Tensor algebra

0.1 Tensor product — first definition

Definition 0.1: Let V and U be vector spaces on $\mathbb{R}$ of dimensions $dimV = m$, $dimU = n$. Consider some basses in V and U denoted as

$$\{e_a \in V; a = 1, \cdots, m\}, \qquad \text{and} \qquad \{f_\alpha \in V; \alpha = 1, \cdots, n\}$$

Denote by

$$e_a \otimes f_\alpha = (e_a, f_\alpha)$$

an ordered set of pairs of basis vectors. Let

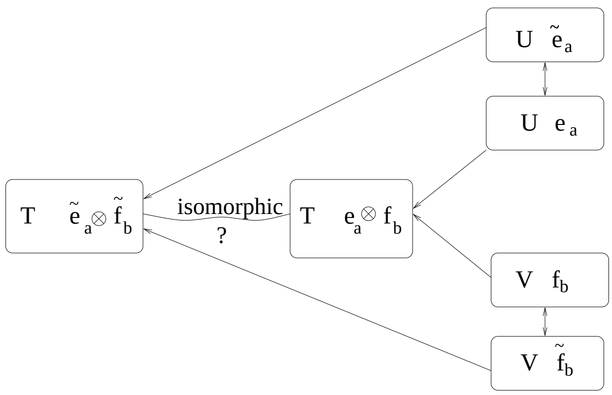
$$T = Span(e_a \otimes f_\alpha),$$

It means that every $t \in T$ is represented as

$$t = t^{a\alpha} e_a \otimes f_\alpha.$$

We refer to T as the *pre-tensor product vector space.* ∎

Remark 0.2: Note, that $e_a \otimes f_\alpha$ is no more that only a symbol. Particularly, we have not fixed how the expression $(2e_a) \otimes f_\alpha$ is connected to $e_a \otimes f_\alpha$. More generally, the question is how the basis $e_a \otimes f_\alpha$ is changed under linear changes of the basses e_a and f_α.



∎

Consider changes of the basses in U and V

$$e_a = A_a{}^b \tilde{e}_b\,, \qquad f_\alpha = B_\alpha{}^\beta \tilde{f}_\beta\,.$$

The general linear relations between tensor basses is

$$e_a \otimes f_\alpha = \chi_{a\alpha}{}^{b\beta} \tilde{e}_a \otimes \tilde{f}_\alpha,$$

where $\chi_{a\alpha}{}^{b\beta}$ is a some matrix.

Definition 0.3:

1) A pre-tensor vector space with the matrix

$$\chi_{a\alpha}{}^{b\beta} = A_a{}^b B_\alpha{}^\beta$$

is called an ordinary *tensor product* of the spaces U and V and denoted as $U \otimes V$.

2) For the matrix

$$\chi_{a\alpha}{}^{b\beta} = \sigma^p A_a{}^b B_\alpha{}^\beta \qquad p \in \mathbb{Q}$$

we have a *pseudo tensor product of class* p of the spaces U and V.

3) If $\det A_a{}^b = \det B_\alpha{}^\beta = \Delta$ and the matrix of the transformations is

$$\chi_{a\alpha}{}^{b\beta} = |\Delta|^{-p} A_a{}^b B_\alpha{}^\beta$$

we have *tensor density of the weight* p.

4) If $\det A_a{}^b = \det B_\alpha{}^\beta = \Delta$ and the matrix of the transformations is

$$\chi_{a\alpha}{}^{b\beta} = \frac{\Delta}{|\Delta|} A_a{}^b B_\alpha{}^\beta$$

we have W-tensor (twisted tensor, odd tensor, impair tensor).

5) If $\det A_a{}^b = \det B_\alpha{}^\beta = \Delta$ and the vector spaces are considered on $\mathbb{C}$ *tensor* Δ*-densities of the weight* p *and anti-weight* q are defined with the matrix of the transformations

$$\chi_{a\alpha}{}^{b\beta} = \Delta^{-p} \bar{\Delta}^{-q} A_a{}^b B_\alpha{}^\beta\,.$$

∎

The vector spaces U and V are mostly (possible always) used to be of the same dimensions. We denote the basis of the space V as e_a and the basis of the dual space V^* as ϑ^a. The duality relation is denoted as

$$e_a \rfloor \vartheta^b = \delta^b_a.$$

It is a bilinear relation: For $v \in V$ and $w \in V^*$,

$$v \rfloor w = v^a w_b (e_a \rfloor \vartheta^b) = v^a w_a$$

If the basis of the vector space V is changed as $e_a = A_a{}^b \tilde{e}_b$ and the dual basis is changed as $\vartheta^a = B^a{}_b \tilde{\vartheta}^b$, the dual relations between the basses yields

$$\delta^b_a = e_a \rfloor \vartheta^b = A_a{}^m B^b{}_n \tilde{e}_m \rfloor \tilde{\vartheta}^n = A_a{}^m B^b{}_n \delta^n_m = A_a{}^m B^b{}_m$$

Thus B is the reciprocal matrix.

Definition 0.4:

Tensors of the type $(2,0)$ are the elements of the tensor space $V \otimes V$, it means that its elements are represented as

$$t = t^{ab} e_a \otimes e_b \,.$$

Tensors of the type $(0,2)$ are the elements of the tensor space $V^* \otimes V^*$, it means that its elements are represented as

$$t = t_{ab} \vartheta^a \otimes \vartheta^b \,.$$

Tensors of the type $(1,1)$ are the elements of the tensor space $V \otimes V^*$, or $V^* \otimes V$, they are represented as

$$t = t_a{}^b \vartheta^a \otimes e_b \,, \qquad \text{or} \qquad t = t^a{}_b e_a \otimes \vartheta^b \,.$$

Tensors of the type (p,q) are the elements of the tensor space

$$V \otimes \cdots V \otimes V^* \cdots V^*,$$

they are represented as

$$t = t_{a_1 \cdots a_p}{}^{b_1 \cdots b_q} \vartheta^{a_1 \cdots a_p} \otimes e_{b_1 \cdots b_q} \,.$$

∎

0.2 Tensor product — second definition

Definition 0.5: Let U and V be vector spaces on $\mathbb{R}$.

Denote by $F(V, U)$ the vector space free generated by the elements of the Cartesian product $V \times U$, it means that $F(V, U)$ is a set of all final linear combinations of the pairs (v, u) , $v \in V$, $u \in U$.

Denote by $R(V, U)$ the subspace of $F(V, U)$ included the elements of the form

$$(v_1 + v_2, u) - (v_1, u) - (v_2, u)\,,$$

$$(v, u_1 + u_2) - (v, u_1) - (v, u_2)\,,$$

$$(av, u) - a(v, u)\,,$$

$$(v, au) - a(v, u)\,,$$

where $a \in \mathbb{R}$, $v, v_1, v_2 \in V$ and $u, u_1, u_2 \in U$.

Factor-space

$$V \otimes U = F(V, U)/R(V, U)$$

is called tensor product of V and U. ∎

0.3 Tensor product — third definition

Definition 0.6: The tensor product $U \otimes V$ of the vector spaces U and V on $\mathbb{R}$ is the vector space of all bilinear (linear in every argument) maps

$$U \times V \to \mathbb{R}\,.$$

In general, a tensor of the type (p, q) is a multilinear map

$$t\,:\,\underbrace{V^* \times \cdots \times V^*}_{p \text{ times}} \times \underbrace{V \times \cdots \times V}_{q \text{ times}} \to \mathbb{R}\,.$$

Particular, for $v \in V$, $w \in V^*$,
- a tensor of the type $(1, 0)$ (vector) is a map

$$v\,:\,V^* \to \mathbb{R}\,, \qquad v(w) \in \mathbb{R}$$

- a tensor of the type $(0, 1)$ (1-form) is a map

$$v\,:\,V \to \mathbb{R}\,, \qquad w(v) \in \mathbb{R}$$

- a tensor of the type $(2, 0)$ is a map

$$t : V^* \times V^* \to \mathbb{R}, \qquad t(w_1, w_2) \in \mathbb{R}$$

- a tensor of the type $(0, 2)$ is a map

$$v : V \times V \to \mathbb{R}, \qquad t(v_1, v_2) \in \mathbb{R}$$

- a tensor of the type $(1, 1)$ (of two types) is a map

$$v : V \times V^* \to \mathbb{R}, \qquad t(w, v) \in \mathbb{R}$$

▮

0.4 Tensor product — fourth definition

Definition 0.7: Tensor is a set of numbers that changes under a transformations $\widetilde{e}_a = A^m{}_a e_m$ of the basis by the "tensor rule"

$$\widetilde{t}^{a_1 \cdots a_p}{}_{b_1 \cdots b_q} = A^{a_1}{}_{m_1} \cdots A^{a_p}{}_{m_p} (A^{-1})^{b_1}{}_{n_1} \cdots (A^{-1})^{b_q}{}_{n_q} t^{m_1 \cdots m_p}{}_{n_1 \cdots n_q}$$

▮

0.5 Tensor operations

1. Addition of tensors

Let tensors T and T' be of the same type (r, s). $T + T'$ is the tensor of the type (r, s) such that for all $X_i \in V$, $i = 1, \cdots s$, $w_j \in V^*$, $j = 1, \cdots r$

$$\begin{aligned} &(T + T')(w_1, \cdots, w_r, X_1, \cdots, X_s) = \\ &T(w_1, \cdots, w_r, X_1, \cdots, X_s) + T'(w_1, \cdots, w_r, X_1, \cdots, X_s) \end{aligned}$$

In components,

$$(T + T')^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} = T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} + T'^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}$$

2. Multiplication of a tensor by a scalar

$$(\alpha T)(w_1, \cdots, w_r, X_1, \cdots, X_s) = \alpha T(w_1, \cdots, w_r, X_1, \cdots, X_s)$$

In components,

$$(\alpha T)^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} = \alpha T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}$$

3. Tensor product Let tensors T and T' be of the types (r,s) and (p,q). Tensor product is the tensor of the type $(r+p, s+q)$, which maps the elements $X_1, \cdots X_{s+q} \in V$, $w_1, \cdots, w_{r+p} \in V^*$ into the number

$$(T \otimes T')(X_1, \cdots X_{s+q}, w_1, \cdots, w_{r+p}) =$$
$$T(X_1, \cdots X_s, w_1, \cdots, w_r) T'(X_{s+1}, \cdots X_{s+q}, w_{r+1}, \cdots, w_{r+p})$$

In components,

$$(T \otimes T')^{a_1 \cdots a_{r+p}}{}_{b_1 \cdots b_{s+q}} = T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s} T'^{a_{r+1} \cdots a_{r+p}}{}_{b_{s+1} \cdots b_{s+q}}$$

4. Contraction of a tensor

Let tensor T be of the type (r,s). T' is the tensor of the type $(r-1, s-1)$ such that for all $X_i \in V$, $i = 1, \cdots s-1$, $w_j \in V^*$, $j = 1, \cdots r-1$

$$T'(w_1, \cdots, w_{r-1}, X_1, \cdots, X_{s-1}) = T(w_1, \cdots, \vartheta^a, \cdots, w_{r-1}, X_1, \cdots, e_a, \cdots, X_{s-1})$$

In components,

$$T'^{a_1 \cdots a_{r-1}}{}_{b_1 \cdots b_{s-1}} = T^{a_1 \cdots a \cdots a_{r-1}}{}_{b_1 \cdots a \cdots b_{s-1}}$$

So, contraction occurs when a pair of indices (one a subscript, the other a superscript) of a mixed tensor are set equal to each other so that a summation over that index takes place (due to the Einstein summation convention). The result is another tensor whose rank is reduced by 2.

5. Symmetrization of a tensor

Let T be a tensor of type $(2,0)$. Its symmetrization is the tensor $S(T)$ of type $(2,0)$ such that for all $w_1, w_2 \in V^*$

$$S(T) = \frac{1}{2!}(T(w_1, w_2) + T(w_2, w_1))$$

In components,

$$T^{(ab)} = \frac{1}{2!}(T^{ab} + T^{ba})$$

The symmetrization can be applied for arbitrary number of up or dawn indices. For instance,

$$K^a{}_{(bcd)} = \frac{1}{3!}(K^a{}_{(bcd)} + K^a{}_{(bdc)} + K^a{}_{(cbd)} + K^a{}_{(cdb)} + K^a{}_{(dbc)} + K^a{}_{(dcb)})$$

$$K^a{}_{(bc)d} = \frac{1}{2!}(K^a{}_{(bcd)} + K^a{}_{(cbd)})$$

$$K^a{}_{(b|c|)d} = \frac{1}{2!}(K^a{}_{(bcd)} + K^a{}_{(dcb)})$$

5. Antisymmetrization of a tensor

Let T be a tensor of type $(2, 0)$. Its antisymmetrization is the tensor $S(T)$ of type $(2, 0)$ such that for all $w_1, w_2 \in V^*$

$$A(T) = \frac{1}{2!}(T(w_1, w_2) - T(w_2, w_1)$$

In components,

$$T^{[ab]} = \frac{1}{2!}(T^{ab} - T^{ba})$$

The symmetrization can be applied for arbitrary number of up or dawn indices. For instance,

$$K^a{}_{[bcd]} = \frac{1}{3!}(K^a{}_{(bcd)} - K^a{}_{(bdc)} - K^a{}_{(cbd)} + K^a{}_{(cdb)} + K^a{}_{(dbc)} - K^a{}_{(dcb)})$$

$$K^a{}_{[bc]d} = \frac{1}{2!}(K^a{}_{(bcd)} - K^a{}_{(cbd)})$$

$$K^a{}_{[b|c|]d} = \frac{1}{2!}(K^a{}_{(bcd)} - K^a{}_{(dcb)})$$

0.6 Tensor algebra

In abstract algebra, a graded algebra is an algebra over a field, in which there is a consistent notion of the weight of an element. The idea is that the weights of elements should add, when elements are multiplied. One has to allow the 'inconsistent' addition of elements of different weights. A formal definition follows.

Let G be an Abelian group. A G-graded algebra A is an algebra with a direct sum decomposition

$$A = \bigoplus_{i \in G} A_i$$

such that

$$A_i A_j \subseteq A_{i+j}$$

An element of the i-th subspace A_i is said to be a homogeneous, or a pure, element of degree i.

Important example of graded algebras is the tensor algebra $T^\bullet V$ of a vector space V defined as a direct sum of all tensor spaces

$$T^\bullet V = \bigoplus_{p,q} T^p_q$$

The basis of the algebra is the union of all tensor basses

$$1, e_a, \vartheta^a, \cdots$$

and the element is defined as

$$\alpha 1 + \beta^a e_a + \gamma_a \vartheta^a + \cdots .$$