Homework #4

Exercise 1.
Prove that from the following relations between the tensor components

$$\sum_a T^{aa}\,, \qquad \sum_a T_{aa}\,, \qquad \sum_a T_a{}^a$$

only the latter is invariant (does not change under transformations of the basis).
Exercise 2.
Let A_{mn} be the components of an antisymmetric tensor ($A_{mn} = -A_{nm}$), and S_{mn} be the components of a symmetric tensor ($S^{mn} = S^{nm}$). Prove

$$A_{mn}S^{mn} = 0$$

Let V^{mn} be the components of an arbitrary tensor. Prove

$$V^{mn}A_{mn} = \frac{1}{2}(V^{mn} - V^{nm})A_{mn}\,, \qquad V^{mn}S_{mn} = \frac{1}{2}(V^{mn} + V^{nm})S_{mn}$$

Exercise 3.
Prove that a completely antisymmetric tensor of the type $(n, 0)$ or $(0, n)$ on an n dimensional vector space is unique up to a scalar factor.
Exercise 4.
Prove that

$$A_{abc} = A_{(abc)} + A_{[abc]} + \frac{2}{3}(A_{[ab]c} + A_{[cb]a}) + \frac{2}{3}(A_{(ab)c} - A_{c(ab)})$$

Exercise 5.
Consider the tensor defined by the components

$$A_{abcd} = B_{ab}B_{cd} - B_{ac}B_{db}$$

Prove:

$$A_{dabc} = -A_{adbc} = -A_{dacb} = A_{bcda}$$
$$A_{dabc} + A_{dbca} + A_{dcab} = 0$$

Exercise 6.
Prove that if the equation

$$mT_{ab} + nT_{ba} = 0$$

in some basis, it valid in any basis. Prove that if $T_{ab} \neq 0$ than $m = \pm n$.
Exercise 7.
Prove that if the equation

$$mT_{abc} + nT_{bca} + kT_{cab} = 0$$

in some basis, it valid in any basis. Prove that if $T_{abc} \neq 0$ than $m+n+k=0$ or $m=n=k$.

Exercise 8.

If $\phi = S_{ab}u^a u^b$ is a scalar for any vector u so S is a tensor. If $\phi = 0$, so $S_{(ab)} = 0$.

Exercise 9.

Let the components of an antisymmetric tensor P^{ab} satisfy the relation

$$P^{ab} = \frac{1}{2}(A^a B^b - A^b B^a)\,,$$

where A^a and B^a the components of some vectors A, B. Such tensors called bivectors. Prove Plucker's relation for bivectors

$$P^{ab}P^{cd} + P^{ac}P^{db} + P^{ad}P^{bc} = 0.$$