On Graphical Procedures for Multiple Comparisons

Yosef Hochberg, Gideon Weiss, Sergiu Hart

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In a graphical procedure for comparing \( k \) treatment means in a one-way ANOVA, one displays uncertainty intervals around the sample means and judges any pair to be significantly different if and only if their uncertainty intervals do not overlap. A graphical procedure is a Multiple Comparison Procedure (MCP) if and only if it controls the experimentwise error rate. In this paper we consider some new graphical MCP’s for the unbalanced one-way ANOVA design. These procedures are based on different approximations to the Tukey-Kramer (TK) procedure (e.g., Kramer 1956). As such, they constitute alternatives to Gabriel (1978) (and its modification by Andrews, Snee, and Sarner 1980), which is based on approximating a less efficient MCP (the GT2 of Hochberg 1974). Two of the four procedures considered here are based on best and simple upper bounds to all the confidence-interval lengths of the TK method and hence must be conservative. The other two procedures are based on approximations (here too we have the best vs. the simple procedure), but simulations were used to find that their true experimentwise error rates are less than the nominal ones; that is, these procedures are still on the conservative side. The choice of a particular procedure will depend then on the relative importance of simplicity, efficiency, and the security of having a controlled experimentwise error rate.

KEY WORDS: Analysis of variance; Simultaneous pairwise comparisons; Graphical display.

1. INTRODUCTION

Consider a one-way layout design for \( k \) treatments with independent and normally distributed observations all of equal variance. When the numbers of observations in the \( k \) groups are all the same (\( n \), say) Tukey’s (1953) T-method can be used for pairwise comparisons among the unknown means. Let \( \hat{Y}_1 \leq \cdots \leq \hat{Y}_k \) be the \( k \) ordered sample means, and \( s^2 \) be the mean square error based on \( v = k(n - 1) \) df; then the T-method gives the simultaneous confidence intervals

\[
\hat{Y}_i - \hat{Y}_j \pm \frac{s}{\sqrt{n}} Q_{k,v}^{(a)} \quad 1 \leq i \leq j \leq k, \tag{1.1}
\]

where \( Q_{k,v}^{(a)} \) is the upper \( a \)-th quantile of the Studentized range distribution with parameters \( k, v \). This method is amenable to a simple graphical presentation as follows. The ordered \( \hat{Y}_i \)'s are depicted as a function of \( i = 1, \ldots, k \) and intervals of \( \pm (s/\sqrt{n}) Q_{k,v}^{(a)}/2 \) are constructed perpendicularly around them. Next one draws horizontal straight lines from the top of each of the perpendicular intervals to the right. Any pair of treatments whose intervals are not connected by a common horizontal line are significantly different according to the T-method. See Figure 1 for a hypothetical example with \( k = 5 \). In this case treatment 1 is found significantly inferior to all other treatments; treatment 5 is significantly superior to all other treatments and treatments 2 and 4 are significantly different.

For unbalanced ANOVA designs, various MCP’s were proposed—see Stoline (1981) for a recent exposition. The most suitable methods seem to be based on the maximum of the \( k^* = k(k - 1)/2 \) Studentized two-sided \( t \)-statistics for the different pairwise comparisons. On letting \( n = (n_1, \ldots, n_k)' \) denote the vector of sample sizes, one can provide simultaneous \( 1 - \alpha \) confidence intervals for all pairwise comparisons in the form:

\[
\hat{Y}_i - \hat{Y}_j \leq s C(\alpha, k, n) \left( \frac{1}{n_i} + \frac{1}{n_j} \right)^{1/2}, \quad (1.2)
\]

where \( C(\alpha, k, n) \) is the upper \( a \)-th quantile of the distribution of \( \max \{ | T_{ij} | \} \), where

\[
T_{ij} = \frac{\hat{Y}_i - \hat{Y}_j}{s(1/n_i + 1/n_j)^{1/2}}, \quad 1 \leq i < j \leq k. \tag{1.3}
\]

Hochberg (1974) proposed to approximate \( C(\alpha, k, n) \) by \( M_{k,v}^{(a)} \), the upper \( a \)-th quantile of the Studentized absolute modulus distribution with parameters \( k^* \) and \( v = \sum n_i - k \). This approximate procedure has been referred to as GT2. The GT2 is based on well-known probability inequalities that guarantee its conservativeness (for any \( n, \alpha, k \)).

Procedures like (1.2) are not suitable for a graphical display like Figure 1 because different interval lengths are associated with the \( k - 1 \) pairs \( (i, j) \neq i \) for any \( i \). Gabriel (1978) proposed an approximate method for multiple comparisons based on a graphical display such as...
in Figure 1 for the unbalanced case. His method is based on replacing the pairwise norms \( (1/n_i + 1/n_j)^{1/2} \) with the lower bounds \( 1/\sqrt{2n_i} + 1/\sqrt{2n_j} \) (1 \( \leq i < j \leq k \)) and approximating \( c(a, k, n) \) by \( M_{k,a}^{(a)}(\alpha) \), as in the GT2 method. This method can be used for testing all pairwise comparisons based on uncertainty intervals of \( \pm M_{k,a}^{(a)}(\alpha)/\sqrt{2n_i} \) around \( \bar{Y}_i \) as described above for the T-method.

Gabriel also discusses how to use his method for graphing confidence intervals on pairwise differences and on other contrasts. Gabriel’s procedure is generally conservative for low to moderate imbalances but can be quite liberal when imbalance is high. Andrews, Snee, and Sarner (1980) modified Gabriel’s procedure by an improved approximation of the \( (1/n_i + 1/n_j)^{1/2} \)'s based on a higher-order polynomial regression on the \( (1/\sqrt{n_i})'s \).

Now, the TK procedure proposed by Tukey (1953) and by Kramer (1956) is of the form (1.2) with \( C(a, k, n) = Q_{k,a}^{(a)}(\alpha)/\sqrt{2} \) and is apparently conservative in all cases (see, e.g., Dunn 1980). Also, the TK procedure gives, for any \( a, k, \) and \( n \), shorter confidence intervals than the GT2 procedure (see, e.g., Stoline 1981). In view of that we examine here several alternatives to Gabriel’s procedure based on approximations of the TK rather than approximations of the GT2.

Section 2 contains some discussion on the nature of the approximation involved in modifying a given procedure to one that is amenable to graphical presentations like Figure 1. In Section 3 we examine two procedures (best and simple) that are obtained by approximating from above all the \( k^a \) confidence-interval lengths of the TK procedure. In Section 4 we introduce two sharper approximations (again, best and simple) of the TK procedure and examine their experimentwise error rate via simulations. The Appendix contains the mathematical details of the best approximation of Section 3. For related work on graphical display of means the reader is referred to Andrews, Snee, and David (1980) and to Sampson (1980).

2. ON GRAPHICAL PROCEDURES THAT APPROXIMATE A GIVEN MCP

The TK procedure rejects the hypothesis of the equality of the \( i \)th and \( j \)th means if an only if

\[
| \bar{Y}_i - \bar{Y}_j | > (s/\sqrt{2}) Q_{k,a}^{(a)}(X_i + X_j), \quad a_{ij} = \left( \frac{1}{n_i} + \frac{1}{n_j} \right)^{1/2}.
\]  

(2.1)

We are interested in constructing a procedure that approximates (2.1) in some sense and is amenable to a graphical presentation as in Figure 1. To this end we need nonnegative numbers \( X_1, \ldots, X_k \), such that when perpendicular intervals of length \( \pm s Q_{k,a}^{(a)}(X_i) / \sqrt{2} \) are constructed around \( \bar{Y}_i \) (rather than fixed-length intervals as in Figure 1) then the resulting graphical procedure is similar to the original procedure. Note that the graphical procedure can be presented analytically in the form of the rule: reject the \( (i, j) \)th null hypothesis if and only if

\[
| \bar{Y}_i - \bar{Y}_j | > (s/\sqrt{2}) Q_{k,a}^{(a)}(X_i + X_j),
\]

(2.2)

that is, \( X_i + X_j \) approximates \( (1/n_i + 1/n_j)^{1/2} \) for all \( 1 \leq i < j \leq k \).

Can we always find \( X_i \)'s such that the graphical procedure (2.2) is identical with the original procedure (2.1)? The answer is no! Even if we do not constrain the intervals to be centered at the \( \bar{Y}_i \)'s, we can always find (for \( k \geq 4 \)) a set of plausible decisions according to (2.1) such that no \( X_i \)'s exist that will produce a graphical procedure with the same decisions. For example, when \( k = 4, n_1 = n_4 = 1, n_2 = n_3 = \infty, \bar{Y}_1 = 0, \bar{Y}_2 = .7, \bar{Y}_3 = .9, \bar{Y}_4 = 1.6, \) and \( s Q_{4,a}^{(a)}(\alpha)/\sqrt{2} = 1 \) we get from (2.1) significant differences for the pairs (1, 4) and (2, 3) only. This set of decisions cannot be constructed by any graphical procedure because the uncertainty intervals for treatments 1 and 4 must each cover the gap between the uncertainty intervals of treatments 2 and 3 and hence can be intersected by a horizontal line!

Thus, an exact graphical presentation of the MCP (2.1) does not always exist. We note, however, that one can always construct a graphical procedure as in (2.2) which gives decisions identical to (2.1)'s for the rejected pairs only or for the accepted pairs only, by using \( X_i + X_j \leq a_{ij} \) or \( X_i + X_j \geq a_{ij} \) (for all \( i, j \)), respectively. In the following section we examine best and simple upper bounds that produce conservative graphical procedures.

3. ON BEST AND SIMPLE CONSERVATIVE APPROXIMATIONS

The experimentwise error rate of using (2.2) is maximized when the true means are all equal and is then given by

\[
P \left\{ \frac{| \bar{Y}_i - \bar{Y}_j |}{s a_{ij}} > Q_{k,a}^{(a)}(X_i + X_j) \sqrt{2} \left( \frac{X_i + X_j}{a_{ij}} \right) \text{ for some } (i,j) \right\}
\]

(3.1)
It would seem desirable to select minimal \( X_i \)'s that still satisfy \( X_i + X_j \geq a_{ij} \) for all \( i, j \). One must, however, select a criterion for minimization since simultaneous minimization of all \( X_i \)'s cannot be achieved. Two possible criteria are

(i) Minimize \( \sum_{i=1}^{k} X_i \), and

(ii) Minimize \( \sum_{(i,j)} (X_i + X_j)/a_{ij} \)

both under the constraints: \( X_i + X_j \geq a_{ij} \), for all \( i \neq j \).

The first problem is equivalent to minimizing \( \sum_{(i,j)} \sum (X_i + X_j) = 2(k - 1) \sum X_i \). The second criterion is equivalent to minimizing \( 2 \sum_{i=1}^{k} b_i X_i = \sum_{(i,j)} \sum (X_i + X_j)/a_{ij} \), where \( b_i = \sum_{j \neq i} (1/a_{ij}), \, i = 1, \ldots, k \).

Both problems are simple linear programming problems and solutions can easily be obtained by the use of the simplex algorithm. For the first problem we can obtain the \( X_i \)'s more directly as follows (see Appendix for proof). Assume that the treatments are ordered so that \( n_1 \geq n_2 \geq \ldots \geq n_k \), and \( i_1, \ldots, i_l \). Rearrange the indices \( k, \ldots, 1 \) as \( i_1, \ldots, \).

An optimal solution of problem (i) is obtained by solving the \( k \) equations

\[
X_{i_1} = \frac{1}{2}a_{i_1i_2} + \frac{1}{2}a_{i_1i_l} - \frac{1}{2}a_{i_l},
\]

\[
X_{i_j} = a_{i_{j-1}i_j} - X_{i_{j-1}}, \quad j = 2, \ldots, k.
\]

These best bounds are somewhat more complicated than Gabriel's simple approximation. We now obtain simple upper bounds for all \( a_{ij} \)'s in the form of \( X_i + X_j \) based on \( X_i \)'s that are proportional to the \((1/\sqrt{n_i})\)'s. The proportionality coefficient \( C \) is the smallest constant that will satisfy \( C(1/\sqrt{n_i} + 1/\sqrt{n_j}) \geq a_{ij} \) for all \( i \neq j \) with at least one exact equality. Now, for any positive numbers \( U \) and \( V \) we have \( \phi(G)[U + V] = (U^2 + V^2)^{1/2} \), where \( \phi(G) = (1 + G^{1/2})(1 + G^{1/2}) \) and \( G = (U/V)^2 \) or \( G = (V/U)^2 \) since \( \phi(G) = \phi(1/G) \) for all \( G > 0 \). Since \( \phi(G) \) is increasing in \( G \), it follows that the required constant \( C \) is given by \( C = \phi(d) = \phi(1/d) \) where \( d = \min(n_i) \) or \( n \).

Thus, the \( X_i \)'s based on this simple conservative method are given by

\[
X_i = \phi(d)/\sqrt{n_i}, \quad i = 1, \ldots, k.
\]

In Table 1 we give the (rounded) percentage increase in average pairwise confidence interval length due to approximating the TK procedure by the best conservative procedure and by the simple conservative procedure for \( k = 4, 5, 6, \) and a wide range of imbalances. Imbalance is reflected in the configuration of sample sizes. We let \( n_1 \geq n_2 \geq \ldots \geq n_k, \quad d_i = \min(n_i)/n_j \) and thus \( \min(d) = d \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( d = .2 )</th>
<th>( d = \sqrt{2} )</th>
<th>( d = \sqrt{5} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>( d = 1 )</td>
<td>( d = 1 )</td>
<td>( d = 1 )</td>
</tr>
<tr>
<td>5</td>
<td>( d = 1 )</td>
<td>( d = 1 )</td>
<td>( d = 1 )</td>
</tr>
<tr>
<td>6</td>
<td>( d = 1 )</td>
<td>( d = 1 )</td>
<td>( d = 1 )</td>
</tr>
</tbody>
</table>

The specific configurations considered are similar to those in Dunnett's (1980) study of the TK procedure. The first column in the table corresponds to the best conservative procedure and the second and third columns correspond to the simple conservative procedure. A star attached to a configuration indicates that the best solution is exact, that is, it gives \( X_i + X_j = a_{ij} \) for all \( i \neq j \). Note that for \( k = 3 \) our best procedures are always exact.

From Table 1 (and from similar results not given here for \( k \) up to 10 and other values of \( d \)) we can summarize as follows:

1. The performances of the two methods are only slightly affected by \( k \), for \( 4 \leq k \leq 10 \).

2. Larger imbalances result in more conservative approximations. But even for a ratio of 1:25 \( (d = .2) \) in sample sizes, the increases in average confidence interval length due to the best upper-bounds range only up to 7 percent.

3. For sample size ratios of 1:2 \( (d = \sqrt{2}) \) or less (i.e., moderate to small imbalances) the best graphical method produces almost the exact intervals, with an average increase of nearly zero (the results in the table are rounded).

4. The simple approximation gives an average increase of nearly zero only for small imbalances (i.e., sample size ratios between 5:6 to 1:1).

Gabriel and Gheva (1982) introduced a modified version of our simple conservative approximation. To describe this modification let \( N = \max(n_i) \) and \( n = \min(n_i) \).
and define
\[ G_i = \frac{N}{n_i} \text{ if } n_i \leq (nN)^{1/2} \]
\[ = n_i/n \text{ if } n_i > (nN)^{1/2}. \]

In their method the \( X_i \) are taken as \( \phi(G_i)/\sqrt{n_i}, i = 1, \ldots, k \). These are sharper conservative bounds than our simple upper bounds given by (3.3). Gabriel and Gheva compared their intervals with our simple conservative intervals by the ratio of average interval widths (for all pairwise comparisons) and by average of ratios of interval widths, and found substantial improvements in some cases.

In summary then, we may offer the user three conservative procedures: the simple conservative, the Gabriel-Gheva modified simple conservative, and the best conservative procedure. In this order we may say that the last one is least simple, the first is the simplest, and the second one is intermediate. However, the third gives the best intervals, the first gives the widest intervals, and the second procedure is intermediate.

In the following section we examine other approximations for obtaining less conservative graphical MCP’s. These approximations are not based on upper bounds for all \( a_{ij} \)'s; thus, to gain insight on their error rates we used simulations.

4. APPROXIMATE PROCEDURES (NOT BASED ON UPPER BOUNDS)

Under a given pattern of imbalance the true confidence coefficient of the TK procedure seems to increase with the degree of imbalance (see, e.g., Dunnett 1980). This gives hope that procedures of the form (2.2) exist with shorter confidence intervals than those considered in Section 3, whose true confidence coefficient is always not less than \( 1 - \alpha \). We now consider two such procedures that are based on best approximate intervals and on simple approximate intervals.

The best approximate intervals are based on \( X_i \)'s that minimize
\[ \sum_{i \neq j} (X_i + X_j - a_{ij})^2. \]

Such \( X_i \)'s are regression estimators and we can get them by taking first-order derivatives and equating to zero. The solution of the normal equations gives
\[ X_i = (k - 1) \frac{\sum_{j \neq i} a_{ij} - \sum_{1 \leq j < i \leq k} a_{ij}}{(k - 1)(k - 2)}, \]
\[ i = 1, \ldots, k, \quad (4.1) \]
and this solution guarantees
\[ \sum_{i \neq j} (X_i + X_j) = \sum_{i \neq j} a_{ij}. \]

To see that these \( X_i \)'s are nonnegative note that, under the ordering \( n_1 \leq n_2 \leq \cdots \leq n_k \), the minimal \( X_i \) is \( X_1 \), and
\[ (k - 1)(k - 2)X_1 = (k - 2) \sum a_{1j} - \sum a_{1i} \geq (k - 2) \sum n_j^{1/2} - \sum n_i^{1/2} + n_i^{1/2} = 0. \]

Alternative approximations can be considered; for example, minimize
\[ \sum_{1 \leq i < j \leq k} [X_i + X_j]/a_{ij} - 1]^2 \]
or minimize
\[ \sum_{1 \leq i < j \leq k} [\ln(X_i + X_j) - \ln(a_{ij})]^2. \]

However, the best approximate intervals given by (4.1) have simple explicit expressions as in (4.1) and simulations show (as we discuss later) that the resulting procedure is on the conservative side!

The best approximate intervals based on (4.1) involve the computation of all the \( k^2 a_{ij} \)'s. For the sake of completeness we also consider a simpler approximate procedure that does not require the computation of all the \( a_{ij} \)'s.

The simple approximate intervals are based on \( X_i \)'s of the form \( X_i = C/\sqrt{n_i} \), as were the simple upper bounds in Section 3, but now we choose for \( C \) the constant \( C = \phi(d) \) where \( \phi(\cdot) \) is as in Section 3 and \( d \) is the arithmetic average of the \( k(k - 1)/2 \) ratios \( n_j/n_i, 1 \leq i < j \leq k \) when \( n_1 \geq n_2 \geq \cdots \geq n_k \). Note that the arithmetic mean is larger than the geometric and harmonic means; and since \( \phi(\cdot) \) is monotone decreasing in \([0, 1]\), it follows that the use of the arithmetic average will produce narrower intervals than the use of the other averages of the \((n_j/n_i)\)'s. The main question now is with regard to the experimentwise error rates of these approximate procedures. From (4.2) it follows that the average pairwise interval of the best approximate intervals is equal to the corresponding average of the TK intervals. The simple approximate intervals are shorter than the simple upper bounds of Section 3, depending on the fraction \( \phi(d) / \phi(d) \). But the procedures of Section 3 are clearly conservative, and this cannot be said for the procedures of Section 4. What can we say about the (experimentwise) error rate of the procedures based on best approximate intervals or on simple approximate intervals? Note that from (3.1) it follows that if there are pairs \((i, j)\) for which \( (X_i + X_j)/a_{ij} \) comes as low as \( \sqrt{2} \exp(\sigma^2/(\sigma^2 + \sigma^2)) \), then the corresponding graphical procedure will be liberal for that given design. However, in extensive numerical work we found that for all practical purposes this does not happen with the best approximate intervals and the simple approximate intervals. This gave further hope that these graphical procedures are always conservative.

Note that (3.1) being less than or equal to \( \alpha \) for all \( \alpha \) when \( \nu = \infty \), implies that the statistic \( \max_{1 \leq i < j \leq k} \{Y_i - Y_j | / \sigma(X_i + X_j)\} \) is stochastically smaller than the
Table 2. Values of \( X_i \) in Using (2.2) for Upper and Approximate, Best and Simple Graphical Procedures

\( (n_1 \geq n_2 \ldots \geq n_k, a_{ij} = (1/n_i + 1/n_j)^{1/2}, d = \min(n_i)/\max(n_i), d_A = \text{average of all } k^n (n_i/n_j) \)'s

<table>
<thead>
<tr>
<th>Upper Bound</th>
<th>( (k, k - 1, \ldots, h) = (1, k, k - 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor + 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_i )</td>
<td>( \frac{d}{\sqrt{n_i}} ) ( X_i = \frac{d}{\sqrt{n_i}} )</td>
</tr>
<tr>
<td>( X_i = \frac{\left\lfloor \frac{k}{2} \right\rfloor}{\sqrt{n_i}} )</td>
<td>( X_i = \frac{d}{\sqrt{n_i}} )</td>
</tr>
</tbody>
</table>

standardized range of \( k \) independent and equally distributed normal variates. This then implies that (3.1) is less than or equal to \( \alpha \) for all \( \alpha \) and all \( \nu \)!

We estimated the experimentwise error rate of these procedures by simulating runs of 10,000 experiments for \( k = 4, 5, 6 \) and the configurations of Table 1 (which are similar to those in Dunnott 1980). We let \( d = .2, \sqrt{1}, \sqrt{.2}, \sqrt{.5}, \sqrt{1/1.5}, \text{and } \sqrt{1/1.3} \) and used the same set of 10,000 simulated experiments for all configurations and all values of \( d \) (by simple transformation of the \( k \) unit normal variables simulated for each experiment). For each value of \( k \) then, we obtained the relative frequencies of (3.1) for \( \nu = \infty \) and \( \alpha = .99, .95, .90, .80, .50, .20, .10, .05, \) and .01 for the TK method as well as for the two graphical procedures based on the best approximate intervals and on the simple approximate intervals. A typical representative of all runs is displayed in Figure 2 for the case \( \alpha = .10, k = 5, \) and configuration \( (d d d 1 1) \). Similar patterns hold for all the other percentiles, all the other values of \( k \) and all the other configurations in our simulation study. In generating the normal deviates, the IMSL Subroutine GGNML was used. A check on the simulations was provided by comparing the empirical estimates with the theoretical values in the case of balance. These two were in good agreement for all values of \( \alpha \) and \( k \). We used the CDC 6600 computer of Tel Aviv University.

These simulations indicate that all three methods are on the conservative side. This conclusion is strengthened by the apparent increase in the true experimentwise error rate as a function of \( d \) for all configurations and all nominal \( \alpha \)'s considered in our study.

Which procedure should one use when a graphical display is required as in Figure 1? The choice will depend on the relative importance one attaches to simplicity in computations (i.e., not having to compute all \( a_{ij} \)'s) versus a better control of the experimentwise error-rate. For convenience we summarize the four graphical procedures in Table 2.

APPENDIX: OPTIMAL UPPER BOUNDS

Lemma. For \( f \) concave, \( \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k, a_{st} = f(\alpha_s + \alpha_t), s, t = 1, \ldots, k, k > 2 \), define the linear program:

\[
Z = \min \sum_{s=1}^{k} X_s
\]

\[
X_s + X_t \geq a_{st} \quad 1 \leq s < t \leq k. \quad \text{(LP)}
\]

1. Rearrange the indices as \( s_1, s_{k-1}, \ldots, s_1 = 1, k, 2, k - 1, \ldots, [k/2] + 1 \). Then

\[
X_{s_1} = \frac{1}{2} a_{s_1s_2} + \frac{1}{2} a_{s_1s_3} - \frac{1}{2} a_{s_2s_3}
\]

\[
X_{s_j} = a_{s_j,s_{j-1}} - X_{s_{j-1}}, j = 2, \ldots, k, \quad \text{(A.1)}
\]

is an optimal solution to (LP).

2. The optimal value of \( \sum X_s \) is

\[
Z = \begin{cases} 
\sum_{s=1}^{[k/2]} a_{s,k+1-s} & \text{k even} \\
\sum_{s=1}^{(k-3)/2} a_{s,k+1-s} + \sum_{s=k+2}^{(k+3)/2} a_{s,k-1-s} & \text{k odd} 
\end{cases}
\]

(A.2)

3. The solution (A.1) maximizes each of \( X_1, \ldots, X_{[k/2]} \) and minimizes each of \( X_{[k/2]+1}, \ldots, X_k \) among all optimal solutions.
4. If \( f \) is also nonnegative, monotone nondecreasing, \( 0 \leq X_1 \leq \cdots \leq X_k \) for all optimal solutions of (LP), and the solution (A.1) minimizes the variance of \( X_1, \ldots, X_k \) among all optimal solutions.

Proof.

1. The dual to (LP) is the fractional matching problem, and it is easily seen that

\[
X_s + X_{k+1-s} = a_{s,k+1-s} \quad s = 1, 2, \ldots, [k/2],
\]

ensures dual feasibility and is satisfied by (A.1). It remains to show that (A.1) is feasible. We use the concavity of \( f \), which implies

\[
a_{st} + a_{sv} \leq a_{su} + a_{tv} + a_{au}, \quad s \leq t \leq u \leq v.
\]

For \( k = 3 \) the unique optimal solution is of type (A.1), namely:

\[
X_1 = \frac{1}{2} a_{12} + \frac{1}{2} a_{13} - \frac{1}{2} a_{23},
\]

\[
X_2 = \frac{1}{2} a_{12} - \frac{1}{2} a_{13} + \frac{1}{2} a_{23},
\]

\[
X_3 = -\frac{1}{2} a_{12} + \frac{1}{2} a_{13} + \frac{1}{2} a_{23}.
\]

The induction step from \( k - 1 \) to \( k \) follows. Note that if \( X_1, \ldots, X_k \) is the solution (A.1) to (LP), then \( X_2, \ldots, X_k \) is the solution (A.1) to LP for the indices \( k, \ldots, k \). Hence by induction \( X_s + X_t = a_{st} \), \( s < t \leq k \). For \( 2 < t \leq k \), by (A.1), (A.4) and the induction

\[
X_1 + X_t = a_{1k} - X_k + X_t = a_{1k} - a_{2k} + X_2 + X_t
\]

\[
\geq a_{1k} - a_{2k} + a_{2t} \geq a_{1t},
\]

and similarly

\[
X_1 + X_2 = a_{1k} - a_{2k} + X_2 + a_{2k-1} - a_3 + X_3
\]

\[
\geq a_{1k} - a_{2k} + a_{2k-1} - a_3 + a_{23} \geq a_{12}.
\]

2. The optimal value of \( \sum X_s \) is obtained as (A.2) directly from (A.3). We note also that this implies that a feasible solution of (LP) is optimal if and only if it satisfies (A.3).

3. There is nothing to prove for \( k = 3 \). For \( k = 4 \) one can check step 3 by writing down the two basic optimal solutions. Proceeding by induction from \( k - 2, k - 1 \) to \( k \), \( k > 4 \), let \( X_1', \ldots, X_k' \) (\( X_1, \ldots, X_k \)) be any optimal solution (the solution (A.1)). Then \( X_2', \ldots, X_{k-1}' \) (\( X_2, \ldots, X_{k-1} \)) satisfies (A.3) (satisfies (A.1)) and is optimal (is the solution (A.1)) for the problem with indices \( 2, \ldots, k - 1, \) so by induction \( X_s' \leq X_s, s = 2, \ldots, [(k + 1)/2] \) and \( X_s' \geq X_s, s = [k/2] + 1, \ldots, k - 2 \). Also, \( X_2, \ldots, X_k \) is solution (A.1) for the indices \( 2, \ldots, k \), so \( X_2 + \cdots + X_k \) is minimal and therefore \( X_1 \) is maximal, and by \( X_1 + X_k = a_{1k} \), \( X_k \) is minimal.

4. We show first that \( X_1 \leq X_2 \leq \cdots \leq X_k \). We use the monotonicity of \( f \), which implies

\[
a_{st} \leq a_{sv} \quad s \leq u, \quad t \leq v.
\]

Every optimal solution satisfies the inequalities in (LP) and the equalities in (A.3), so that

\[
X_{k+1} + X_{k+1-s} \geq a_{s+k+1-k-s} \geq a_{k+1-k-s} = X_s + X_{k+1-s}, \quad s \neq [(k + 1)/2].
\]

For \( k \) odd, \([(k + 1)/2]\) \( = [k/2] \), and by (A.3)

\[
X_{k+3/2} + X_{k+1/2} = a_{(k+3)/2(k+1)/2} \geq a_{(k-1)/2(k-1)/2} = X_{(k+1)/2} + X_{(k-1)/2}.
\]

For \( k \) even, \([(k + 1)/2]\) \( = [k/2] \), and we have to check that \( X_{k+1/2} \leq X_{k/2} \) for all optimal solutions. But \( X_{k/2} \) is maximal and \( X_{k+1/2} \) is minimal when we take the solution (A.1), so we need only check that solution, which is easily done. This completes the proof that \( X_1 \leq \cdots \leq X_k \) for all optimal solutions.

To show that \( X_1 \geq 0 \) for all optimal solutions when \( f \) is nonnegative, we note that if we define \( s_1, s_2, \ldots, s_1 = 1, \ldots, s_1 = 1, \ldots, s_1 = 1 \), and substitute it into (A.1), we obtain an optimal solution \( X_1, \ldots, X_k \) that minimizes \( X_1, X_1, \ldots, X_1 \), i.e., \( X_1 \) has the same property for indices \( 1, \ldots, k, k \). We now check by (A.4) and (A.5) that \( X_1 \geq 0 \) when \( k = 3 \) and use induction.

Finally, the solution (A.1) minimizes the variance because of step 3, since all optimal solutions have the same mean of \( \sum X_s \) and satisfy \( X_1 \leq \cdots \leq X_k \), thus having the same mean and the same median.

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REFERENCES


