Walras-Bowley Lecture 2003 (extended)

Sergiu Hart

This version: June 2005
Most of this talk is based on joint work with

**Andreu Mas-Colell**

(Universitat Pompeu Fabra, Barcelona)
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All papers – and this presentation – are available on my home page

http://www.ma.huji.ac.il/hart
http://www.ma.huji.ac.il/hart
Papers

http://www.ma.huji.ac.il/hart

- *Economic Essays* (2001)
- *Econometrica* (forthcoming)
PART I

Introduction: Dynamics
Dynamics
Dynamics

“Learning”
“Learning”

- **START**: prior beliefs
- **STEP**:  
  - observe  
  - update (Bayes)  
  - optimize (best-reply)
- **REPEAT**
Dynamics

“Evolution”
Dynamics

“Evolution”

- populations
- each individual $\leftrightarrow$ fixed action (“gene”)
- frequencies of each action in the population $\leftrightarrow$ mixed strategy
Dynamics

“Evolution”
- populations
- each individual ↔ fixed action (“gene”)
- frequencies of each action in the population ↔ mixed strategy

Change:
- **Selection**
  - higher payoff ⇒ higher frequency
- **Mutation**
  - random and relatively rare
“Adaptive Heuristics”
“Adaptive Heuristics”

- “rules of thumb”
- myopic
- simple
- stimulus response, reinforcement
- behavioral, experiments
- non-Bayesian
Dynamics

“Adaptive Heuristics”

- “rules of thumb”
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Example: Fictitious Play
Dynamics

“Adaptive Heuristics”

- “rules of thumb”
- myopic
- simple
- stimulus response, reinforcement
- behavioral, experiments
- non-Bayesian

Example: **Fictitious Play**

(Play optimally against the empirical distribution of past play of the other player)
most of this talk
Can simple adaptive heuristics lead to sophisticated rational behavior?
$N$-person game in strategic (normal) form

Players

$i = 1, 2, ..., N$
$N$-person game in strategic (normal) form

- Players
  
  $i = 1, 2, ..., N$

- For each player $i$: Actions
  
  $s^i$ in $S^i$
\(N\)-person game in strategic (normal) form

- **Players**
  \[ i = 1, 2, \ldots, N \]

- For each player \(i\): **Actions**
  \[ s^i \text{ in } S^i \]

- For each player \(i\): **Payoffs (utilities)**
  \[ u^i(s) \equiv u^i(s^1, s^2, \ldots, s^N) \]
Dynamics

Time

\[ t = 1, 2, \ldots \]
Dynamics

- Time

\[ t = 1, 2, \ldots \]

- At time \( t \) each player \( i \) chooses an action \( s_t^i \) in \( S^i \)
Regret Matching
DON’T YOU FEEL A PANG OF REGRET?

47.15% YIELD

January–May 2003

XYZ  17.92%  4.43%

Nasdaq  Dow Jones

Don’t wait! Ask your broker today

Haaretz – June 3, 2003
**REGRET MATCHING** =

Switch next period to a different action with a probability that is proportional to the regret for that action.
Regret Matching

\[ \text{REGRET MATCHING} = \]
Switch next period to a different action with a probability that is proportional to the \textit{regret} for that action

\[ \text{REGRET} = \text{increase in payoff had such a change always been made in the past} \]
Regret

$U = \text{average payoff up to now}$
Regret

- $U = \text{average payoff up to now}$
- $V(k) = \text{average payoff if action } k \text{ had been played instead of the current action } j$ every time in the past that $j$ was played
\[ U = \text{average payoff up to now} \]

\[ V(k) = \text{average payoff if action } k \text{ had been played instead of the current action } j \text{ every time in the past that } j \text{ was played} \]

\[ R(k) = [V(k) - U]_+ = \text{regret for action } k \]
Regret

- \( U \) = average payoff up to now
- \( V(k) \) = average payoff if action \( k \) had been played instead of the current action \( j \) every time in the past that \( j \) was played
- \( R(k) \equiv R^i_T(j \to k) = \left[ \frac{1}{T} \sum_{t \leq T} s_t^i = j \left( u^i(k, s_t^{-i}) - u^i(s_t) \right) \right]_+ \)

\( R(k) \equiv R^i_T(j \to k) = \left[ \frac{1}{T} \sum_{t \leq T} s_t^i = j \left( u^i(k, s_t^{-i}) - u^i(s_t) \right) \right]_+ \)
Regret

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$R(k) \equiv R^i_T(j \rightarrow k) = \left[ \frac{1}{T} \sum_{t \leq T: s_t^i = j} \left( u^i(k, s_t^{−i}) - u^i(s_t) \right) \right]_+$
Regret

- $U = \text{average payoff up to now}$
- $V(k) = \text{average payoff if action } k \text{ had been played instead of the current action } j \text{ every time in the past that } j \text{ was played}$
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$R(k) \equiv R^i_T(j \rightarrow k) =
\left[ \frac{1}{T} \sum_{t \leq T} s_t^i = j \left( u^i(k, s_t^{\neg i}) - u^i(s_t) \right) \right]^+$
Regret

- $U = \text{average payoff up to now}$
- $V(k) = \text{average payoff if action } k \text{ had been played instead of the current action } j$ every time in the past that $j$ was played
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\left[ \frac{1}{T} \sum_{t \leq T: s_t^i = j} (u^i(k, s_t^{-i}) - u^i(s_t)) \right]_+$$
Regret

- \( U = \) average payoff up to now
- \( V(k) = \) average payoff if action \( k \) had been played instead of the current action \( j \) every time in the past that \( j \) was played
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\[
R(k) \equiv R^i_T(j \rightarrow k) = \\
\left[ \frac{1}{T} \sum_{t \leq T: s_t^i = j} \left( u^i(k, s_t^{-i}) - u^i(s_t) \right) \right]_+
\]
Regret Matching

Next period play:

Switch to action $k$ with a probability that is proportional to the regret $R(k)$ (for $k \neq j$)
Next period play:

- **Switch** to action $k$ with a probability that is proportional to the regret $R(k)$ (for $k \neq j$)
- Play **the same** action $j$ of last period with the remaining probability
Regret Matching

Next period play:

- **Switch** to action $k$ with a probability that is proportional to the regret $R(k)$ (for $k \neq j$)
- Play **the same** action $j$ of last period with the remaining probability

\[
\begin{align*}
\sigma(k) &\equiv \sigma^i_{T+1}(k) = cR(k), \quad \text{for } k \neq j \\
\sigma(j) &\equiv \sigma^i_{T+1}(j) = 1 - \sum_{k \neq j} cR(k)
\end{align*}
\]
Regret Matching

Next period play:

- **Switch** to action $k$ with a probability that is proportional to the regret $R(k)$ (for $k \neq j$)
- Play the same action $j$ of last period with the remaining probability

\[
\sigma(k) \equiv \sigma^i_{T+1}(k) = cR(k), \quad \text{for } k \neq j
\]

\[
\sigma(j) \equiv \sigma^i_{T+1}(j) = 1 - \sum_{k \neq j} cR(k)
\]

- $c$ = a fixed positive constant (so that the probability of not switching is $> 0$)
Regret Matching Theorem

Theorem

If all players play Regret Matching then the joint distribution of play converges to the set of CORRELATED EQUILIBRIA of the game.
Joint distribution of play $z_T = $ 
The relative frequencies that the $\mathcal{N}$-tuples of actions have been played up to time $T$
Joint distribution of play $z_T$

The relative frequencies that the $N$-tuples of actions have been played up to time $T$
Joint distribution of play $z_T = T = 1$
Joint distribution of play $z_T$ =

The relative frequencies that the $N$-tuples of actions have been played up to time $T$

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$T = 1$

$z_1$
Joint distribution of play $z_T = \cdot \cdot \cdot$

The relative frequencies that the $N$-tuples of actions have been played up to time $T$

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<th>0</th>
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<th>1/2</th>
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<tr>
<td>1/2</td>
<td>0</td>
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$T = 2$

$z_2$
Joint distribution of play \( z_T \)

The relative frequencies that the \( N \)-tuples of actions have been played up to time \( T \)

\[
\begin{array}{c|c|c}
  & * & * \\
* & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
  & 0 & 2/3 \\
1/3 & 0 & 0 \\
\end{array}
\]

\( T = 3 \)

\( z_3 \)
Joint distribution of play $z_T = T = 10$

The relative frequencies that the $N$-tuples of actions have been played up to time $T$

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<th>3/10</th>
<th>0</th>
<th>2/10</th>
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<tbody>
<tr>
<td>1/10</td>
<td>3/10</td>
<td>1/10</td>
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Joint Distribution of Play

**Note 1:** The fact that the players randomize independently at each period does not imply that the joint distribution is independent!

\[ T = 10 \]

\[
\begin{array}{c|c|c}
\ast & \ast & \ast \\
\hline
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\end{array}
\]

\[
\begin{array}{c|c|c}
3/10 & 0 & 2/10 \\
1/10 & 3/10 & 1/10 \\
\end{array}
\]
Note 1: The fact that the players randomize independently at each period does not imply that the joint distribution is independent!

Note 2: Players observe the joint distribution (the history of play)
Joint Distribution of Play

**Note 1:** The fact that the players randomize *independently at each period* does not imply that the *joint distribution is independent*!

**Note 2:** Players *observe* the *joint distribution* (the history of play)

**Note 3:** Players *react to* the *joint distribution* (patterns, “coincidences”, communication, signals, ...)
A **Correlated Equilibrium** is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974)
A **Correlated Equilibrium** is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974)

- **Examples:**
A **Correlated Equilibrium** is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974)

- **Examples:**
  - Independent signals
A Correlated Equilibrium is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974)

- **Examples:**
  - Independent signals $\iff$ Nash equilibrium
A **Correlated Equilibrium** is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974)

- **Examples:**
  - Independent signals $\iff$ Nash equilibrium
  - Public signals ("sunspots")
A Correlated Equilibrium is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974)

- **Examples:**
  - Independent signals $\iff$ Nash equilibrium
  - Public signals (“sunspots”) $\iff$ convex combinations of Nash equilibria
A **Correlated Equilibrium** is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974)

- **Examples:**
  - Independent signals $\iff$ Nash equilibrium
  - Public signals ("sunspots") $\iff$ convex combinations of Nash equilibria
  - Butterflies play the Chicken Game ("Speckled Wood" *Pararge aegeria*)
Correlated Equilibria

"Chicken" game

<table>
<thead>
<tr>
<th></th>
<th>LEAVE</th>
<th>STAY</th>
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<tbody>
<tr>
<td>LEAVE</td>
<td>5,5</td>
<td>3,6</td>
</tr>
<tr>
<td>STAY</td>
<td>6,3</td>
<td>0,0</td>
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Correlated Equilibria

"Chicken" game

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<td>3, 6</td>
</tr>
<tr>
<td>STAY</td>
<td>6, 3</td>
<td>0, 0</td>
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A Nash equilibrium
Correlated Equilibria

**"Chicken" game**

<table>
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<tr>
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<tbody>
<tr>
<td>LEAVE</td>
<td>5, 5</td>
<td>3, 6</td>
</tr>
<tr>
<td>STAY</td>
<td>6, 3</td>
<td>0, 0</td>
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another **Nash equilibrium**
**Correlated Equilibria**

"Chicken" game

<table>
<thead>
<tr>
<th></th>
<th>LEAVE</th>
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<tbody>
<tr>
<td>LEAVE</td>
<td>5, 5</td>
<td>3, 6</td>
</tr>
<tr>
<td>STAY</td>
<td>6, 3</td>
<td>0, 0</td>
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</table>

A (publicly) correlated equilibrium
Correlated Equilibria

"Chicken" game

<table>
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<tr>
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<td>0,0</td>
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<td>S</td>
<td>1/3</td>
<td>0</td>
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</tbody>
</table>

another correlated equilibrium

- after signal L play LEAVE
- after signal S play STAY
A **Correlated Equilibrium** is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974).

**Examples:**
- Independent signals $\iff$ Nash equilibrium
- Public signals (“sunspots”) $\iff$ convex combinations of Nash equilibria
- Butterflies play the Chicken Game (“Speckled Wood” *Pararge aegeria*)
A Correlated Equilibrium is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974)

**Examples:**
- Independent signals $\iff$ Nash equilibrium
- Public signals (“sunspots”) $\iff$ convex combinations of Nash equilibria
- Butterflies play the Chicken Game (“Speckled Wood” *Pararge aegeria*)
- Boston Celtics’ front line
Correlated Equilibrium

Signals (public, correlated) are unavoidable
Correlated Equilibrium

- Signals (public, correlated) are **unavoidable**
- **Bayesian Rationality** ⇔ **Correlated Equilibrium** (Aumann 1987)
Correlated Equilibrium

- Signals (public, correlated) are unavoidable
- **Bayesian Rationality** ⇔ **Correlated Equilibrium** (Aumann 1987)

A joint distribution \( z \) is a correlated equilibrium if

\[
\sum_{s^{-i}} u(j, s^{-i}) z(j, s^{-i}) \geq \sum_{s^{-i}} u(k, s^{-i}) z(j, s^{-i})
\]

for all \( i \in N \) and all \( j, k \in S^i \)
Theorem

If all players play Regret Matching then the joint distribution of play converges to the set of CORRELATED EQUILIBRIA of the game.
Regret Matching Theorem

- $\text{CE} = \text{set of correlated equilibria}$
- $z_T = \text{joint distribution of play up to time } T$

$$\text{distance}(z_T, \text{CE}) \to 0 \quad \text{as } T \to \infty \quad (\text{a.s.})$$
Regret Matching Theorem

- $\text{CE} =$ set of correlated equilibria
- $z_T =$ joint distribution of play up to time $T$

$$\text{distance}(z_T, \text{CE}) \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty \quad (\text{a.s.})$$

$z_T$ is approximately a correlated equilibrium (or $z_T$ is a correlated approximate equilibrium) from some time on (for all large enough $T$)
Regret Matching Theorem

Proof

\( z_T \) is a correlated equilibrium \iff there is no regret:

\[ R^i_T(j \rightarrow k) = 0 \] for all players and all actions
Regret Matching Theorem

Proof

- $z_T$ is a correlated equilibrium ⇔ there is no regret:
  $R^i_T(j \rightarrow k) = 0$ for all players and all actions

- Regret Matching
  ⇒ all regrets converge to 0
  (Proof: Blackwell Approachability for the vector of regrets + approximate eigenvector probabilities by transition probabilities)
Regret Matching Theorem

Proof

- \( z_T \) is a correlated equilibrium \iff there is no regret:
  \[ R^i_T(j \rightarrow k) = 0 \]
  for all players and all actions

- Regret Matching
  \[ \Rightarrow \text{all regrets converge to 0} \]
  (Proof: Blackwell Approachability for the vector of regrets + approximate eigenvector probabilities by transition probabilities)

Note: \( z_T \) converges to the set CE, not to a point
Approachability Setup:

- Payoffs are $m$-dimensional vectors (in $\mathbb{R}^m$)
- Repeated game
### Approachability

#### Approachability Setup:
- Payoffs are $m$-dimensional vectors (in $\mathbb{R}^m$)
- Repeated game

<table>
<thead>
<tr>
<th>Other players ($-i$)</th>
<th>Player $i$</th>
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($m = 2$)
Let $H \subset \mathbb{R}^m$ be a half-space. Then:
Approachability Theorem 1

Let $H \subset \mathbb{R}^m$ be a **half-space**. Then:

- Player $i$ can guarantee that the long-run average payoff converges to $H$
Approachability Theorem 1

Let $H \subset \mathbb{R}^m$ be a half-space. Then:

- Player $i$ can guarantee that the long-run average payoff converges to $H$

$\iff$

- Player $i$ can guarantee that the one-shot payoff lies in $H$
Approachability Theorem 1

Let $H \subset \mathbb{R}^m$ be a *half-space*. Then:

- Player $i$ can guarantee that the *long-run average payoff* converges to $H$

$H$ is “APPROACHABLE”

$\iff$

- Player $i$ can guarantee that the *one-shot payoff* lies in $H$
Approachability Theorem 1

Let $H \subset \mathbb{R}^m$ be a half-space. Then:

- Player $i$ can guarantee that the long-run average payoff converges to $H$ 
  $H$ is “APPROACHABLE”

- Player $i$ can guarantee that the one-shot payoff lies in $H$ 
  $H$ is “ENFORCEABLE”
Approachability Theorem 1

Let $H \subset \mathbb{R}^m$ be a half-space. Then:

- Player $i$ can guarantee that the long-run average payoff converges to $H$

  $H$ is “APPROACHABLE”

- Player $i$ can guarantee that the one-shot payoff lies in $H$

  $H$ is “ENFORCEABLE”

Proof: Real payoffs ($m = 1$): Minimax Theorem + Law of Large Numbers
Approachability Theorem 2

Let $C \subset \mathbb{R}^m$ be a convex set. Then:
Approachability Theorem 2

Let $C \subset \mathbb{R}^m$ be a convex set. Then:

Player $i$ can guarantee that the long-run average payoff converges to $C$. 
Approachability Theorem 2

Let $C \subset \mathbb{R}^m$ be a convex set. Then:

- Player $i$ can guarantee that the long-run average payoff converges to $C$

$C$ is “APPROACHABLE”
Let $C \subset \mathbb{R}^m$ be a convex set. Then:

- Player $i$ can guarantee that the long-run average payoff converges to $C$

$C$ is “APPROACHABLE”

- Every half-space $H$ that contains $C$ is approachable
Let $C \subset \mathbb{R}^m$ be a \textbf{convex} set. Then:

- Player $i$ can guarantee that the long-run average payoff converges to $C$

$C$ is “\textbf{APPROACHABLE}”

- Every half-space $H$ that contains $C$ is approachable

- Every half-space $H$ that contains $C$ is enforceable
Approachability

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Approachability
Note: The intersection of approachable sets need not be an approachable set.
Note: The intersection of approachable sets need not be an approachable set.

\[
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0
\end{pmatrix}
\begin{pmatrix}
0 \\
1
\end{pmatrix}
\]
Note: The intersection of approachable sets need **not** be an approachable set.

But: If **all** half-spaces containing $C$ are approachable, then their intersection ($= C$) is approachable.
Correlating device: the *history* of play
Correlating device: the **history** of play

Other procedures leading to correlated equilibria:

- Foster–Vohra 1997
  *Calibrated Learning*: best-reply to calibrated forecasts

- Fudenberg–Levine 1999
  *Conditional Smooth Fictitious Play Eigenvector strategy*
Remarks

- Correlating device: the **history** of play
- Other procedures leading to correlated equilibria:
  - Foster–Vohra 1997
    *Calibrated Learning*: best-reply to calibrated forecasts
  - Fudenberg–Levine 1999
    *Conditional Smooth Fictitious Play*
    *Eigenvector strategy*

**Not heuristics!**
Behavioral aspects of Regret Matching:
Behavioral aspects of Regret Matching:
- Commonly used rules of behavior
Behavioral Aspects

Behavioral aspects of Regret Matching:

- Commonly used rules of behavior
- Never change a winning team
Behavioral aspects of **Regret Matching**:

- **Commonly used** rules of behavior
  - Never change a winning team
  - The higher would have been the payoff from another action – the higher the tendency to switch to it
Behavioral aspects of **Regret Matching**:

- **Commonly used** rules of behavior
  - Never change a winning team
  - The higher would have been the payoff from another action – the higher the tendency to switch to it
  - Small probability of switching (the “status quo bias”)

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Behavioral Aspects

Behavioral aspects of Regret Matching:

- **Commonly used** rules of behavior
  - Never change a winning team
  - The higher would have been the payoff from another action – the higher the tendency to switch to it
  - Small probability of switching (the “status quo bias”)

- **Stimulus-response, reinforcement**
Behavioral aspects of Regret Matching:

- **Commonly used** rules of behavior
  - Never change a winning team
  - The higher would have been the payoff from another action – the higher the tendency to switch to it
  - Small probability of switching (the “status quo bias”)

- **Stimulus-response, reinforcement**

- **No beliefs** (defined directly on actions)
  - **No best-reply** (better-reply ?)
Behavioral Aspects

Similar to models of learning, experimental and behavioral economics:
Behavioral Aspects

Similar to models of learning, experimental and behavioral economics:

- Bush–Mosteller 1955
- ...

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Behavioral Aspects

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N. Camille et al.,
“The Involvement of the Orbitofrontal Cortex in the Experience of Regret”
Science May 2004 (304: 1167–1170)
PART III

Generalized Regret Matching
How special is Regret Matching?
Questions

- How special is Regret Matching?
- Why does conditional smooth fictitious play work?
Questions

- How special is Regret Matching?
- Why does conditional smooth fictitious play work?
- Any connections?
Generalized Regret Matching

Regret Matching = Switching probabilities are proportional to the regrets: \( \sigma(k) = cR(k) \)
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Generalized Regret Matching = Switching probabilities are a function of the regrets:

$$\sigma(k) = f(R(k))$$
Generalized Regret Matching

Regret Matching = Switching probabilities are proportional to the regrets: \( \sigma(k) = cR(k) \)

Generalized Regret Matching = Switching probabilities are a function of the regrets:

\[ \sigma(k) = f(R(k)) \]

- \( f \) is a sign-preserving function:
  \[ f(0) = 0, \text{ and } x > 0 \Rightarrow f(x) > 0 \]
- \( f \) is a Lipschitz continuous function

(in fact, much more general: \( f_{k,j} \), potential)
Generalized Regret Matching

**Theorem**

If all players play Generalized Regret Matching then the joint distribution of play converges to the set of correlated equilibria of the game.
Theorem
If all players play Generalized Regret Matching then the joint distribution of play converges to the set of correlated equilibria of the game.

Special Cases

Play probabilities proportional to the $m$-th power of the regrets

\[ f(x) = cx^m, \text{ for } m \geq 1 \]
Special Cases

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$f(x) = cx^m$, for $m \geq 1$

- $m = 1$: Regret Matching
Special Cases

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- $m = 1$: Regret Matching
- $m = \infty$: Positive probability only to actions with maximal regret
Special Cases

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Conditional Fictitious Play
Special Cases

Play probabilities proportional to the $m$-th power of the regrets

$$f(x) = cx^m, \text{ for } m \geq 1$$

- $m = 1$: Regret Matching
- $m = \infty$: Positive probability only to actions with maximal regret $\iff$ Conditional Fictitious Play
- But: Not continuous
Special Cases

Play probabilities proportional to the \( m \)-th power of the regrets
\( (f(x) = cx^m, \text{ for } m \geq 1) \)

- \( m = 1 \): Regret Matching
- \( m = \infty \): Positive probability only to actions with maximal regret ⇔ Conditional Fictitious Play
- But: Not continuous
- Therefore: Smooth Conditional Fictitious Play
Unknown Game
The case of the "Unknown game":

- The player knows only
  - Its own set of actions
  - Its own past actions and payoffs
Unknown Game

The case of the “Unknown game”:

- The player knows only
  - Its own set of actions
  - Its own past actions and payoffs

- The player does not know the game
  (other players, actions, payoff functions, history of other players’ actions and payoffs)
Proxy Regret

- **Unknown game** $\Rightarrow$ **Unknown regret**
  (The player does not know what the payoff would have been if he had played a different action $k$)
Proxy Regret

- **Unknown game** $\Rightarrow$ **Unknown regret**
  (The player does not know what the payoff would have been if he had played a different action $k$)

- “**Proxy Regret**” for $k$: Use the payoffs received when $k$ has been actually played in the past
Proxy Regret

- **Unknown game** $\Rightarrow$ **Unknown regret**
  (The player does not know what the payoff would have been if he had played a different action $k$)

- **“Proxy Regret”** for $k$: Use the payoffs received when $k$ has been actually played in the past

**Theorem**. If all players play strategies based on **proxy regret**, then the joint distribution of play converges to the set of correlated equilibria of the game
Nash Equilibrium

Question:
Adaptive heuristics $\rightarrow$ Nash equilibria?
Nash Equilibrium

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In **SPECIAL** classes of games: **YES**
Nash Equilibrium

Question:
Adaptive heuristics $\rightarrow$ Nash equilibria?

In **SPECIAL** classes of games: YES
Fictitious play, Regret-based, ...

- Two-person zero-sum games
- Two-person potential games
- Supermodular games
- ...
Nash Equilibrium

Question:
Adaptive heuristics → Nash equilibria?

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- In **GENERAL** games: **NO**
Nash Equilibrium

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Adaptive heuristics $\rightarrow$ Nash equilibria?

• In **SPECIAL** classes of games: YES
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  - Two-person zero-sum games
  - Two-person potential games
  - Supermodular games
  - ...

• In **GENERAL** games: NO

  WHY?
General dynamic for 2-person games:

\[
\dot{x}(t) = F ( x(t) , y(t) ; u^1 , u^2 ) \\
\dot{y}(t) = G ( x(t) , y(t) ; u^1 , u^2 )
\]

\[
x(t) \in \Delta(S^1), \quad y(t) \in \Delta(S^2)
\]
General dynamic for 2-person games:

\[ \dot{x}(t) = F \left( x(t), y(t) ; u^1, u^2 \right) \]
\[ \dot{y}(t) = G \left( x(t), y(t) ; u^1, u^2 \right) \]

**Uncoupled** dynamic:

\[ \dot{x}(t) = F \left( x(t), y(t) ; u^1 \right) \]
\[ \dot{y}(t) = G \left( x(t), y(t) ; u^2 \right) \]

\[ x(t) \in \Delta(S^1), \quad y(t) \in \Delta(S^2) \]
Uncoupled Dynamics

- "Adaptive" ("rational") dynamics
  (best-reply, better-reply, payoff-improving, monotonic, fictitious play, regret-based, replicator dynamics, ...)

- are uncoupled
Uncoupled Dynamics

- “Adaptive” (“rational”) dynamics
  (best-reply, better-reply, payoff-improving, monotonic, fictitious play, regret-based, replicator dynamics, ...)

- are uncoupled

- Uncoupledness is a natural informational condition
Nash-Convergent Dynamics

- Consider a **family of games**, each having a **unique Nash equilibrium**
  (no “coordination problems”)

Nash-Convergent Dynamics

Consider a family of games, each having a unique Nash equilibrium (no “coordination problems”)

A dynamic is Nash-convergent if it always converges to the unique Nash equilibrium

Regularity conditions: The unique Nash equilibrium is a stable rest-point of the dynamic
There exist no uncoupled dynamics which guarantee Nash convergence
There exist no **uncoupled** dynamics which guarantee **Nash convergence**

There are simple families of games whose **unique Nash equilibrium**

is **unstable**

for every **uncoupled** dynamic
“Adaptive” ("rational") dynamics
(best-reply, better-reply, payoff-improving, monotonic, fictitious play, regret-based, replicator dynamics, ...)
are uncoupled
“Adaptive” ("rational") dynamics
(best-reply, better-reply, payoff-improving, monotonic, fictitious play, regret-based, replicator dynamics, ...)

are uncoupled

⇒ cannot always converge to Nash equilibria
Nash vs Correlated

Correlated equilibria $\leftrightarrow$ Uncoupled dynamics
Nash vs Correlated

Correlated equilibria ↔ Uncoupled dynamics

Nash equilibria ↔ Coupled dynamics
Correlated equilibria $\leftrightarrow$ Uncoupled dynamics

Nash equilibria $\leftrightarrow$ Coupled dynamics

“Law of Conservation of Coordination”

There must be coordination either in the equilibrium concept or in the dynamic
Where Do We Go From Here?
Where Do We Go From Here?

- Dynamics and equilibria
  - Which equilibria?
  - Which dynamics?
Where Do We Go From Here?

- Dynamics and equilibria
  - Which equilibria?
  - Which dynamics?
- Correlated equilibria: theory and practice
  - Coordination
  - Communication
  - Bounded complexity
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- Experiments, empirics ↔ Theory
Where Do We Go From Here?

- Dynamics and equilibria
  - Which equilibria?
  - Which dynamics?
- Correlated equilibria: theory and practice
  - Coordination
  - Communication
  - Bounded complexity
- Experiments, empirics ↔ Theory
- **Joint distribution** of play
  (instead of just the *marginals*)
There is a simple adaptive heuristic always leading to correlated equilibria
There is a simple adaptive heuristic always leading to correlated equilibria (Regret Matching)
There is a simple adaptive heuristic always leading to correlated equilibria
(Regret Matching)

There are many adaptive heuristics always leading to correlated equilibria
Summary

- There is a simple adaptive heuristic always leading to correlated equilibria (Regret Matching)

- There are many adaptive heuristics always leading to correlated equilibria (Generalized Regret Matching)
There is a simple adaptive heuristic always leading to correlated equilibria (Regret Matching)

There are many adaptive heuristics always leading to correlated equilibria (Generalized Regret Matching)

There can be no adaptive heuristics always leading to Nash equilibria
There is a simple adaptive heuristic always leading to correlated equilibria (Regret Matching)

There are many adaptive heuristics always leading to correlated equilibria (Generalized Regret Matching)

There can be no adaptive heuristics always leading to Nash equilibria (Uncoupledness)
Summary

ADAPTIVE HEURISTICS
Summary

ADAPTIVE HEURISTICS

BEHAVIORAL
Summary

Adaptive Heuristics

Correlated Equilibria

Behavioral
Summary

Adaptive Heuristics

Behavioral

Correlated Equilibria

Regret Matching
Summary

Generalized Regret Matching
Summary

BEHAVIORAL

ADAPTIVE HEURISTICS

CORRELATED EQUILIBRIA

Uncoupledness
Can simple adaptive heuristics lead to sophisticated rational behavior?
Can simple adaptive heuristics lead to sophisticated rational behavior?

YES!
Can simple adaptive heuristics lead to sophisticated rational behavior?

YES!

in time ...
Summary – Macro

Behavioral

Adaptive Heuristics

Rational
ADAPTIVE HEURISTICS
A Little Rationality Goes a Long Way
ADAPTIVE HEURISTICS
(A Little Rationality Goes a Long Way)
Rationality Takes Time
ADAPTIVE HEURISTICS
(A Little Rationality Goes a Long Way)
Rationality Takes Time

Regret ? ...
ADAPTIVE HEURISTICS
(A Little Rationality Goes a Long Way)
Rationality Takes Time

Regret? ... No Regret!