Homework #8

Exercise 1.
In a unholonomic basis e_a, the components of the Riemann tensor are defined as

$$\Big[\nabla_i, \nabla_j\Big] e_k - \nabla_{[e_i,e_j]} e_k = R^l{}_{kij} e_l$$

(a) Prove that in a coordinate (holonomic) basis

$$R^l{}_{kij} = \Gamma^l{}_{kj,i} - \Gamma^l{}_{ki,j} + \Gamma^m{}_{kj}\Gamma^l{}_{mi} - \Gamma^m{}_{ki}\Gamma^l{}_{mj}$$

(b) Prove that in a unholonomic basis with $[e_i, e_j] = C^m{}_{ij} e_m$

$$R^l{}_{kij} = \Gamma^l{}_{kj,i} - \Gamma^l{}_{ki,j} + \Gamma^m{}_{kj}\Gamma^l{}_{mi} - \Gamma^m{}_{ki}\Gamma^l{}_{mj} - C^m{}_{ij}\Gamma^l{}_{km}$$

(c) Prove that for a general connection

$$R^l{}_{k(ij)} = 0$$

(d) Prove that for a torsion-free connection

$$R^l{}_{[kij]} = 0$$

(e) Prove that the Riemann tensor of a torsion-free connection has $\frac{1}{3}n^2(n^2-1)$ independent components.

(e) Prove the Bianchi identity

$$R^l{}_{k[ij;m]} = 0$$

How much identities we have here?

Exercise 2.
Prove that for Riemann tensor of the Levi-Civita connection

$$R_{ijkl} = R_{klij}$$

Prove that the Riemann tensor has now $\frac{1}{12}n^2(n^2-1)$ independent components.

Exercise 3

Define the tensor Ricci as

$$R_{kl} = R^i{}_{kil}$$

and the curvature scalar as

$$R = g^{kl} R_{kl}$$

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(a) Prove that R_{kl} of the Levi-Civita connection is symmetric.
(b) Prove the Bianchi identity

$$\left(R^{ij} - \frac{1}{2} R g^{ij} \right)_{;j} = 0$$