Homework #7

Exercise 1.
Let V be a 3 dimensional space with Euclidean metric $g_{ab} = diag(+1, +1, +1)$. Consider two vectors u and v and their standard vector product $u \times v$. Let α, β be 1-forms corresponded to u, v.

Define a product, say $(\alpha \diamond \beta)$ (in exterior form notation), that corresponds to $u \times v$?

Exercise 2.
The following differential operators defined on three dimensional vectors.

$$1) \quad \text{div} \qquad 2) \quad \text{rot} \qquad 3) \quad \text{grad}$$

Define the corresponding exterior form derivative operators on 1-forms.

Exercise 3
Using the differential form representation, prove the identities

$$
\begin{aligned}
&1) \quad \text{div}(f\vec{A}) = \text{grad}f \cdot \vec{A} + f\text{div}\vec{A} \\
&2) \quad \text{rot}(f\vec{A}) = \text{grad}f \times \vec{A} + f\text{rot}\vec{A} \\
&3) \quad \text{div}(\vec{A} \times \vec{B}) = \vec{B} \cdot \text{rot}\vec{A} - \vec{A} \cdot \text{rot}\vec{B} \\
&4) \quad \text{rot}(\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\text{div}\vec{B}) - \vec{B}(\text{div}\vec{A}) \\
&5) \quad \text{grad}(\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} + (\vec{A} \cdot \nabla)\vec{B} + \vec{A} \times (\text{rot}\vec{B}) + \vec{B} \times (\text{rot}\vec{A}) \\
&6) \quad \text{rot}(\text{grad}f) = 0 \\
&7) \quad \text{div}(\text{rot}\vec{\text{B}}) = 0
\end{aligned}
$$

Exercise 4
Define the Hodge-de Rham Laplacian on differential forms as

$$\triangle = *d * d + d * d*$$

Check the identities

$$d\triangle = \triangle d$$

$$*\triangle = \triangle *$$

Check that, for a function f,

$$\triangle f = \mathrm{div(grad\,f) = f_{xx} + f_{yy} + f_{zz}}\,,$$

while, for a vector $\vec{A}$,

$$\triangle\vec{A} = (\mathrm{graddiv}\vec{A}) - \mathrm{rot}(\mathrm{rot}\vec{A})$$