1 From Aristotle to Galilei

When a pure mathematician is asked about the mathematical model of physical world $\mathcal{M}$ his first answer probably will be: A vector space isomorphic to $\mathbb{R}^3$. Taking into account the time coordinate, which is independent on the space, he will say: $\mathcal{M}$ is a direct product of two vector spaces

$$\mathcal{M} = U \times V\,, \qquad U \simeq \mathbb{R}, \qquad V \simeq \mathbb{R}^3,, \tag{1}$$

where U represents the time, while V is for the space. Consider the vector spaces endowed with two norms

$$\mathcal{M} = \{U \times V\,, \tau\,, \ell\}\,, \tag{2}$$

where the *time interval* is

$$\tau = |t| \qquad \text{for} \quad \text{every} \quad t \in U \tag{3}$$

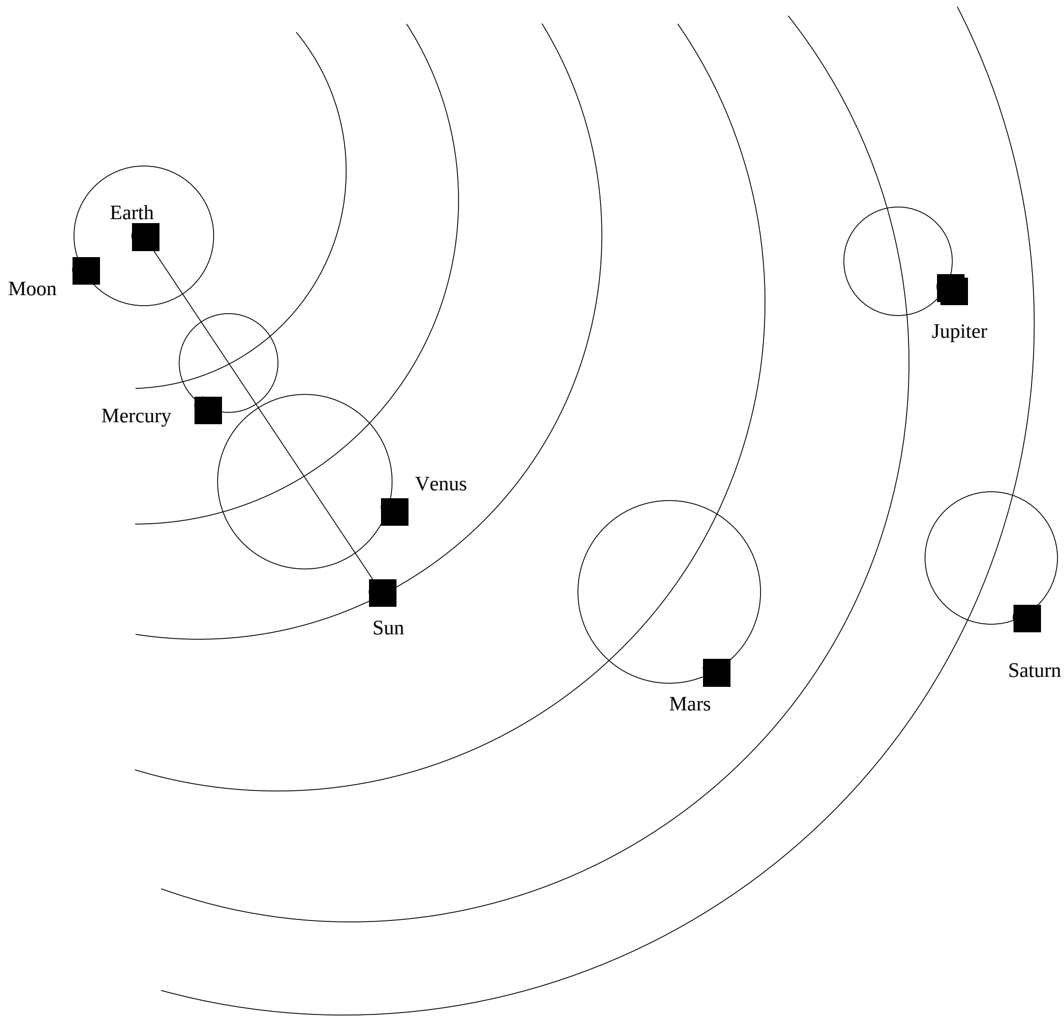
and the *space interval* is

$$\ell = (x\,, x)^{1/2} \qquad \text{for} \quad \text{every} \quad x \in V\,. \tag{4}$$

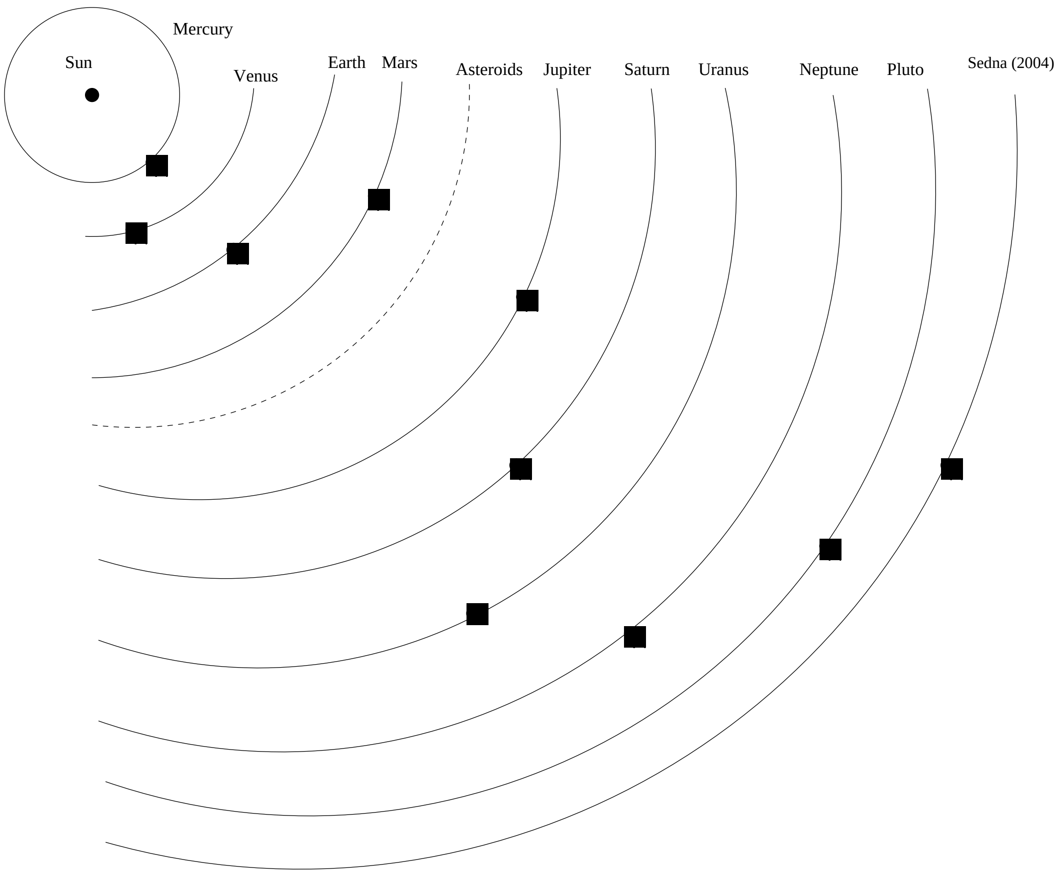
The structure (2) is in a correspondence with the phylosophical constructions of Aristotle so we refere to it as the *Aristotle spacetime.*

Circles are special curves in (2) since they invariant under rotations. This fact brings us to the Ptolemy and Copernicus planet system.

Ptolemy planet system Each planet moves uniformly in a circle, epicycle, while the center of this curve was in uniform circular motion around the Earth.



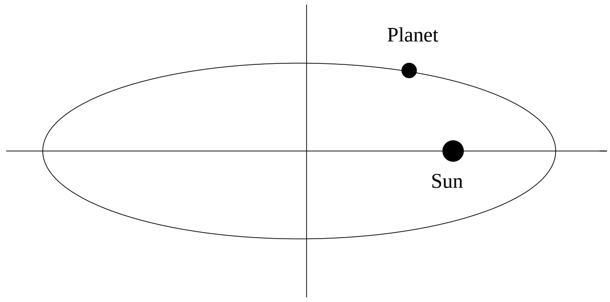
Copernicus planet system



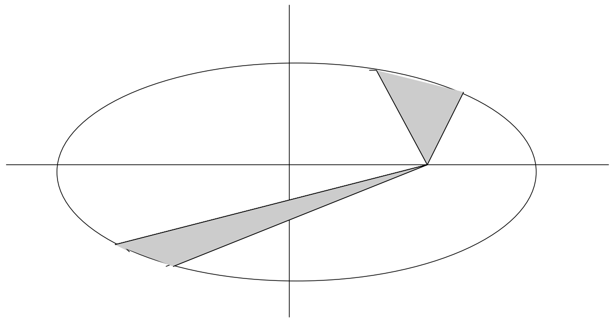
The Sun is stationary in the center. Around it the planets revolved uniformly as before. Epicycles where still required, but their number was smaller than before.

Kepler's Laws

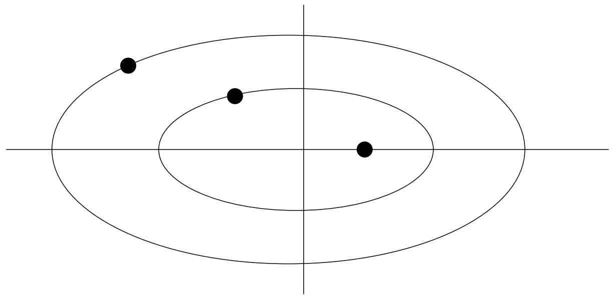
Law 1. The orbit of a planet about the Sun is an ellipse with the Sun's center of mass at one focus.



Law 2. A line joining a planet and the Sun swaps out equal areas in equal intervals of time.



Law 3. The squares of the periods of the planets are proportional to the cubes of their semimajor axes



$$\frac{T_a^2}{T_b^2} = \frac{R_a^3}{R_b^3}\,. \tag{5}$$

2 Galilei spacetime

Definition 2.1: Affine n dimensional space A.

Let A be a non-empty set and let V be an n dimensional vector space over a field $K(\mathbb{R})$. Define an addition operation $P + v \in A$ for $P \in A$ and $v \in V$ such that

1) $P + 0 = P$,

2) $(P + v) + u = P + (v + u)$,

3) For any $Q \in A$ there exists a unique $v \in V$ such that $Q = P + v$.

▮

Definition 2.2: Galilean structure.

1) The universe — an affine space $A = A^4$ with a 4 dimensional vector space V. Events — the points in A. The space of parallel displacements $V \simeq \mathbb{R}^4$.

2) Time — a linear map

$$t : V \to \mathbb{R}$$

3) The time interval between points $P, Q \in A$

$$\tau = |t(P - Q)|$$

P and Q simultaneous events iff $t(P - Q) = 0$.

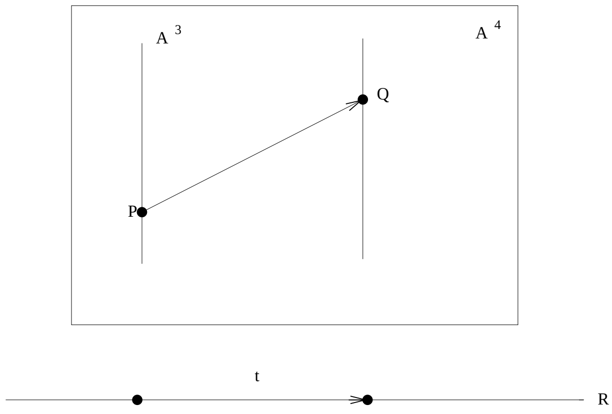
The set of events simultaneous with a given event forms a 3 dimensional subspace of A — space of simultaneous events.

4) The distance between simultaneous events

$$d(P, Q) = ||P - Q|| \, .$$

▮

Definition 2.3: The Galileian group - the group of all linear (affine) transformations of galileian spacetime which preserve the galileian structure, i.e., the map t, the time interval τ, and the distance d. ▮



Examples of galileian transformations.

1) T-transformation $g_1(t, x) = (-t, x)$

2) P-transformation $g_2(t, x) = (t, -x)$

3) Translations $g_3(t, x) = (t + s, x + a)$

4) Rotations $g_4(t, x) = (t, Gx)$, $G \in SO(3)$

5) Uniform motion $g_4(t, x) = (t, x + vt)$

3 Motion of a matter point

A physical body is approximated by a point in $\mathbb{R}^3$.
Motion of a point is a differentiable map

$$x : \mathbb{R} \to \mathbb{R}^3 \qquad x = x(t)$$

Velocity

$$\dot{x} = \frac{dx}{dt}$$

Acceleration

$$\ddot{x} = \frac{d^2x}{dt^2}$$

Motion of n points

$$^{(i)}x : \mathbb{R} \to \mathbb{R}^N \qquad ^{(i)}x = {}^{(i)}x(t) \qquad N = 3n$$

Newton's principle of determinicity

The initial state $\{^{(i)}x(0)\,,{}^{(i)}\dot{x}(0)\}$ of a mechanical system uniquely determines its motion.

Newton's equation for n points

$$^{(i)}\ddot{x} = {}^{(i)}F({}^{(j)}x, {}^{(j)}\dot{x}, t) \qquad i, j = 1, \cdots, n$$

Constraints imposed by the invariance transformations

- t-translation $t \to t + s$

$$^{(i)}\ddot{x} = {}^{(i)}F({}^{(j)}x, {}^{(j)}\dot{x})$$

- x-translation $x \to x + a$

$$^{(i)}\ddot{x} = {}^{(i)}F({}^{(j)}x - {}^{(k)}x, {}^{(j)}\dot{x})$$

- Uniform motion $x \to x + vt$

$$^{(i)}\ddot{x} = {}^{(i)}F({}^{(j)}x - {}^{(k)}x, {}^{(j)}\dot{x} - {}^{(k)}\dot{x})$$

- Rotations $x \to Gx\,, \qquad G \in SO(3)$

$$GF(x, \dot{x}) = F(Gx, G\dot{x})$$

Examples of Newton's equations

- A stone falling to the earth

$$m\ddot{z} = -mg\,, \qquad g = 9.8m/s^2$$

$$m\ddot{z} = -\frac{dU}{dz}\,, \qquad U = mgz\,.$$

- A stone falling to the earth from the great height

$$m\ddot{z} = -mg\frac{r_0^2}{(r_0 + z)^2}\,, \qquad r_0 --- \text{the radius of the earth.}$$

$$m\ddot{z} = -\frac{dU}{dz}\,, \qquad U = \frac{mgr_0^2}{r_0 + z}\,.$$

Aristotle on the Motion of Bodies

I call absolutely light that whose nature is to move always upward, and heavy whose nature is to move always downward, if there is no interference. … the natural motion of the earth as a whole, like that of its parts, is towards the center of the Universe: that is the reason it is now lying at the center … light bodies like fire, whose motion is contrary to that of the heavy, move to the extremity of the region which surrounds the centre.

Aristotle, *On the Heavens*, W.K.C. Guthrie, trans., Loeb Classical Library, Harvard University Press, Cambridge, 1939.

Aristotle on Falling Bodies

We see that bodies which have a greater impulse either of weight or of lightness, if they are alike in other respects, move faster over an equal space, and in the ratio which their magnitudes bear to each other. Therefore, they will also move through the void with this ratio of speed. But that is impossible; for why should one move faster? (In moving through *plena** it must be so; for the greater divides them faster by its force. For a moving thing cleaves the medium either by its shape, or by the impulse which the body that is carried along or is projected possesses.) Therefore all will possess equal velocity. But that is impossible.

Aristotle, *Physics, The Works of Aristotle Translated into English*, Vol 2, W.D. Ross, ed., Oxford University Press, New York 1930.

* The *plena* is the hypothetical medium that filled empty space according to the ancient Greeks who held that empty space as such could not exist. It is similar to the 19th century concept of the *aether*.

Galileo on Local Motion

THIRD DAY

Change of Position [De Motu Locali]

My purpose is to set forth a very new science dealing with a very anciant subject. There is, in nature, perhaps nothing older than motion, concerning which the books written by philosophers are neither few nor small; nevertheless I have discovered by experiment some properties of it which are worth knowing and which have not hitherto been either observed or demonstrated.

Some superficial observations have been made, as, for instance, that the free motion of a heavy falling body is continuously accelerated; but to just what extent this acceleration occurs has not yet been announced; for so far as I know, no one has yet pointed out that the distances traversed, during equal intervals of time, by a body falling from rest, stand to one another in the same ratio as the odd numbers beginning with unity.

It has been observed that missiles and projectiles describe a curved path of some sort; however no one has pointed out the fact that this path is a parabola. But this and other facts, not few in number or less worth knowing, I have succeeded in proving; and what I consider more important, there have been opened up to this vast and most excellent science, of which my work is merely the beginning, ways and means by which other minds more acute than mine will explore its remote corners.

This discussion is divided into three parts; the first part deals with motion which is steady or uniform; the second treats of motion as we find it accelerated in nature; the third deals with the so-called violent motions and with projectiles.

UNIFORM MOTION

In dealing with steady or uniform motion, we need a single definition which I give as follows:

DEFINITION

By steady or uniform motion, I mean one in which the distances traversed by the moving particle during any equal intervals of time, are themselves equal.

CAUTION

We must add to the old definition (which defined steady motion simply as one in which equal distances are traversed in equal times) the word “any”, meaning by this, all equal intervals of time; for it may happen that the moving body will traverse equal distances during some equal intervals of time and yet the distances traversed during some small portion of these time-intervals may not be equal, even though the time-intervals be equal.

Galileo Galilei, Dialogues Concerning Two New Sciences, Henry Crew and Alfonso de Salvio, trans., Macmillan, New York, 1914.