Maxwell equations and their symmetry

0.1 Maxwell equations

Definition 0.1: The electric and magnetic phenomena are described by two independent vector fields:

$$E:\mathbb{R}^4\to\mathbb{R}^4\,,\qquad E=E(t,x)\ ---\ \text{electric field}\,.$$

$$H:\mathbb{R}^4\to\mathbb{R}^4\,,\qquad H=H(t,x)\ ---\ \text{magnetic field}\,.$$

▮

Remark 0.2: Note that 3-vectors (vectors of 3 components) are correct defined objects in Galileian spacetime. Which other objects are defined? ▮

Differential operations

Definition 0.3: Gradient of a scalar field $\varphi:\mathbb{R}^3\to\mathbb{R}$ is a map grad $:\mathbb{R}\to\mathbb{R}^3$

$$\text{grad}\,\varphi=e_1\frac{\partial\varphi}{\partial x^1}+e_1\frac{\partial\varphi}{\partial x^1}+e_1\frac{\partial\varphi}{\partial x^1}=e_a\frac{\partial\varphi}{\partial x^a}$$

▮

Definition 0.4: Divergence of a vector field $H:\mathbb{R}^3\to\mathbb{R}^3$ is a map div $:\mathbb{R}^3\to\mathbb{R}$

$$\text{div}\,H=\text{div}\,(e_aH^a)=\frac{\partial H^1}{\partial x^1}+\frac{\partial H^2}{\partial x^2}+\frac{\partial H^3}{\partial x^3}=\frac{\partial H^a}{\partial x^a}$$

▮

Definition 0.5: Rotor (Curl) of a vector field $H:\mathbb{R}^3\to\mathbb{R}^3$ is a map rot $:\mathbb{R}^3\to\mathbb{R}^3$

$$\text{rot}\,H=\begin{vmatrix} e_1 & e_2 & e_3\\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3}\\ H^1 & H^2 & H^3\end{vmatrix}\,,$$

where the determinant is to be evaluated by the first row. ▮

Definition 0.6: Laplacian of a scalar field $\varphi:\mathbb{R}^3\to\mathbb{R}$ is a map $\triangle:\mathbb{R}\to\mathbb{R}$

$$\triangle\varphi=\text{div}\,\text{grad}\,\varphi$$

▮

Maxwell's system

Definition 0.7: The electric and magnetic fields satisfy the system of partial differential equations

$$
\begin{aligned}
&(1) \qquad && \operatorname{div} E = \rho\,, \\
&(2) \qquad && \operatorname{rot} E + \frac{\partial H}{\partial t} = 0\,, \\
&(3) \qquad && \operatorname{div} H = 0\,, \\
&(4) \qquad && \operatorname{rot} H - \frac{\partial E}{\partial t} = j\,,
\end{aligned}
$$

where $\rho : \mathbb{R}^3 \to \mathbb{R}$ is a scalar field, called *electric charge density*, while $J : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field, called *electric current.* ∎

Remark 0.8: The derivative operators $(\operatorname{grad}, \operatorname{div}, \operatorname{rot})$ are introduced only for fields on $\mathbb{R}^3$. The Galileian space is 4 dimensional. However, because of the fixed time map, the time coordinate is only as a parameter. So the differential operators can be considered also fields defined on the Galileian space, as they used in the Maxwell system. ∎

Remark 0.9: Usually the support of the electric charge density and the electric current are points and lines in $\mathbb{R}^3$. In most points they can be taken to be zero. ∎

Problem 0.10: What are the symmetries of the Maxwell system and how they are related to the group of symmetries of the Galileian space? ∎

Theorem 0.11:
1) For an arbitrary vector field $M : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field $A : \mathbb{R}^3 \to \mathbb{R}^3$ is defined such that

$$\operatorname{div} M = 0 \qquad \text{iff} \qquad M = \operatorname{rot} A\,.$$

2) For an arbitrary vector field $N : \mathbb{R}^3 \to \mathbb{R}^3$ a scalar field $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is defined such that

$$\operatorname{rot} N = 0 \qquad \text{iff} \qquad N = \operatorname{grad} \varphi\,.$$

Theorem 0.12: *Maxwell system is equivalent to the system of Poisson type equations*

$$
\begin{aligned}
\frac{\partial^2 \varphi}{\partial t^2} - \triangle \varphi = \rho\,, \\
\frac{\partial^2 A}{\partial t^2} - \triangle A = j\,,
\end{aligned}
$$

where the vector potential A is defined from

$$H = \operatorname{rot} A\,,$$

the scalar potential is defined from

$$E = -\frac{\partial A}{\partial t} + \operatorname{grad}\varphi$$

and they satisfied the Lorenz gauge condition

$$\operatorname{div} A + \frac{\partial \varphi}{\partial t} = 0$$

Remark 0.13: The Laplacian of the vector is considered here in a component-wise form

$$\triangle A = (\triangle A^a) e_a \,.$$

∎

0.2 Symmetries of Maxwell equations

Instead of looking for the symmetries of the Maxwell system itself we will consider the equation for the scalar potential

$$(*) \qquad \frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial z^2} = \rho \,.$$

Recall that in the affine space $\{A, V\}$ we have two types of the the transformations:
(1) The shift of the basis point

$$O \to O + m \,, \qquad O \in A \,, \quad m \in V$$

(2) The change of the basis in V

$$e_\alpha \to h_\alpha{}^\beta e_\beta \,, \qquad h_\alpha{}^\beta \in GL(4, \mathbb{R})$$

Corresponding change of the coordinates of a point in A^4 is

$$\tilde{x}^\beta = x^\alpha h_\alpha{}^\beta + m^\beta$$

The shift of the basis point does not changes the equation (*).
Consider different subgroups of $GL(4, \mathbb{R})$.
1) rotations

$$\tilde{t} = t \,, \qquad \tilde{x} = ax + by \,,$$
$$\tilde{z} = z \,, \qquad \tilde{y} = cx + dy \,,$$

Correspondingly,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = (a^2 + b^2)\frac{\partial^2 \varphi}{\partial \tilde{x}^2} + 2(ac + bd)\frac{\partial^2 \varphi}{\partial \tilde{x} \partial \tilde{y}} + (c^2 + d^2)\frac{\partial^2 \varphi}{\partial \tilde{y}^2}$$

Thus the invariance of the equation (*) requires

$$a^2 + b^2 = 1 \,, \quad c^2 + d^2 = 1 \,, \quad ac + bd = 0 \,.$$

The solution of this system is

$$\begin{aligned}\tilde{x} &= \quad \cos\theta\ x + \sin\theta\ y \\ \tilde{y} &= -\sin\theta\ x + \cos\theta\ y,\end{aligned}$$

i.e., the corresponding matrix is from $O(2)$.

2) (t, x) - transformations (pseudo-rotations, busts)

$$\tilde{t} = at + bx\,, \qquad \tilde{x} = ct + dx\,,$$

$$\tilde{y} = y\,, \qquad \tilde{z} = z\,,$$

Correspondingly,

$$\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^2\varphi}{\partial x^2} = (a^2 - b^2)\frac{\partial^2\varphi}{\partial\tilde{t}^2} + 2(ac - bd)\frac{\partial^2\varphi}{\partial\tilde{t}\partial\tilde{x}} + (c^2 - d^2)\frac{\partial^2\varphi}{\partial\tilde{x}^2}\,.$$

Thus the invariance of the equation (*) requires

$$a^2 - b^2 = 1\,, \quad c^2 - d^2 = 1\,, \quad ac - bd = 0\,.$$

The solution of this system is

$$\begin{aligned}\tilde{t} &= \cosh\theta\ t + \sinh\theta\ x \\ \tilde{x} &= \sinh\theta\ t + \cosh\theta\ x\,.\end{aligned}$$

It is easy to check that such matrices compose a group, which is called *the group of pseudo-rotations* $O(1,1)$.

Denote

$$\sinh\theta = \frac{v}{\sqrt{1-v^2}}\,, \cosh\theta = \frac{1}{\sqrt{1-v^2}}$$

Thus, we derive the *Lorentz transformations*

$$\begin{aligned}\tilde{t} &= \frac{1}{\sqrt{1-v^2}}(t + xv) \\ \tilde{x} &= \frac{1}{\sqrt{1-v^2}}(t + xv)\,.\end{aligned}$$

Recall that the Galileian transformations are

$$\begin{aligned}\tilde{t} &= t \\ \tilde{x} &= t + xv\,.\end{aligned}$$

The result: The transformation of the Galileian spacetime are not the invariant transformations of the Maxwell equations. The pure Galileian transformations are replaced by new Lorentz transformations. Consequently, also the Galileian spacetime has to be replaced by some other model.

0.3 Minkowski spacetime

Definition 0.14: Minkowski vector space M is a vector space $V \sim \mathbb{R}^4$ endowed with a bilinear for (Minkowski "vector product") $(\cdot,\cdot) : V \times V \to \mathbb{R}$ such that for two arbitrary vectors $u = u^\alpha e_\alpha \in V$, $v = v^\alpha e_\alpha \in V$

$$(u, v) = u^0 v^0 - u^1 v^1 - u^2 v^2 - u^3 v^3 = \eta_{\alpha\beta} u^\alpha v^\beta \,,$$

where

$$\eta_{\alpha\beta} = diag(1, -1, -1, -1).$$

$\blacksquare$

The square of a vector

$$(u, u) = (u^0)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2$$

is not positive defined, so different types of vectors are defined

- Spacelike vectors with $(u, u) < 0$
- Timelike vectors with $(u, u) > 0$
- Null (light) vectors with $(u, u) = 0$

Under a transformation of a basis $e_\alpha = A_\alpha{}^\beta \tilde{e}_\beta$ the components of a vector $u = u^\alpha e_\alpha$ change as $\tilde{u}^\beta = u^\alpha A_\alpha{}^\beta$.

Proposition 0.15: *The transformation of the basis preserved the Minkowski vector product are satisfied*

$$\eta_{\mu\nu} A_\alpha{}^\mu A_\beta{}^\nu = \eta_{\alpha\beta} \,.$$

Proposition 0.16: *The set of the transformation (with positive determinant) preserved the Minkowski vector product is with group 6 generators, which includes the rotations (3 generators) and the Lorentz transformations (3 generators). The group is denoted as $SO(1,3)$.*

Definition 0.17: Null cone in the Minkowski vector space M is a set of all null vectors. $\blacksquare$

Definition 0.18: Minkowski spacetime is an affine space A^4 with associated Minkowski vector space. $\blacksquare$

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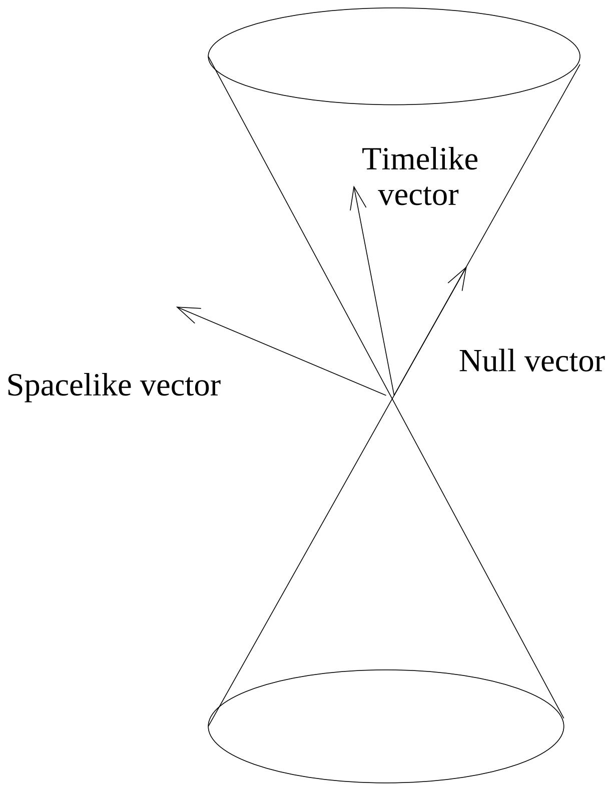
Timelike
vector
Spacelike vector
Null vector