Surely You’re Using The Sure-Thing Principle!

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Surely You’re Using
The Sure-Thing Principle!

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*Conditioning and the Sure-Thing Principle*

Center for Rationality DP-393, June 2005
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*Agreeing on Decisions*
2005
Papers

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The first paper and this presentation are available on my home page

http://www.ma.huji.ac.il/hart
Conditioning and the Sure-Thing Principle

Robert J. Aumann
Sergiu Hart
Motty Perry
Savage’s Sure-Thing Principle

Next election, 2 candidates:
DEMOCRAT, REPUBLICAN
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Assume:
- If the DEMOCRAT were to lose (DL),
  I would BUY the property
Savage’s Sure-Thing Principle

Next election, 2 candidates: DEMOCRAT, REPUBLICAN

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- If the REPUBLICAN were to lose (RL), I would BUY the property
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Either DL or RL must hold
Savage’s Sure-Thing Principle

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**Conclusion:** I should BUY the property
The Sure-Thing Principle

Next election, 3 candidates:
DEMOCRAT, REPUBLICAN, INDEPENDENT
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Example:
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- I should **buy** the property if and only if the chance that the **INDEPENDENT** wins is > 50%
The Sure-Thing Principle?

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- I should **buy** the property if and only if the chance that the INDEPENDENT wins is \( > 50\% \)
- \( \text{Prob (DEMOCRAT wins)} = 30\% \)
- \( \text{Prob (REPUBLICAN wins)} = 30\% \)
- \( \text{Prob (INDEPENDENT wins)} = 40\% \)
The Sure-Thing Principle?

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- \( \text{Prob (REPUBLICAN wins)} = 30\% \)
- \( \text{Prob (INDEPENDENT wins)} = 40\% \)

**Conclusion:** I should **NOT buy** the property
(40% < 50%)
The Sure-Thing Puzzle

\[ \text{Prob (INDEPENDENT wins | DEMOCRAT loses)} = \frac{40\%}{30\% + 40\%} = \frac{4}{7} > 50\% \]

so I would buy the property if I knew DL
The Sure-Thing Puzzle

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so I would buy the property if I knew RL
The Sure-Thing Puzzle

Prob (INDEPENDENT wins | DEMOCRAT loses) = \frac{40\%}{30\% + 40\%} = \frac{4}{7} > 50\%
so I would buy the property if I knew DL

Prob (INDEPENDENT wins | REPUBLICN loses) = \frac{40\%}{30\% + 40\%} = \frac{4}{7} > 50\%
so I would buy the property if I knew RL

Either DL or RL holds, so ??
The Sure-Thing Puzzle

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  so I would buy the property if I knew RL

- Either DL or RL holds, so ? ?

When there are 3 candidates

DL and RL are NOT disjoint events
The Sure-Thing Puzzle

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  so I would **buy** the property if I knew RL

Either DL or RL holds, so ? ?

When there are 3 candidates

**DL and RL are NOT disjoint events**

I should **NOT** buy!
Bayes’ Rule

\[ \text{Prob}(E | A) = \alpha \]
\[ \text{Prob}(E | B) = \alpha \]
Bayes’ Rule

\[ \text{Prob} \left( \frac{E}{A} \right) = \alpha \]

\[ \text{Prob} \left( \frac{E}{B} \right) = \alpha \]

Conclusion:

\[ \text{Prob} \left( \frac{E}{A \cup B} \right) = \alpha \]
Bayes’ Rule

\[ \text{Prob} \left( E \mid A \right) = \alpha \]

\[ \text{Prob} \left( E \mid B \right) = \alpha \]

\[ A \cap B = \phi \]

Conclusion:

\[ \text{Prob} \left( E \mid A \cup B \right) = \alpha \]
Bayes’ Rule

\[ \text{Prob}(E | A) = \alpha \]
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Conclusion:
\[ \text{Prob}(E | A \cup B) = \alpha \]

Proof:
\[ \text{Prob}(E | A \cup B) \]
Bayes’ Rule

- \( \text{Prob}(E | A) = \alpha \)
- \( \text{Prob}(E | B) = \alpha \)
- \( A \cap B = \emptyset \)

Conclusion:

\[ \text{Prob}(E | A \cup B) = \alpha \]

Proof:

\[ \text{Prob}(E | A \cup B) = \]
\[ \text{Prob}(E | A) \cdot \text{Prob}(A | A \cup B) + \]
\[ \text{Prob}(E | B) \cdot \text{Prob}(B | A \cup B) \]
Bayes’ Rule

- \( \text{Prob} \left( E \mid A \right) = \alpha \)
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Conclusion:

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Proof:

\[
\text{Prob} \left( E \mid A \cup B \right) = \\
\text{Prob} \left( E \mid A \right) \cdot \text{Prob} \left( A \mid A \cup B \right) + \\
\text{Prob} \left( E \mid B \right) \cdot \text{Prob} \left( B \mid A \cup B \right) = \alpha
\]
The *Logical Sure-Thing Principle* (LSTP):
The *Logical Sure-Thing Principle* (**LSTP**):

$p, q, q'$: propositions
The *Logical Sure-Thing Principle* (LSTP):

\( p, q, q': \) propositions

- Assume:
  
  \[ q \rightarrow p \]
  
  \[ q' \rightarrow p \]
The *Logical Sure-Thing Principle* (LSTP):

$p, q, q':$ propositions

- Assume:
  - $q \rightarrow p$
  - $q' \rightarrow p$

- Conclusion: $q \lor q' \rightarrow p$
Logical Sure-Thing Principle

The **Logical Sure-Thing Principle (LSTP)**:

\[ p, q, q': \text{propositions} \]

- Assume:
  - \( q \rightarrow p \)
  - \( q' \rightarrow p \)

- **Conclusion:** \( q \lor q' \rightarrow p \)

... whether \( q \) and \( q' \) are compatible or not.
The **Logical Sure-Thing Principle** (LSTP):

\[ p, q, q': \text{ propositions} \]

- **Assume:**
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**LSTP** is a theorem of Propositional Calculus
The *Logical Sure-Thing Principle* (**LSTP**):
The *Logical Sure-Thing Principle* (LSTP):

\[ P, Q, Q' : \text{sets} \]
The *Logical Sure-Thing Principle* (LSTP):

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Assume:

- \[ Q \subset P \]
- \[ Q' \subset P \]
The *Logical Sure-Thing Principle* (**LSTP**):

\[ P, Q, Q' : \text{sets} \]

- **Assume:**
  - \( Q \subseteq P \)
  - \( Q' \subseteq P \)

- **Conclusion:** \( Q \cup Q' \subseteq P \)
Logical Sure-Thing Principle

The **Logical Sure-Thing Principle** (**LSTP**):

\[ P, Q, Q' : \text{sets} \]

- Assume:
  - \[ Q \subset P \]
  - \[ Q' \subset P \]

- **Conclusion:** \[ Q \cup Q' \subset P \]

... whether the sets \( Q \) and \( Q' \) are disjoint or not.
The Sure-Thing Principle of decision theory (STP) is NOT the Sure-Thing Principle of logic (LSTP)!
STP and LSTP

The Sure-Thing Principle of decision theory (STP) IS NOT the Sure-Thing Principle of logic (LSTP)!

- STP is a property of *rational decision-making*
The Sure-Thing Principle of decision theory (STP) is not the Sure-Thing Principle of logic (LSTP)!

- **STP** is a property of *rational decision-making*
- **STP** is not a logical necessity
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- Savage ’54: **STP** is an “extralogical principle”
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- **STP** is a property of *rational decision-making*
- **STP** is not a logical necessity
- Savage ’54: **STP** is an “extralogical principle”

**QUESTION**: What is the role of *disjointness* in **STP**?
The probability of a person who tested positive for HIV to survive 5 years is 40%
The probability of a person who tested positive for HIV to survive 5 years is 40%

Your friend tells you that he just tested positive for HIV
Conditional Probability

The probability of a person who tested positive for HIV to survive 5 years is 40%.

Your friend tells you that he just tested positive for HIV.

What is your probability for his surviving 5 years?
Conditional Probability

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What is your probability for his surviving 5 years?

**ANSWER:** 40%
Conditional Probability

- The probability of a person who tested positive for HIV to survive 5 years is 40%
- Your friend tells you that he just tested positive for HIV

What is your probability for his surviving 5 years?

**ANSWER: 40%**

you know that he tested positive for HIV
Conditional Probability

The probability of a person who tested positive for HIV to survive 5 years is 40%.

Your friend tells you that he just tested positive for HIV.

What is your probability for his surviving 5 years?

**ANSWER:** 40%

you know that he tested positive for HIV

and

you know that he told you so
Conditional Probability

Conditioning should be done on:

- The information you have obtained
Conditional Probability

Conditioning should be done on:

- The information you have obtained
- The way the information was obtained

*and*
Conditional Probability

Conditioning should be done on:

- The information you have obtained
  
  ("event" $E$)

  and

- The way the information was obtained
  
  ("signal" $s$)
Conditional Probability

Conditioning should be done on:

- The information you have obtained
  
  \( \text{"event" } E \)

  and

- The way the information was obtained
  
  \( \text{"signal" } s \)

\[ E = \text{your friend tested positive} \]
\[ s = \text{your friend told you that he tested positive} \]
Conditional Probability

- The signal $s$ implies the information $E$ ($s$ is sufficient for $E$)

- $E$ = your friend tested positive
- $s$ = your friend told you that he tested positive
Conditional Probability

- The signal $s$ implies the information $E$ ($s$ is sufficient for $E$)

- Conditioning on $E$ should be done only when $s$ is necessary and sufficient for $E$

- $E = \text{your friend tested positive}$

- $s = \text{your friend told you that he tested positive}$
Conditional Probability

- The signal $s$ implies the information $E$ ($s$ is sufficient for $E$)

- Conditioning on $E$ should be done only when $s$ is necessary and sufficient for $E$

For example, when you are the lab technician:

$s$ if and only if $E$

$E = $ your friend tested positive

$s = $ your friend told you that he tested positive
Set of *states* of the world: $\Omega$
Epistemological Model

- Set of states of the world: $\Omega$
- Event: $E \subset \Omega$
Epistemological Model

- Set of *states* of the world: $\Omega$
- *Event*: $E \subset \Omega$
- *Information* of the DECISION MAKER (DM): A partition $\mathcal{K}$ of $\Omega$
Epistemological Model

- Set of states of the world: $\Omega$
- Event: $E \subset \Omega$

Information of the DECISION MAKER (DM):
A partition $\mathcal{K}$ of $\Omega$

- Atom of $\mathcal{K}$: information set or ken
Epistemological Model

- Set of *states* of the world: $\Omega$
- *Event*: $E \subset \Omega$
- *Information* of the DECISION MAKER (DM):
  A partition $\mathcal{K}$ of $\Omega$
  - Atom of $\mathcal{K}$: *information set* or *ken*
  - When the true state is $\omega \in \Omega$, DM knows only that the true state is in that ken $K(\omega) \in \mathcal{K}$ to which $\omega$ belongs

*ken* = “the range of perception, understanding, or knowledge” (Merriam-Webster)
Epistemological Model

Equivalently:

A *signalling function* $\sigma : \Omega \rightarrow S$
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- The kens are $\sigma^{-1}(s) = \{\omega : \sigma(\omega) = s\}$ for $s \in S$
Epistemological Model

Equivalently:

- A signalling function \( \sigma : \Omega \rightarrow S \)
- The kens are \( \sigma^{-1}(s) = \{ \omega : \sigma(\omega) = s \} \)
  for \( s \in S \)
- DM knows \( E \) when the signal \( s \) implies \( E \):
  \[ \sigma^{-1}(s) \subseteq E \]
Equivalently:

A signalling function \( \sigma : \Omega \rightarrow S \)

The kens are \( \sigma^{-1}(s) = \{ \omega : \sigma(\omega) = s \} \) for \( s \in S \)

DM *knows* \( E \) when the signal \( s \) implies \( E \):

\[ \sigma^{-1}(s) \subseteq E \]

*Conditioning: on the signal* \( s \)

\[ \text{Prob}(A \mid \sigma^{-1}(s)) \] or \[ \text{Prob}(A \mid K(\omega)) \]
Epistemological Model

- Equivalently:
  - A signalling function $\sigma : \Omega \rightarrow S$
  - The kens are $\sigma^{-1}(s) = \{\omega : \sigma(\omega) = s\}$ for $s \in S$

- DM knows $E$ when the signal $s$ implies $E$:
  $$\sigma^{-1}(s) \subset E$$

- Conditioning: on the signal $s$

  $\text{Prob}(A \mid \sigma^{-1}(s))$ or $\text{Prob}(A \mid K(\omega))$

  (not $P(A \mid E)$ nor $P(A \mid KE)$)
The Sure-Thing Principle

Given a signalling function
The Sure-Thing Principle

Given a signalling function

If the decision maker makes the same decision no matter what signal he gets
The Sure-Thing Principle

Given a signalling function

If the decision maker makes the same decision no matter what signal he gets

Then he can make that same decision without getting any signal
The 3 Candidates

D wins

I wins

R wins
The 3 Candidates

Events: RL, DL
The 3 Candidates

Events: RL, DL

signals: “RL”, “DL”
The 3 Candidates

Events: RL, DL

signals: “RL”, “DL”

NOT disjoint

DISJOINT
Sure-Thing Principle - Summary

One must use *all the information*, including the way the information was received (the signal)
Sure-Thing Principle - Summary

One must use *all the information*, including the way the information was received (the signal)

⇒ *disjointness*
Sure-Thing Principle - Summary

- One must use *all the information*, including the way the information was received (the signal)
  \[ \Rightarrow \text{disjointness} \]

- The *Sure-Thing Principle of Decision Theory* is **NOT** the *Sure-Thing Theorem of Logic*
Part II

Agreeing on Decisions

Robert J. Aumann
Sergiu Hart
The Agreement Theorem (Aumann 1976):
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If two people have the same prior
The Agreement Theorem (Aumann 1976):

*If two people have the same prior and their posteriors are common knowledge*
The Agreement Theorem (Aumann 1976):

If two people have the same prior

and their posteriors are common knowledge

then their posteriors must be equal
The Decision Agreement Theorem
(Cave 1983, Bacharach 1985):
The Decision Agreement Theorem

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If two people have the same decision function
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(Cave 1983, Bacharach 1985):

*If two people have the same decision function*

*and their decisions are common knowledge*
The Decision Agreement Theorem
(Cave 1983, Bacharach 1985):

*If two people have the same decision function*

*and their decisions are common knowledge*

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The Decision Agreement Theorem
(Cave 1983, Bacharach 1985):

If two people have the same decision function
which satisfies the Sure-Thing Principle
and their decisions are common knowledge
then their decisions must be equal
The *Sure-Thing Principle* for a decision function:
The *Sure-Thing Principle* for a decision function:

- If the decision is $\delta$ when one knows that $A$ happened, and also when one knows that $B$ happened.
The *Sure-Thing Principle* for a decision function:

- If the decision is $\delta$ when one knows that $A$ happened, and also when one knows that $B$ happened

- and $A$ and $B$ are mutually exclusive (disjoint)
Decision Sure-Thing Principle

The *Sure-Thing Principle* for a decision function:

- If the decision is $\delta$ when one knows that $A$ happened, and also when one knows that $B$ happened
- and $A$ and $B$ are mutually exclusive (disjoint)
- then the decision is $\delta$ when one knows that either $A$ or $B$ happened, without knowing which one
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$$D(A) = D(B) = \delta \quad \text{and} \quad A \cap B = \emptyset$$

$$\implies D(A \cup B) = \delta$$
A murder has been committed. Alice and Bob ...
Another Detective Story

A murder has been committed. Alice and Bob ...

The Decision Agreement Theorem appears in the lecture notes of Aumann on *Interactive Epistemology* (1989)
Another Detective Story

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Another Detective Story

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Why?
Another Detective Story

A murder has been committed. Alice and Bob ...

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Why?

**Moses and Nachum (1990)**
Another Detective Story

A murder has been committed. Alice and Bob ...

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Why?

Moses and Nachum (1990):

*The union of kens is not a ken!*
The union of kens is not a ken:
\textbf{The union of kens is not a ken:}

The union of kens cannot be a ken in the same partition.
The union of kens is not a ken:

- The union of kens cannot be a ken in the same partition
- But can’t one consider other partitions?
**Union of Kens**

*The union of kens is not a ken:*

- The *union* of kens **cannot** be a ken in the same partition
- But can’t one consider other partitions?
- **Union of kens** $\leftrightarrow$ **loss of information**
  $\leftrightarrow$ **confusion:**
The union of kens is not a ken:

- The union of kens cannot be a ken in the same partition
- But can’t one consider other partitions?

Union of kens ⇔ loss of information  ⇔ confusion:

from: Alice knows $A$ or Alice knows $A'$
to: Alice knows $\{A$ or $A'\}$
The union of kens is not a ken:

- The union of kens cannot be a ken in the same partition
- But can’t one consider other partitions?

Union of kens $\leftrightarrow$ loss of information
  $\leftrightarrow$ confusion:

  *from*: Alice knows $A$ or Alice knows $A'$
  *to*: Alice knows $\{A \text{ or } A'\}$

Affects also other people:

  *from*: Bob knows that $\{A$ knows $A$ or $A'$
  *to*: Bob knows $\ ?$
The union of kens is not a ken:

- The union of kens cannot be a ken in the same partition
- But can’t one consider other partitions?

- Union of kens $\leftrightarrow$ loss of information $\leftrightarrow$ confusion:
  - from: Alice knows $A$ or Alice knows $A'$
  - to: Alice knows \{ $A$ or $A'$ \}

- Affects also other people:
  - from: Bob knows that \{ Alice knows $A$ or Alice knows $A'$ \}
  - to: Bob knows ? decides ?
Use the *syntactic approach* (sentences, logic) rather than the *semantic approach* (partitions)
Solution

Use the *syntactic approach* (sentences, logic) rather than the *semantic approach* (partitions)

Formalize and prove syntactically
The *Decision Agreement Theorem*
Solution

Use the **syntactic approach** (sentences, logic) rather than the **semantic approach** (partitions)

- Formulate and prove syntactically
  The **Decision Agreement Theorem**

- Define the **union of kens**
Use the *syntactic approach* (sentences, logic) rather than the *semantic approach* (partitions)

- Formalize and prove syntactically
  The **Decision Agreement Theorem**
- Define the *union of kens* = “ken-fusion”
Use the *syntactic approach* (sentences, logic) rather than the *semantic approach* (partitions)

- Formalize and prove syntactically
  The **Decision Agreement Theorem**

- Define the *union of kens* \(= \text{“ken-fusion”} \) ("confusion")
Two Assumptions

[A1] All knowledge is elementary

- The alphabet is *rich enough* to be able to express everything relevant for the decision — facts and signals
Two Assumptions

[A1] All knowledge is elementary

- The alphabet is *rich enough* to be able to express everything relevant for the decision — facts and signals

- In the 3-candidates example, it must express when exactly the signals “DL” and “RL” are received — otherwise the decision may not be well-defined
Two Assumptions

[A2] *Substantive decisions*

- The decisions depend on *substantive knowledge* only — not on knowledge per se
Two Assumptions

[A2] Substantive decisions

- The decisions depend on substantive knowledge only — not on knowledge per se.
- Knowledge of facts is substantive knowledge; knowledge about knowledge (other’s or one’s own) that does not have factual implications is not relevant to the decision.
[A2] Substantive decisions

- The decisions depend on substantive knowledge only — not on knowledge per se
- Knowledge of facts is substantive knowledge; knowledge about knowledge (other’s or one’s own) that does not have factual implications is not relevant to the decision
- For example: probabilities
Two Assumptions

[A2] Substantive decisions

- The decisions depend on *substantive knowledge* only — not on knowledge per se.
- Knowledge of facts is substantive knowledge; knowledge about knowledge (other’s or one’s own) that does not have factual implications is not relevant to the decision.
- For example: probabilities
- The book author example
The Model

- $\mathcal{X}$: the alphabet
- $\mathcal{E}$, the algebra generated by $\mathcal{X}$: the elementary formulas
- $N$: the set of players
The Model

- $\mathcal{X}$: the *alphabet*
- $\mathcal{E}$, the algebra generated by $\mathcal{X}$: the *elementary* formulas
- $\mathcal{N}$: the set of *players*
- $\Delta$: the set of *decisions*
- $D: \mathcal{E} \rightarrow \Delta$: decision function, the same for all $i \in \mathcal{N}$
The Model

- $\mathcal{X}$: the alphabet
- $\mathcal{E}$, the algebra generated by $\mathcal{X}$: the elementary formulas
- $\mathcal{N}$: the set of players
- $\Delta$: the set of decisions
- $D: \mathcal{E} \rightarrow \Delta$: decision function, the same for all $i \in \mathcal{N}$

[STP]: If $\vdash \neg (e' \land e'')$ and $D(e') = D(e'') = \delta$ then $D(e' \lor e'') = \delta$
The Model

$d_i^\delta$: a symbol for “the decision of $i$ is $\delta$”
The Model

- \( d_i^\delta \): a symbol for “the decision of \( i \) is \( \delta \)”
- for all \( i \in N \) and \( \delta, \delta' \in \Delta \) with \( \delta \neq \delta' \)

\[ \vdash d_i^\delta \Rightarrow \neg d_i^{\delta'} \]

- \( \mathcal{D} = \{ d_i^\delta : i \in N, \delta \in \Delta \} \)
The Model

- $d_i^\delta$: a symbol for “the decision of $i$ is $\delta$”
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$$\vdash d_i^\delta \Rightarrow \neg d_i^{\delta'}$$

- $\mathcal{D} = \{d_i^\delta : i \in N, \delta \in \Delta\}$

Apply the universal canonical construction of Aumann (1999) to generate the *syntax* $\mathcal{G}(N, \mathcal{X} \cup \mathcal{D})$
[A1] All knowledge is elementary:

$$\vdash \forall c \in E \, (k_i e \leftrightarrow c)$$

for every $$i \in N$$, $$e \in E$$, and $$m \in N$$
The Model

[A1] All knowledge is elementary:

\[ \vdash \forall c \in \mathcal{C} k^m_i (k_i e \iff c) \]

for every \( i \in \mathbb{N}, e \in \mathcal{C}, \) and \( m \in \mathbb{N} \)

[A2] Substantive decisions:

\[ \vdash k_i^* e \Rightarrow d_i^D(e) \]

for every \( i \in \mathbb{N} \) and \( e \in \mathcal{C}, \) where

\[ k_i^* e := (\forall c \in \mathcal{C}) (k_i c \iff \vdash (e \Rightarrow c)) \]

(i knows exactly e)
The Result

For a formula $f$, let $c(f)$ denote the list of formulas $\{k^m f : m \in \mathbb{N}\}$ (= “$f$ is *commonly known*”)
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($= \text{“} f \text{ is } \text{commonly known} \text{”}$)

The Decision Agreement Theorem

Assume [STP], [A1] and [A2].

Let $i, j \in \mathbb{N}$ and $\delta, \delta' \in \Delta$. 
For a formula $f$, let $c(f)$ denote
the list of formulas $\{k^m f : m \in \mathbb{N}\}$
(= “$f$ is commonly known”)

The Decision Agreement Theorem

Assume [STP], [A1] and [A2].

Let $i, j \in \mathbb{N}$ and $\delta, \delta' \in \Delta$.

Then $c(d_i^\delta \land d_j^\delta')$ implies $\delta = \delta'$. 
The End?

Sure-Thing!