Homework #6

Exercise 1.
Consider the antisymmetric tensor F_{ab} (Minkowski tensor of the electromagnetic field). The expression

$$T^a{}_b = F^{am}F_{bm} - \frac{1}{4}\delta^a_b F^{mn}F_{mn}$$

is called *the energy-momentum tensor of electromagnetic field.* Prove

$$1) \qquad T^a{}_a = 0 \qquad ---\text{traceless tensor}$$

$$2) \qquad T^a{}_b = \frac{1}{2}(F^{am}F_{bm} + \check{F}^{am}\check{F}_{bm}),$$

where $\check{F}_{ab} = \frac{1}{2}\varepsilon_{abmn}F^{mn}$. Prove that the tensor $T_{ab} = g_{am}T^m{}_b$ is symmetric.

Exercise 2.
Let two p-forms P, Q be given. Define a product

$$\{P, Q\} = \frac{1}{p!}g^{i_1j_1}\cdots g^{i_pj_p}P_{i_1\cdots i_p}Q_{i_1\cdots i_p}$$

Prove

$$P \wedge *Q = \{P, Q\}\,\text{vol}$$

Exercise 3.
Prove the identities of the Hodge map

$$*^2 = (-1)^{p(n-p)+i}$$

$$*w_1 \wedge w_2 = *w_2 \wedge w_1$$

$$*w \wedge \vartheta^a = -g^{ab} * (e_b \rfloor w)$$

Exercise 4.
Define tensor Riemann in the form

$$R_{abc}{}^d = 2\partial_{[a}\Gamma^d{}_{b]c} - 2\Gamma^m{}_{[a|c|}\Gamma^d{}_{b]m},$$

where $\Gamma^p{}_{mn} = \Gamma^p{}_{nm}$.

Prove the symmetries for $R_{abcd} = R_{abc}{}^m g_{md}$

$$R_{abcd} = R_{[ab]cd}$$

$$R_{[abc]d} = 0$$

$$R_{abcd} = R_{ab[cd]}$$

$$R_{abcd} = R_{cdab}$$

How much independent components does this tensor have?