

$$A(e_a \otimes \varepsilon_b) = e_a \otimes \varepsilon_b - e_b \otimes \varepsilon_a$$

Consider a linear space

$$A(T_0^2) = Span(A(e_a \otimes \varepsilon_b)\,.$$

This is a subset of T_0^2 of *antisymmetric tensors* .

Proposition 0.2: *1. Every antisymmetric tensor is represented as*

$$T = \frac{1}{2!} T^{ab} A(e_a \otimes \varepsilon_b)\,,$$

where $T^{ab} = -T^{ba}$.

2. The dimension of $S(T_0^2)$ *is* $n(n-1)/2$.

3. The space $A(T_0^2)$ *does not depend on the basis used for its definition, i.e., if a tensor is antisymmetric in some basis it is antisymmetric in every basis.*

An antisymmetric tensor of type $(p,0)$ defined analogously by the basis

$$A(e_{a_1} \otimes \cdots \otimes e_{a_p})$$

where A is the operator of odd permutations. Every antisymmetric tensor is represented as

$$T = \frac{1}{p!} T^{a_1 \cdots a_p} A(e_{a_1} \otimes \cdots \otimes e_{a_p})\,.$$

An antisymmetric tensor of type $(0,p)$ defined by the basis

$$A(\vartheta^{a_1} \otimes \cdots \otimes \vartheta^{a_p})$$

and represented as

$$T = \frac{1}{p!} T_{a_1 \cdots a_p} A(\vartheta^{a_1} \otimes \cdots \otimes \vartheta^{a_p})\,.$$

Proposition 0.3: *An antisymmetric tensor of type* $(p,0)$ *or* $(0,p)$ *with* $p > n$ *is zero (*$dim V = n$*).*

0.3 Exterior forms

An exterior p-form is an antisymmetric tensor of type $(0,p)$. A basis of exterior forms is denoted as

$$\vartheta^{a_1} \wedge \cdots \wedge \vartheta^{a_p} = A(\vartheta^{a_1} \otimes \cdots \otimes \vartheta^{a_p}) \,.$$

Exterior p-form - another definition An exterior p-form is an antisymmetric linear map

$$w : \underbrace{V \times \cdots \times V}_{p \text{ factors}} \to \mathbb{R} \,,$$

i.e.,

$$w(v_1\,, \cdots, v_a\,, \cdots, v_b\,, \cdots, v_p) = -w(v_1\,, \cdots, v_b\,, \cdots, v_a\,, \cdots, v_p)$$

The linear space of all p-forms is denoted as $\Lambda^p(V)$. Its dimension is

$$dim \Lambda^p(V) = \frac{n!}{p!(n-p)!}$$

For $n = 4$ the dimensions of spaces of p-forms are

$$\begin{aligned} p &= 0\,,1\,,2\,,3\,,4 \\ dim &= 1\,,4\,,6\,,4\,,1\,. \end{aligned}$$

Exterior product

Definition 0.4: Exterior product (wedge product, Grassmann product) of a p-form and a q-forms is a linear mapping

$$\wedge : (\Lambda^p(V), \Lambda^q(V)) \to \Lambda^{p+q}(V)\,, \qquad (\alpha, \beta) \to \alpha \wedge \beta\,,$$

such that

$$(\alpha \wedge \beta)(v_1\,, \cdots, v_{p+q}) = \frac{1}{p!q!} \sum_{\pi} (\text{sign}\,\pi)\pi[\alpha(v_1\,, \cdots, v_p)\beta(v_{p+1}\,, \cdots, v_{p+q})$$

∎

In particular, if α and β are 1-forms

$$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$$

Proposition 0.5:

$$
\begin{array}{ll}
1) & (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \\
2) & (\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma \\
3) & k(\alpha \wedge \beta) = (k\alpha) \wedge \beta = \alpha \wedge (k\beta) \\
4) & \alpha \wedge \beta = (-1)^{pq} \beta \wedge \gamma
\end{array}
$$

Definition 0.6: Interior product (inner product, contraction) of a vector and a p-form is a linear mapping

$$\rfloor : V \times \Lambda^p(V) \to \Lambda^{p-1}(V), ,$$

such that

$$(X \rfloor w)(v_1\,, \cdots, v_{p-1}) = w(X\,, v_1\,, \cdots, v_{p-1})$$

∎

For $p = 0$, $X \rfloor w = 0$.
For $p = 1$, $X \rfloor w = w(X)$, particular $e_a \rfloor \vartheta^b = \delta^a_b$.

Proposition 0.7:

$$
\begin{array}{ll}
1) & v \rfloor (\alpha + \beta) = v \rfloor \alpha + v \rfloor \beta \\
2) & (v + u) \rfloor \alpha = v \rfloor \alpha + u \rfloor \alpha \\
3) & v \rfloor u \rfloor w = -u \rfloor v \rfloor w \\
4) & v \rfloor (\alpha \wedge \beta) = (v \rfloor \alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (v \rfloor \beta)
\end{array}
$$

Definition 0.8: Grassmann (exterior) algebra is a $\mathbb{Z}$-graded algebra defined on the direct sum

$$\Lambda^\bullet(V) = \bigoplus_{p=0}^{n} \Lambda^p(V)$$

∎

Vector spaces with pseudo-scalar product

Definition 0.9: *Euclidean metric* on a vector space V is a symmetric tensor of type $(2, 0)$ such that for some basis $\{e_a\}$ on V

$$g(e_a, e_b) = g_{ab} = diag(+1, +1, \cdots, +1)\,.$$

Lorentzian metric on a vector space V is a symmetric tensor of type $(2,0)$ such that for some basis $\{e_a\}$ on V

$$g(e_a, e_b) = g_{ab} = diag(-1, +1, \cdots, +1)\,.$$

Pseudo-Euclidean metric on a vector space V is a symmetric tensor of type $(2,0)$ such that for some basis $\{e_a\}$ on V

$$g(e_a, e_b) = g_{ab} = diag(-1, \cdots, -1, +1, \cdots, +1)\,.$$

▮

Definition 0.10: *Scalar product* of two vectors $X, Y \in V$

$$(X, Y) = g(X, Y)$$

▮

Musical isomorphisms

Definition 0.11: Define isomorphism (flat)

$$^\flat : V \to V^* \qquad \text{for}\, X \in V \quad {}^\flat : X \mapsto X^\flat \in V^*$$

such that for every $Y \in V$

$$Y \rfloor X^\flat = g(X, Y)$$

▮

Definition 0.12: Define isomorphism (sharp)

$$^\sharp : V^* \to V \qquad \text{for}\, w \in V^* \quad {}^\sharp : w \mapsto w^\sharp \in V$$

such that for every $Y \in V$

$$Y \rfloor w = g(w^\sharp, Y)$$

▮

Definition 0.13: Scalar product of 1-forms $\alpha, \beta \in V^*$

$$\hat{g}(\alpha, \beta) = g(\alpha^\sharp, \beta^\sharp)$$

▮

Notations:

$$g(e_a, e_b) = g_{ab} \qquad\qquad \hat{g}(\vartheta^a, \vartheta^b) = g^{ab}$$

Proposition 0.14:

$$g^{ab} g_{ac} = \delta^b_c$$

Metrical volume form

Definition 0.15: Let ϑ^a be a basis in V^*. Define a volume form

$$vol = \vartheta^1 \wedge \cdots \wedge \vartheta^n = \frac{1}{n!}\varepsilon_{a_1 \cdots a_n}\vartheta^{a_1} \wedge \cdots \wedge \vartheta^{a_n}$$

▮

Proposition 0.16: *Volume form is a twisted tensor.*

Hodge map

Definition 0.17: Hodge map is a linear map

$$* : \Lambda^p \to \Lambda^{n-p}\,, \qquad \text{where} \qquad n = dimV$$

that satisfies

$$*(w \wedge \phi) = \phi^\sharp \rfloor * w$$

for every w — p-form, and ϕ — 1-form. In addition,

$$*1 = vol = \frac{1}{n!}\varepsilon_{a_1 \cdots a_n}\vartheta^{a_1} \wedge \cdots \wedge \vartheta^{a_n}$$

▮

Example 0.18:

$$*\vartheta^a = \frac{1}{3!}g^{am}\varepsilon_{mnpq}\vartheta^n \wedge \vartheta^p \wedge \vartheta^q$$

In Lorentzian metric with signature $(+,-,-,-)$, for orthonormal coframe ϑ^a (Check!)

$$*\vartheta^0 = \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$$
$$*\vartheta^1 = \vartheta^0 \wedge \vartheta^2 \wedge \vartheta^3$$
$$*\vartheta^2 = -\vartheta^0 \wedge \vartheta^1 \wedge \vartheta^3$$
$$*\vartheta^3 = \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2$$

▮

Proposition 0.19:

$$*^2 = (-1)^{p(n-p)+i}$$

$$*w_1 \wedge w_2 = *w_2 \wedge w_1$$

$$*w \wedge \vartheta^a = -g^{ab} * (e_b \rfloor w)$$