1 Action functional, Euler-Lagrange equations

Definition 1.1: Consider a curve

$$\gamma : \mathbb{R} \to R^n\,, \qquad x = x(t)\,.$$

The action functional on the curve is defined as

$$\Phi(\gamma) = \int_{t_2}^{t_1} L(\gamma) dt\,,$$

thus $L(\gamma)$ is a function on the infinity space of differential curves to $\mathbb{R}$. ∎

Example 1.2: The length of a curve $\gamma = \{x(t)\,|t_1 \le t \le t_1\}$ is a functional

$$\ell = \int_{t_1}^{t_2} \sqrt{1 + \dot{x}^2(t)} dt\,.$$

∎

Variation of a curve

$$\tilde{\gamma} = \gamma + h \qquad \tilde{x}(t) = x(t) + h(t)$$

Definition 1.3: Functional $\Phi(\gamma)$ is differentiable if

$$\Phi(\gamma + h) - \Phi(\gamma) = F(\gamma, h) + R(\gamma, h)\,,$$

where F is a linear function of h, while $R(\gamma, h) = \mathcal{O}(h^2)$ in the sense that $|h| < \epsilon$ and $|dh/dt| < \epsilon$ yield $|R| < C\epsilon^2$. ∎

Theorem 1.4: *For $L(x, \dot{x}, t) \qquad x(t) : \mathbb{R} \to R^n$ a differential function from 3 variables, the functional*

$$\Phi(\gamma) = \int_{t_2}^{t_1} L(x, \dot{x}, t) dt\,,$$

is differentiable and

$$F(h) = \int_{t_2}^{t_1} \left[\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \right] h dt + \left(\frac{\partial L}{\partial \dot{x}} h \right) \Big|_{t_2}^{t_1}$$

Lemma 1.5: *If a continuous function $f(t)$ satisfies*

$$\int_{t_1}^{t_2} f(t)h(t)dt = 0$$

for any continuous function $h(t)$ with $h(t_1) = h(t_2) = 0$, then $f(t) = 0$.

Theorem 1.6: *The curve $x = x(t)$ is an extremal of the functional*

$$\Phi(\gamma) = \int_{t_2}^{t_1} L(x, \dot{x}, t)dt\,,$$

if and only if, along the curve,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0\,.$$

For a one point in the 3 dimensional space $x = x^i e_i \in \mathbb{R}^3$.
For N points in the 3 dimensional space $x = x^i e_i \in \mathbb{R}^{3N}$.

2 Energy

Theorem 2.1: *For a Lagrangian, which is not depends on time t explicitly,*

$$L = L(x, \dot{x})$$

the scalar (real function) of the energy

$$E := \frac{\partial L}{\partial \dot{x}}\,\dot{x} - L$$

is preserved along the trajectory $x = x(t)$, that means E does not depend of time.

Proof: For a shift of the time coordinate $t \to t + \tau$,

$$\delta L \ := \ L(x(t+\tau), \dot{x}(t+\tau)) - L(x, \dot{x}) = \Big(\frac{\partial L}{\partial x}\,\dot{x} + \frac{\partial L}{\partial \dot{x}}\,\ddot{x}\Big)\tau + \mathcal{O}(\tau^2)$$

Using the Euler-Lagrange equation

$$\begin{aligned} \delta L \ &= \ \Big(\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big)\dot{x} + \frac{\partial L}{\partial \dot{x}}\,\ddot{x}\Big)\tau + \mathcal{O}(\tau^2) = \\ &\quad \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\dot{x}\Big)\tau + \mathcal{O}(\tau^2)\,. \end{aligned}$$

The shift of the time coordinate is an invariance transformation of the Lagrangian, so $\delta L = 0$ for arbitrary τ, particularly in every order of τ. Thus

$$\frac{dE}{dt} = \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\dot{x}\Big) = 0\,.$$

∎

Remark 2.2: In all the expression above the scalar product of the vectors is subtended. For instance the expression for the energy, in the components, is

$$E = \frac{\partial L}{\partial \dot{x}^i}\, \dot{x}^i - L$$

Observe the positions of the indices (down-up)! The summation in i is subtended. ∎

For a Lagrangian of the form

$$L = T(\dot{x}) - U(x)$$

the energy takes the form

$$E = \frac{\partial T}{\partial \dot{x}}\dot{x} - T + U$$

If $T(\dot{x})$ is the homogeneous function of the order 2, than

$$E = T + U$$

- the total energy is the sum of the *kinetic energy* T and the *potential energy* U.

3 Momentum

Theorem 3.1: *For a Lagrangian, which is not depends on time x explicitly,*

$$L = L(\dot{x}, t)$$

the vector of the momentum

$$p := \frac{\partial L}{\partial \dot{x}}$$

is preserved along the trajectory $x = x(t)$, that means p does not depend of time.

Proof: For a shift of the space coordinate $x \to x + m$, where $x, m \in \mathbb{R}^3$

$$\frac{\partial L}{\partial x} = 0$$

Using the Euler-Lagrange equation (it means restricting this equation to th trajectory !)

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) = 0 \quad ===> \quad \frac{dp}{dt} = 0\,.$$

∎

Remark 3.2: In the components

$$p_i = \frac{\partial L}{\partial \dot{x}^i}$$

We can write it as a usual vector

$$p^i = \delta^{ij} \frac{\partial L}{\partial \dot{x}^i}$$

▮

Remark 3.3: Observe the relation

$$E = p_i x^i - L = \delta_{ij} p^i x^i - L$$

For the Lagrangian rewrites as $L = L(x, p)$, it is called *Hamiltonian*

$$H = px - L(x, p)\,.$$

▮

4 Examples

1-point system
Lagrangian is

$$L = \frac{m\dot{x}^2}{2}\,.$$

The equation of motion (EL) is

$$m\ddot{x} = 0\,.$$

The energy is

$$E = \frac{m\dot{x}^2}{2}\,.$$

The momentum is

$$p = mx\,.$$

2-point system
Lagrangian is

$$L = \frac{m_1\dot{x}^2}{2} + \frac{m_2\dot{y}^2}{2} - \frac{km_1m_2}{||x-y||}\,.$$

The equations of motion are

$$m_1\ddot{x} = -\frac{km_1m_2(x-y)}{||x-y||^2}\,.$$

$$m_2\ddot{y} = \frac{km_1m_2(x-y)}{||x-y||^2}\,.$$

The total energy is

$$E = \frac{m_1\dot{x}^2}{2} + \frac{m_2\dot{y}^2}{2} + \frac{km_1m_2}{||x-y||}\,.$$

The forces are

$$^{(1)}F = -^{(2)}F$$

- the third law of Newton.
The momentum is not preserved.