1 Groups of transformations

1.1 Vector space $V = \mathbb{R}^2$

Basis $e_i = \{e_1, e_2\}$

$$x = x^1 e_1 + x^2 e_2 = \sum_{i=1}^{2} x^i e_i = x^i e_i$$

Einstein's summation notation:
1) In every expression, the indices can appear only once (free indices) or twice (indices of summation).
2) An index of summation appears in each term both as a subscript and as a superscript and the sigma summation symbol is omitted.
3) The same free indices have to appear in both sides of an equation in the same position.

The transformation of the basis

$$e_i = h_i{}^j \tilde{e}_j$$

Thus

$$x = x^i e_i = x^i h_i{}^j \tilde{e}_j = \tilde{x}^j \tilde{e}_j \,.$$

This yields the transformation of coordinates

$$\tilde{x}^j = x^i h_i{}^j \,.$$

The transformations are reversible

$$\det h \neq 0$$

Levi-Civita symbol

$$\varepsilon_{ij} = \varepsilon^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Proposition 1.1: *The determinant can be represented as*

$$\det(h) = \frac{1}{2!}\varepsilon^{ij}\varepsilon_{mn}h_i{}^m h_j{}^n$$

Proposition 1.2: *The group of matrices*

$$H = \{h_i{}^j \mid \det h \neq 0\} = GL(2, \mathbb{R})$$

is a disjoint union of two sets

$$H_1 = \{h_i{}^j \mid \det(h) > 0\} \qquad --- \text{a subgroup}$$

$$H_2 = \{h_i{}^j \mid \det(h) < 0\} \qquad --- \text{non a subgroup}$$

Proposition 1.3: *H_1 is a normal subgroup of H: $hgh^{-1} \in H_1$ for every $g \in H_1$ and for every $h \in H$.*

1.2 Vector space $V = \mathbb{R}^n$

Basis $e_i = \{e_1, e_2 \cdots e_n\}$

$$x = x^i e_i$$

The transformation of the basis

$$e_i = h_i{}^j \tilde{e}_j$$

yields the transformation of coordinates

$$\tilde{x}^j = x^i h_i{}^j\,.$$

The transformations are reversible

$$\det(h) \neq 0$$

Definition 1.4: Levi-Civita symbol

$$\varepsilon_{i_1\cdots i_n} = \varepsilon_{j_1\cdots j_n} = \begin{cases} 1 & \text{if } (i_1\cdots i_n) \text{ is an even permutation of } (1,\cdots,n) \\ -1 & \text{if } (i_1\cdots i_n) \text{ is an odd permutation of } (1,\cdots,n) \\ 0 & \text{if } (i_1\cdots i_n) \text{ is not a permutation of } (1,\cdots,n) \end{cases}$$

∎

Proposition 1.5: *The determinant can be represented as*

$$\det(h) = \frac{1}{n!}\varepsilon^{i_1\cdots i_n}\varepsilon_{j_1\cdots j_n}h_{i_1}^{j_1}\cdots h_{i_n}^{j_n}$$

1.3 Vector space $V = \mathbb{R}^2$ with a scalar product

Euclidean scalar product

$$(x, y) = x^1y^1 + x^2y^2 = \sum_{i=1}^{2} x^iy^i = \delta_{ij}x^iy^j$$

$$\delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(x, x) \geq 0 \quad ===> |x| = (x, x)^{1/2} \qquad \text{norm}$$

Scalar product under transformation of the basis

$$(x, y) = \delta_{mn}\tilde{x}^m\tilde{x}^n = \left(\delta_{mn}h_i{}^m h_j{}^n\right)x^iy^j$$

Transformation of a basis that preserves the scalar product satisfied the equation

$$\delta_{mn}h_i{}^m h_j{}^n = \delta_{ij} \quad <===> \quad h^T h = I$$

The transformations form a group, denoted as $O(2, \mathbb{R})$.

Two solutions of the equation

$$h_1 = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \in SO(2, \mathbb{R})$$

$$h_1 = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \in SO(2, \mathbb{R})$$

$$h_2 = \begin{pmatrix} \cos\phi & -\sin\phi \\ -\sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1.4 Vector space $V = \mathbb{R}^3$ with a scalar product

Euclidean scalar product

$$(x, y) = \delta_{ij}x^iy^j$$

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transformation of a basis that preserves the scalar product satisfied the equation

$$\delta_{mn} h_i{}^m h_j{}^n = \delta_{ij} \quad <===> \quad h^T h = I$$

The transformations form a group, denoted as $O(3, \mathbb{R})$.

Theorem 1.6: Euler
Every matrix $h \in SO(3)$ is represented as $g = g_1 g_2 g_3$, where

$$g_1 = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ -\sin\phi & -\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & -\sin\theta & -\cos\theta \end{pmatrix}$$

$$g_3 = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ -\sin\psi & -\cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 Space-time

2.1 Vector space $V = \mathbb{R}^4$

Basis $e_\alpha = \{e_0, e_1\, e_2, e_n\}$

$$x = x^\alpha e_\alpha = x^0 e_0 + x^i e_i\,, \qquad \alpha = 0,1,2,3 \qquad i = 1,2,3$$

The transformation of the basis

$$e_\alpha = h_\alpha{}^\beta \tilde{e}_\beta$$

yields the transformation of coordinates

$$\tilde{x}^\beta = x^\alpha h_\alpha{}^\beta\,.$$

The transformations are reversible

$$\det(h) \neq 0 \qquad ===> \qquad h = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

2.2 Aristotle space-time

Vector space is a direct product $\mathbb{R}\times\mathbb{R}^3$. The transformation of basses in two vector spaces are independent.

$$h = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & h_i{}^j \end{pmatrix}, \qquad h_i{}^j \in GL(3,\mathbb{R})$$

The transformations preserved both norms

$$||t|| = |t|, \qquad \text{for} \quad t \in \mathbb{R}$$

$$||x|| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \qquad \text{for} \quad x \in \mathbb{R}^3$$

are

$$h = \begin{pmatrix} \pm 1 & 0 \\ 0 & h_i{}^j \end{pmatrix}, \qquad h_i{}^j \in SO(3,\mathbb{R})$$

2.3 Affine space

Let $\mathcal{A}$ be an affine space with an associated vector space $V \simeq \mathbb{R}^4$. For a fixed basis point $O \in \mathcal{A}$, an arbitrary point $P \in \mathcal{A}$ is represented as

$$P = O + x = O + x^\alpha e_\alpha, \qquad x \in V$$

Two different types of transformations appear in the affine space.

- Change of the basis point

$$O = O' + m, \qquad m \in V$$

- Transformation of the basis

$$e_\alpha = h_\alpha{}^\beta e_\beta, \qquad h_\alpha{}^\beta \in GL(4,\mathbb{R})$$

Corresponding transformation of the components — affine transformations

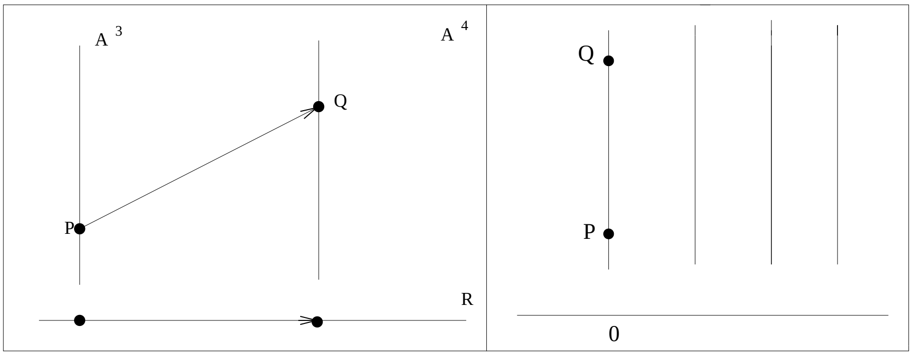
$$\tilde{x}^\beta = x^\alpha h_\alpha{}^\beta + m^\beta$$

Proposition 2.1: *Affine transformations form a group.*

2.4 Affine space + a fixed linear map

Consider an affine space $\mathcal{A}$ endowed with a linear map

$$t : \mathcal{A} \times \mathcal{A} \to \mathbb{R} \qquad <===> \qquad t : V \to \mathbb{R}$$



$$P - Q = x^\alpha e_\alpha = x^i e_i \,, \qquad \alpha = 0, 1, 2, 3 \qquad i = 1, 2, 3$$

For every two simultaneous events (right figure)

$$t(P - Q) = t(x^i e_i) = x^i t(e_i) = 0 \qquad ===>$$

$$t(e_i) = 0$$

We require this equation to preserve also after a change of the basis

$$t(\tilde{e}_i) = 0$$

Thus

$$t(e_i) = t({h_i}^\alpha \tilde{e}_\alpha) = {h_i}^0 t(\tilde{e}_0) + {h_i}^j t(\tilde{e}_j) = 0 \,.$$

Consequently

$${h_i}^0 = 0 \,.$$

The matrix of transformation of the basis that preserves the linear map t is of the form

$$h = \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

2.5 Galileo spacetime

Galileo spacetime = Affine space + a linear map+norm in $\mathbb{R}$ + norm in $\mathbb{R}^3$.
Events — points in Galileo spacetime.
Norm in $\mathbb{R}$

$$\tau = |t(P-Q)|$$

For two non-simultaneous events

$$\tau = |t(x^\alpha e_\alpha)| = |x^0||t(e_0)|$$

Change of the basis gives

$$\tau = |x^0||{h_0}^0||t(\tilde{e}_0)|$$

Requirement

$$|t(e_0)| = |t(\tilde{e}_0)|$$

gives

$$|{h_0}^0| = 1$$

Norm in $\mathbb{R}^3$
For two simultaneous events

$$d = |d(P-Q)| = (x,x)^{1/2} = \delta_{ij}x^i x^j$$

Change of the basis gives

$$d = \delta_{mn}{h_i}^m {h_j}^n x^i x^j$$

Invariance of the norm gives

$$\delta_{mn}{h_i}^m {h_j}^n = \delta_{ij} \qquad ===> \qquad {h_i}^m \in O(3,\mathbb{R}).$$

Proposition 2.2: *The matrix of transformation of the basis that preserves the Galilean structure is of the form*

$$h = \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ v_1 & * & * & * \\ v_2 & * & * & * \\ v_3 & * & * & * \end{pmatrix},$$

where the matrix 3×3 *is from* $O(3,\mathbb{R})$*, while* $v_i \in \mathbb{R}$*. The dimension of the Galileian group is* $10 = 1+3+3+3$.