Comparing Risks by Acceptance and Rejection

Sergiu Hart

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Dean Foster and Sergiu Hart
"An Operational Measure of Riskiness"
www.ma.huji.ac.il/hart/abs/risk.html
Dean Foster and Sergiu Hart
"An Operational Measure of Riskiness"
www.ma.huji.ac.il/hart/abs/risk.html

Dean Foster and Sergiu Hart
"A Reserve-Based Axiomatization of the Measure of Riskiness" (2008)
www.ma.huji.ac.il/hart/abs/risk-ax.html
Sergiu Hart
"A Simple Riskiness Order Leading to the Aumann–Serrano Index of Riskiness" (2008)
www.ma.huji.ac.il/hart/abs/risk-as.html
Sergiu Hart
"A Simple Riskiness Order Leading to the Aumann–Serrano Index of Riskiness" (2008)
www.ma.huji.ac.il/hart/abs/risk-as.html

Sergiu Hart
"Comparing Risks by Acceptance and Rejection" (2009)
www.ma.huji.ac.il/hart/abs/risk-u.html
Gamble ("Risky Asset")

\[ g = \begin{array}{c}
1/2 \\
1/2 \\
\end{array} \quad \begin{array}{c}
\text{+$120} \\
\text{-$100} \\
\end{array} \]
Gamble ("Risky Asset")

\[ g = \frac{1}{2} \cdot +$120 + \frac{1}{2} \cdot -$100 \]

Net gains and losses
Gamble ("Risky Asset")

\[ g = \begin{cases} \frac{1}{2} & \text{+$120} \\ \frac{1}{2} & \text{-$100} \end{cases} \]

- *Net gains and losses*
- *Positive expectation*
Gamble (“Risky Asset”)

\[ g = \frac{1}{2} \cdot +$120 + \frac{1}{2} \cdot -$100 \]

- Net gains and losses
- Positive expectation
- Some losses
Gamble ("Risky Asset")

\[ g = \frac{1}{2} + \frac{1}{2} \]

- Net gains and losses
- Positive expectation
- Some losses
- Pure risk (known probabilities)
Comparing Risks
Comparing Risks

Let $g$ and $h$ be gambles
Comparing Risks

Let $g$ and $h$ be gambles

Question:
Comparing Risks

Let $g$ and $h$ be gambles

**Question:**
When is $g$ LESS RISKY THAN $h$?
Let $g$ and $h$ be gambles

**Question:**
When is $g$ LESS RISKY THAN $h$?

**Answer:**
Let $g$ and $h$ be gambles

**Question:**

When is $g$ LESS RISKY THAN $h$ ?

**Answer:**

When RISK-AVERSE decision-makers are LESS AVERSE to $g$ than to $h$ !
Comparing Risks

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$” = ?
Comparing Risks: Take 1

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$” =

$g$ is PREFERRED to $h$
“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$” =

g is PREFERRED to $h$

$E[u(w + g)] \geq E[u(w + h)]$

for every (concave) utility $u$ and wealth $w$
"risk-averse decision-makers are LESS AVERSE to $g$ than to $h$" =

$g$ is PREFERRED to $h$\[
E[u(w + g)] \geq E[u(w + h)]
\]

for every (concave) utility $u$ and wealth $w$\[
\]

$g$ STOCHASTICALLY DOMINATES $h$ (2nd-degree)
Comparing Risks: Take 1

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$” =

$g$ is PREFERRED to $h$

$E[u(w + g)] \geq E[u(w + h)]$

for every (concave) utility $u$ and wealth $w$

$g$ STOCHASTICALLY DOMINATES $h$ (2nd-degree)

$g \succeq_s h$
Stochastic Dominance

$g$ STOCHASTICALLY DOMINATES $h$ (1st-degree)

$g \geq_{S1} h$
Stochastic Dominance

\( g \) STOCHASTICALLY DOMINATES \( h \) (1st-degree)

\[ g \geq_{S1} h \]

\[
\begin{align*}
\text{1/2} & \quad +$200 \\
\text{1/2} & \quad -$100 \\
\end{align*}
\]

\[
\begin{align*}
\text{1/2} & \quad +$150 \\
\text{1/2} & \quad -$100 \\
\end{align*}
\]
Stochastic Dominance

\( g \) STOCHASTICALLY DOMINATES \( h \) (1st-degree)

\( g \geq_{S1} h \)

\[ \begin{array}{c}
\frac{1}{2} \\
+ \$200 \\
\frac{1}{2}
\end{array} \geq_{S1} \begin{array}{c}
\frac{1}{2} \\
- \$100 \\
\frac{1}{2}
\end{array} \]

\[ \begin{array}{c}
\frac{1}{2} \\
+ \$150 \\
\frac{1}{2}
\end{array} \geq_{S1} \begin{array}{c}
\frac{1}{2} \\
- \$100 \\
\frac{1}{2}
\end{array} \]

\( g' \geq h' \)

\( g \sim g' \)

\( h \sim h' \)
Stochastic Dominance

\[ g \text{ STOCHASTICALLY DOMINATES } h \text{ (1st-degree)} \]

\[ g \geq_{S1} h \]

\[ \begin{array}{c}
\frac{1}{2} \quad \frac{1}{2} \\
+ \$200 \quad - \$100 \\
\end{array} \]

\[ \begin{array}{c}
\frac{1}{2} \quad \frac{1}{2} \\
+ \$200 \quad - \$120 \\
\end{array} \]

- \( g' \geq h' \)
- \( g \sim g' \)
- \( h \sim h' \)
Stochastic Dominance

\( g \) STOCHASTICALLY DOMINATES \( h \) (1st-degree)

\[ g \geq_{S1} h \]

\[
\begin{array}{c}
\frac{1}{2} \\
\downarrow \quad \downarrow \quad \downarrow \\
+200 \quad -100 \quad +200 \\
\frac{1}{2} \quad \frac{1}{3} \quad \frac{2}{3}
\end{array}
\]

\[ g' \geq h' \]

\[ g \sim g' \]

\[ h \sim h' \]
$g$ STOCHASTICALLY DOMINATES $h$ (2nd-degree)

$g \geq_{S2} h$
$g$ STOCHASTICALLY DOMINATES $h$ (2nd-degree)

$$g \succeq_{S2} h$$

- $g$:
  - $1/2$: +$200$
  - $1/2$: -$100$

- $h$:
  - $1/4$: +$250$
  - $1/4$: +$150$
  - $1/2$: -$100$

$g \succeq_{S2} h$
$g$ STOCHASTICALLY DOMINATES $h$ (2nd-degree)

$g \geq_{S2} h$

from $g$ to $h$: a MEAN-PRESERVING SPREAD
Stochastic Dominance

\( g \) STOCHASTICALLY DOMINATES \( h \) (2nd-degree)

\[ g \succeq_{S2} h \]

from \( g \) to \( h \): a MEAN-PRESERVING SPREAD

\[ g \succeq_{S2} h = g \succeq_{S1} + \text{mean-preserving spreads} \]
Stochastic Dominance

Problem
Stochastic Dominance

PROBLEM

Stochastic dominance is a very partial order


**Problem**

Stochastic dominance is a very partial order: most pairs of gambles cannot be compared.
Stochastic dominance is a very partial order:
most pairs of gambles cannot be compared.
Stochastic dominance is a **very partial** order: most pairs of gambles **cannot be compared**.

1/2  +$150  1/2
  |    |    |
  V  V  V
1/2  −$100

1/4  +$500  3/4
  |    |    |
  V  V  V
3/4  −$100
Acceptance and Rejection

Let $g$ be a gamble.
Acceptance and Rejection

Let $g$ be a gamble.

$g$ is ACCEPTED by a decision-maker with utility $u$ at wealth $w$ if

$$
E[u(w + g)] > u(w)
$$
Let $g$ be a gamble.

- $g$ is ACCEPTED by a decision-maker with utility $u$ at wealth $w$ if

  $$E[u(w + g)] > u(w)$$

- $g$ is REJECTED by a decision-maker with utility $u$ at wealth $w$ if

  $$E[u(w + g)] \leq u(w)$$
Comparing Risks

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$”
“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$” = $g$ is REJECTED LESS than $h$
Comparing Risks: Take 2

“risk-averse decision-makers are LESS AVERSE to \( g \) than to \( h \)”

\[ g \text{ is REJECTED LESS than } h \]

IF \( g \) is rejected by \( u \) at \( w \)
THEN \( h \) is rejected by \( u \) at \( w \)

for every (concave) utility \( u \) and wealth \( w \)
“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$” =

$g$ is REJECTED LESS than $h$

IF $E[u(w + g)] \leq u(w)$
THEN $E[u(w + h)] \leq u(w)$

for every (concave) utility $u$ and wealth $w$
Comparing Risks: Take 2

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$”

$g$ is REJECTED LESS than $h$

IF $g$ is rejected by $u$ at $w$
THEN $h$ is rejected by $u$ at $w$

for every (concave) utility $u$ and wealth $w$
Comparing Risks: Take 2

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$”

$g$ is REJECTED LESS than $h$

---

IF $g$ is rejected by $u$ at $w$
THEN $h$ is rejected by $u$ at $w$

for every (concave) utility $u$ and wealth $w$

---

$g$ ACCEPTANCE DOMINATES $h$
Comparing Risks: Take 2

“risk-averse decision-makers are less averse to \( g \) than to \( h \)”

\[ g \text{ is rejected less than } h \]

IF \( g \) is rejected by \( u \) at \( w \)
THEN \( h \) is rejected by \( u \) at \( w \)

for every (concave) utility \( u \) and wealth \( w \)

\[ g \text{ acceptance dominates } h \]

\[ g \succeq_A h \]
Comparing Risks

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$”
Comparing Risks: Take 3

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$” =

$g$ is WEALTH-UNIFORMLY REJECTED LESS than $h$
Comparing Risks: Take 3

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$”

$g$ is WEALTH-UNIFORMLY REJECTED LESS than $h$

IF $g$ is rejected by $u$ at all $w$
THEN $h$ is rejected by $u$ at all $w$

for every (concave) utility $u$
Comparing Risks: Take 3

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$”

$g$ is WEALTH-UNIFORMLY REJECTED LESS than $h$

\[
\text{IF } \mathbb{E}[u(w + g)] \leq u(w) \text{ for all } w \\
\text{THEN } \mathbb{E}[u(w + h)] \leq u(w) \text{ for all } w
\]

for every (concave) utility $u$
Comparing Risks: Take 3

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$” =

$g$ is WEALTH-UNIFORMLY REJECTED LESS than $h$

IF $g$ is rejected by $u$ at all $w$
THEN $h$ is rejected by $u$ at all $w$

for every (concave) utility $u$
“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$” =

$g$ is WEALTH-UNIFORMLY REJECTED LESS than $h$

IF $g$ is rejected by $u$ at all $w$
THEN $h$ is rejected by $u$ at all $w$

for every (concave) utility $u$

$g$ WEALTH-UNIFORMLY DOMINATES $h$
“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$”

$g$ is WEALTH-UNIFORMLY REJECTED LESS than $h$

IF $g$ is rejected by $u$ at all $w$
THEN $h$ is rejected by $u$ at all $w$

for every (concave) utility $u$

$g$ WEALTH-UNIFORMLY DOMINATES $h$

$g \geq_{wu} h$
Comparing Risks

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$”
Comparing Risks: Take 4

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$” =

$g$ is UTILITY-UNIFORMLY REJECTED LESS than $h$
Comparing Risks: Take 4

“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$”

$g$ is UTILITY-UINFORMLY REJECTED LESS than $h$

IF $g$ is rejected by all $u$ at $w$
THEN $h$ is rejected by all $u$ at $w$

for every wealth $w$
“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$”

$g$ is UTILITY-UNIFORMLY REJECTED LESS than $h$

IF $\mathbb{E} [u(w + g)] \leq u(w)$ for all $u$

THEN $\mathbb{E} [u(w + h)] \leq u(w)$ for all $u$

for every wealth $w$
“risk-averse decision-makers are LESS AVERSE to $g$ than to $h$"  

$g$ is UTILITY-UNFORMLY REJECTED LESS than $h$

IF  $g$ is rejected by all $u$ at $w$
THEN  $h$ is rejected by all $u$ at $w$

for every wealth $w$
"risk-averse decision-makers are LESS AVERSE to \( g \) than to \( h \)"

\[ g \text{ is UTILITY-UNIFORMLY REJECTED LESS than } h \]

IF \( g \) is rejected by \textbf{all} \( u \) at \( w \)

THEN \( h \) is rejected by \textbf{all} \( u \) at \( w \)

for every wealth \( w \)

\[ g \text{ UTILITY-UNIFORMLY DOMINATES } h \]
Comparing Risks: Take 4

“risk-averse decision-makers are LESS AVERSE to \( g \) than to \( h \)” =

\( g \) is UTILITY-UNIFORMLY REJECTED LESS than \( h \)

IF \( g \) is rejected by all \( u \) at \( w \)
THEN \( h \) is rejected by all \( u \) at \( w \)

for every wealth \( w \)

\( g \) UTILITY-UNIFORMLY DOMINATES \( h \)

\( g \geq_{uu} h \)
Comparing Risks by Rejection

\[ g \text{ is LESS RISKY than } h \]
\[ \iff g \text{ is REJECTED LESS than } h \]
Comparing Risks by Rejection

$g$ is LESS RISKY than $h$

$\iff g$ is REJECTED LESS than $h$

<table>
<thead>
<tr>
<th>REJECTED =</th>
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<tbody>
<tr>
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</tbody>
</table>
Comparing Risks by Rejection

\[ g \text{ is LESS RISKY than } h \]
\[ \iff g \text{ is REJECTED LESS than } h \]

\[ g \succeq_A h \]

<table>
<thead>
<tr>
<th>REJECTED =</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ g \succeq_A h ] REJECTED by ( u ) at ( w )</td>
</tr>
</tbody>
</table>

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Comparing Risks by Rejection

$g$ is LESS RISKY than $h$

$\Leftrightarrow$ $g$ is REJECTED LESS than $h$

<table>
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<th>REJECTED =</th>
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<tbody>
<tr>
<td>$g \succeq_A h$</td>
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</tr>
<tr>
<td>$g \succeq_{wu} h$</td>
<td>REJECTED by $u$ at ALL $w$</td>
</tr>
</tbody>
</table>
Comparing Risks by Rejection

$g$ is **LESS RISKY** than $h$

$\iff g$ is **REJECTED LESS** than $h$

<table>
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<td>$g \geq_{uu} h$</td>
<td>REJECTED by <strong>ALL</strong> $u$ at $w$</td>
</tr>
</tbody>
</table>
Comparing “Comparing Risks”

\[ g \geq s \ h \]
Comparing “Comparing Risks”

$g \geq_S h$

\[\downarrow\]

$g \geq_A h$
Comparing “Comparing Risks”

\[ g \geq_{S} h \]

\[ \Downarrow \]

\[ g \geq_{A} h \]

\[ \Downarrow \]

\[ g \geq_{WU} h \]

\[ \Downarrow \]

\[ g \geq_{UU} h \]
Riskiness Orders: Results
Riskiness Orders: Results

WEALTH-UNIFORM DOMINANCE:
Riskiness Orders: Results

WEALTH-UNIFORM DOMINANCE:

UTILITY-UNIFORM DOMINANCE:
Riskiness Orders: Results

- **WEALTH-UNIFORM DOMINANCE**: is a *complete* order

- **UTILITY-UNIFORM DOMINANCE**: 
Riskiness Orders: Results

**WEALTH-UNIFORM DOMINANCE:**
- is a *complete* order:
  - for every $g, h$ either $g \succeq_{wu} h$ or $h \succeq_{wu} g$

**UTILITY-UNIFORM DOMINANCE:**
WEALTH-UNIFORM DOMINANCE:
- is a complete order

UTILITY-UNIFORM DOMINANCE:
Riskiness Orders: Results

WEALTH-UNIFORM DOMINANCE:
- is a complete order

UTILITY-UNIFORM DOMINANCE:
- is a complete order
Riskiness Orders: Results

**WEALTH-UNIFORM DOMINANCE:**
- is a *complete* order
- is equivalent to the order induced by the *Aumann–Serrano index of riskiness*

**UTILITY-UNIFORM DOMINANCE:**
- is a *complete* order
WEALTH-UNIFORM DOMINANCE:
- is a complete order
- is equivalent to the order induced by the Aumann–Serrano index of riskiness:
  \[ g \succeq_{wu} h \iff R^{AS}(g) \leq R^{AS}(h) \]

UTILITY-UNIFORM DOMINANCE:
- is a complete order
Riskiness Orders: Results

**WEALTH-UNIFORM DOMINANCE:**
- is a *complete* order
- is equivalent to the order induced by the *Aumann–Serrano index of riskiness*:
  \[ g \succeq_{wu} h \iff R^{AS}(g) \leq R^{AS}(h) \]

**UTILITY-UNIFORM DOMINANCE:**
- is a *complete* order
- is equivalent to the order induced by the *Foster–Hart measure of riskiness*
Riskiness Orders: Results

WEALTH-UNIFORM DOMINANCE:
- is a complete order
- is equivalent to the order induced by the Aumann–Serrano index of riskiness:
  \[ g \succeq_{wu} h \iff R^{AS}(g) \leq R^{AS}(h) \]

UTILITY-UNIFORM DOMINANCE:
- is a complete order
- is equivalent to the order induced by the Foster–Hart measure of riskiness:
  \[ g \succeq_{uu} h \iff R^{FH}(g) \leq R^{FH}(h) \]
RAS and R^FH
The Aumann–Serrano index of riskiness $R^{AS}$ is given by:

$$E \left[ 1 - \exp \left( - \frac{1}{R^{AS}(g)} g \right) \right] = 0$$
\( R^{\text{AS}} \) and \( R^{\text{FH}} \)

**Aumann–Serrano index of riskiness** \( R^{\text{AS}} \):

\[
E \left[ 1 - \exp \left( -\frac{1}{R^{\text{AS}}(g)} g \right) \right] = 0
\]

**Foster–Hart measure of riskiness** \( R^{\text{FH}} \):

\[
E \left[ \log \left( 1 + \frac{1}{R^{\text{FH}}(g)} g \right) \right] = 0
\]
\( \text{Aumann–Serrano index of riskiness } R_{\text{AS}} : \)

\[
E \left[ 1 - \exp \left( - \frac{1}{R_{\text{AS}}(g)} g \right) \right] = 0
\]

( 1 / the CRITICAL RISK-AVERSION coefficient )

\( \text{Foster–Hart measure of riskiness } R_{\text{FH}} : \)

\[
E \left[ \log \left( 1 + \frac{1}{R_{\text{FH}}(g)} g \right) \right] = 0
\]
Aumann–Serrano index of riskiness $R^{AS}$:

$$E \left[ 1 - \exp \left( -\frac{1}{R^{AS}(g)} g \right) \right] = 0$$

(1 / the CRITICAL RISK-AVERSION coefficient)

Foster–Hart measure of riskiness $R^{FH}$:

$$E \left[ \log \left( 1 + \frac{1}{R^{FH}(g)} g \right) \right] = 0$$

( the CRITICAL WEALTH LEVEL )
Aumann–Serrano Index

*Aumann–Serrano index of riskiness* $R^{\text{AS}}$:

$$E \left[ 1 - \exp \left( - \frac{1}{R^{\text{AS}}(g)} g \right) \right] = 0$$
Aumann–Serrano index of riskiness $R_{AS}$:

$$E \left[ 1 - \exp \left( - \frac{1}{R_{AS}(g)} g \right) \right] = 0$$

Let $\alpha^* \equiv \alpha^*(g)$ be the Arrow–Pratt coefficient of absolute risk-aversion of that agent $u(x) = -\exp(-\alpha^* x)$ with constant absolute risk aversion (CARA) who is indifferent between accepting and rejecting $g$. 
Aumann–Serrano index of riskiness $R^{\text{AS}}$:

$$E \left[ 1 - \exp \left( - \frac{1}{R^{\text{AS}}(g)} g \right) \right] = 0$$

Let $\alpha^* \equiv \alpha^*(g)$ be the Arrow–Pratt coefficient of absolute risk-aversion of that agent $u(x) = - \exp(-\alpha^*x)$ with constant absolute risk aversion (CARA) who is indifferent between accepting and rejecting $g$.

Then $R^{\text{AS}}(g) = 1/\alpha^*$
Aumann–Serrano Index

Aumann–Serrano index of riskiness $R^{AS}$:

$$E \left[ 1 - \exp \left( - \frac{1}{R^{AS}(g)} g \right) \right] = 0$$

Let $\alpha^* \equiv \alpha^*(g)$ be the Arrow–Pratt coefficient of absolute risk-aversion of that agent $u(x) = -\exp(-\alpha^* x)$ with constant absolute risk aversion (CARA) who is indifferent between accepting and rejecting $g$

Then $R^{AS}(g) = 1/\alpha^*$

(JPE 2008)
Foster–Hart measure of riskiness $R^{FH}$:

$$
E \left[ \log \left( 1 + \frac{1}{R^{FH}(g)} g \right) \right] = 0
$$
Foster–Hart Measure

**Foster–Hart measure of riskiness** $R^{FH}$:

$$E \left[ \log \left( 1 + \frac{1}{R^{FH}(g)} g \right) \right] = 0$$

- If: each gamble $g$ is rejected when the wealth $W < R^{FH}(g)$
- Then: no-bankruptcy is guaranteed
Foster–Hart measure of riskiness $R^{FH}$:

$$E \left[ \log \left( 1 + \frac{1}{R^{FH}(g)} g \right) \right] = 0$$

- If: each gamble $g$ is rejected when the wealth $W < R^{FH}(g)$
  Then: no-bankruptcy is guaranteed

- $R^{FH}$ is the minimal scale-invariant threshold that guarantees no-bankruptcy
Foster–Hart Measure

**Foster–Hart measure of riskiness** $R_{FH}^*$:

$$E \left[ \log \left( 1 + \frac{1}{R_{FH}^*(g)} \right) \right] = 0$$

- If: each gamble $g$ is rejected when the wealth $W < R_{FH}^*(g)$
- Then: no-bankruptcy is guaranteed

- $R_{FH}^*$ is the **minimal** scale-invariant threshold that guarantees no-bankruptcy

*(JPE 2009)*
\( R^{\text{AS}} \) and \( R^{\text{FH}} \)

**Aumann–Serrano index of riskiness** \( R^{\text{AS}} \):

\[
E \left[ 1 - \exp \left( - \frac{1}{R^{\text{AS}}(g)} g \right) \right] = 0
\]

(1 / the **CRITICAL RISK-AVERSION** coefficient)

**Foster–Hart measure of riskiness** \( R^{\text{FH}} \):

\[
E \left[ \log \left( 1 + \frac{1}{R^{\text{FH}}(g)} g \right) \right] = 0
\]

( the **CRITICAL WEALTH LEVEL** )
Riskiness Orders

\[ g \gtrseq_{S} h \]

\[ g \gtrseq_{A} h \]

\[ g \gtrseq_{WU} h \quad g \gtrseq_{UU} h \]
Riskiness Orders

\[ g \succeq_S h \]

\[ g \succeq_A h \]

\[ g \succeq_{wu} h \]

\[ g \succeq_{uu} h \]

* = complete order
Riskiness Orders

\[ g \succeq_S h \]

\[ g \succeq_A h \]

\[ g \succeq_{\text{WU}} h \quad \text{and} \quad g \succeq_{\text{UU}} h \]

\[ R^{\text{AS}}(g) \leq R^{\text{AS}}(h) \quad \text{and} \quad R^{\text{FH}}(g) \leq R^{\text{FH}}(h) \]

\[ * = \text{complete order} \]
Technical Details

GAMBLE $g$: 


Technical Details

GAMBLE $g$:
- a real-valued random variable
- $E\ [g] > 0$
- $P\ [g < 0] > 0$
- finitely many values
GAMBLE $g$:
- a real-valued random variable
- $\mathbb{E}[g] > 0$
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- finitely many values

UTILITY $u$:
Technical Details

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- $E[g] > 0$
- $P[g < 0] > 0$
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**UTILITY** $u$:
- $u : \mathbb{R}_+ \rightarrow \mathbb{R}$
- strictly increasing
- concave
UTILITY $u$ (continued):
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rejection decreases with wealth:
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  $g$ rejected at $w \implies$
  
  $g$ rejected at $w'$, for $w' < w$
UTILITY $u$ (continued):

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or: DARA (condition 2 of Arrow 1965)
Technical Details

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Technical Details

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Technical Details

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- every gamble is sometimes rejected:
  - for every \( g \) there is \( w \) where \( g \) is rejected
  - or: \( u(0^+) = -\infty \)
Acceptance Dominance

\( p \ast g = \text{the } p\text{-DILUTION of } g = \)
Acceptance Dominance

\[ p \ast g = \text{the } p\text{-DILUTION of } g = g \text{ with probability } p, \text{ and } 0 \text{ with probability } 1 - p \]
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Theorem.
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**Theorem.**

$g$ ACCEPTANCE DOMINATES $h$
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  - \( g \) accepted with probability \( p \), and
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\]

**Theorem.**

\( g \) ACCEPTANCE DOMINATES \( h \)

\iff

there exist \( p, q \in (0, 1] \) such that

\( p \ast g \) STOCHASTICALLY DOMINATES \( q \ast h \)
Acceptance Dominance
Acceptance Dominance

For every $g$ with $\mathbb{E}[g] > 0$

$g \geq_A 2g$
Acceptance Dominance

For every $g$ with $\mathbb{E}[g] > 0$

- $g \succeq_{A} 2g$

- $g \succ_{S} 2g$
For every $g$ with $\mathbb{E}[g] > 0$

- $g \geq_{A} 2g$

- $g \not\geq_{s} 2g : \mathbb{E}[g] < \mathbb{E}[2g]$
Acceptance Dominance

For every $g$ with $E[g] > 0$

- $g \geq_A 2g$:
  - $2u(w + x) \geq u(w + 2x) + u(w)$

- $g <_S 2g$:
  - $E[g] < E[2g]$
For every $g$ with $E[g] > 0$

- $g \succeq_A 2g :$
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  - $2E[u(w + g)] \geq E[u(w + 2g)] + u(w)$

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Acceptance Dominance

For every $g$ with $E[g] > 0$

- $g \geq_A 2g :$
  - $2u(w + x) \geq u(w + 2x) + u(w)$
  - $2E[u(w + g)] \geq E[u(w + 2g)] + u(w)$
  - **IF** $E[u(w + g)] \leq u(w)$ **THEN** $E[u(w + 2g)] \leq u(w)$

- $g \not\geq_S 2g :$
  - $E[g] < E[2g]$
Summary
$g$ is **LESS RISKY** than $h$ whenever risk-averse agents are **LESS AVERSE** to $g$ than to $h$
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- rejection of different gambles should be compared whenever it is **SUBSTANTIVE**: **UNIFORM** over a range of decisions
Summary

- $g$ is LESS RISKY than $h$ whenever risk-averse agents are LESS AVERSE to $g$ than to $h$
- AVERSION to a gamble: REJECTION
- rejection of different gambles should be compared whenever it is SUBSTANTIVE: UNIFORM over a range of decisions

$g$ is less risky than $h$
whenver
$g$ is uniformly rejected less than $h$
by risk-averse agents
\( g \succeq_S h \)

\[ \Downarrow \]

\( g \succeq_A h \)

\( g \succeq_{\text{WU}} h \)

\( g \succeq_{\text{uu}} h \)

\( R^{\text{AS}}(g) \leq R^{\text{AS}}(h) \)

\( R^{\text{FH}}(g) \leq R^{\text{FH}}(h) \)

* = complete order
Summary

ORDINAL approach to riskiness
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**Summary**

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Summary

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- **STATUS QUO**: current wealth $w$
  (in addition to the utility $u$)

- $g \succeq_A \lambda g$ for every $\lambda > 1$
  (ALL risk-averse agents reject $\lambda g$ more than $g$)
ORDINAL approach to riskiness
(Aumann–Serrano and Foster–Hart: “cardinal”)

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Summary

- **ORDINAL** approach to riskiness
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Intrinsic Risk

CAUTION
THIS SIGN HAS SHARP EDGES
DO NOT TOUCH THE EDGES OF THIS SIGN

ALSO, THE BRIDGE IS OUT AHEAD