Dynamics and Equilibrium

Sergiu Hart

Presidential Address, GAMES 2008 (July 2008)

Revised and Expanded (February 2009)
Papers

- Hart and Mas-Colell, *J Econ Theory* 2001
- Hart and Mas-Colell, *Amer Econ Rev* 2003
- Hart and Mas-Colell, *Games Econ Behav* 2006
- Hart and Mansour, *Games Econ Behav* 2009
- Hart, *Center for Rationality DP* 2008
Papers

- Hart and Mas-Colell, *J Econ Theory* 2001
- Hart and Mas-Colell, *Amer Econ Rev* 2003
- Hart and Mas-Colell, *Games Econ Behav* 2006
- Hart and Mansour, *Games Econ Behav* 2009
- Hart, *Center for Rationality DP* 2008

http://www.ma.huji.ac.il/hart
Nash Equilibrium
Nash Equilibrium

EQUILIBRIUM POINT:

John Nash, Ph.D. Dissertation, Princeton 1950
EQUILIBRIUM POINT:

"Each player’s strategy is optimal against those of the others."

John Nash, Ph.D. Dissertation, Princeton 1950
FACT

There are no general, natural dynamics leading to Nash equilibrium
FACT

There are no \textit{general}, natural dynamics leading to Nash equilibrium

"general"
FACT

There are no general, natural dynamics leading to Nash equilibrium

"general" : in all games
FACT

There are no *general*, natural dynamics leading to Nash equilibrium

"general" : in all games rather than: in specific classes of games
FACT

There are no \textit{general}, natural dynamics leading to Nash equilibrium

\textit{"general"} : in all games rather than: in specific classes of games:

- two-person zero-sum games
- two-person potential games
- supermodular games
- . . .
FACT

*There are no general, natural dynamics leading to Nash equilibrium*
FACT

There are no general, natural dynamics leading to Nash equilibrium

"leading to Nash equilibrium"
FACT

There are no general, natural dynamics leading to Nash equilibrium

"leading to Nash equilibrium": at a Nash equilibrium (or close to it) from some time on
FACT

There are no general, natural dynamics leading to Nash equilibrium
FACT

There are no general, natural dynamics leading to Nash equilibrium

"natural"
There are no general, natural dynamics leading to Nash equilibrium

"natural": 
FACT

There are no general, natural dynamics leading to Nash equilibrium

"natural":

- adaptive (reacting, improving, ...)
There are no general, *natural* dynamics leading to Nash equilibrium

- "natural":
  - adaptive (reacting, improving, ...)
  - simple and efficient
There are no general, *natural* dynamics leading to Nash equilibrium

"natural":
- adaptive (reacting, improving, ...)
- simple and efficient:
  - computation (performed at each step)
There are no general, natural dynamics leading to Nash equilibrium

"natural":
- adaptive (reacting, improving, ...)
- simple and efficient:
  - computation (performed at each step)
  - time (how long to reach equilibrium)
FACT

There are no general, natural dynamics leading to Nash equilibrium

"natural":
- adaptive (reacting, improving, ...)
- simple and efficient:
  - computation (performed at each step)
  - time (how long to reach equilibrium)
  - information (of each player)
FACT

There are no general, natural dynamics leading to Nash equilibrium

"natural":
- adaptive (reacting, improving, ...)
- simple and efficient:
  - computation (performed at each step)
  - time (how long to reach equilibrium)
  - information (of each player)

bounded rationality
Dynamics that are **NOT** "natural":

Dynamics that are NOT "natural":

- **exhaustive search**
  (deterministic or stochastic)
Exhaustive Search
Exhaustive Search

\[ E = mc^2 \]
Exhaustive Search

\[ E = mc^2 \]
\[ E = mb^2 \]
Exhaustive Search
Exhaustive Search

\[ E = mc^2 \]

\[ E = ma^2 \]

\[ E = mb^2 \]
Exhaustive Search
Exhaustive Search

\[ E = mc^2 \]

\[ E = ma^2 \]

\[ E = mb^2 \]

[Image of Albert Einstein writing on a blackboard]
Einstein’s Manuscript

Albert Einstein, 1912

On the Special Theory of Relativity (manuscript)
Dynamics that are **NOT** "natural":

- **exhaustive search**
  (deterministic or stochastic)
Dynamics that are NOT "natural":

- exhaustive search (deterministic or stochastic)
- using a mediator
Dynamics that are **NOT** "natural":

- exhaustive search (deterministic or stochastic)
- using a **mediator**
- broadcasting the private information and then performing **joint** computation
Dynamics that are **NOT** "natural":

- **exhaustive search** (deterministic or stochastic)
- using a **mediator**
- **broadcasting** the private information and then performing **joint** computation
- **fully rational learning** (prior beliefs on the strategies of the opponents, Bayesian updating, optimization)
FACT

There are no general, \textit{natural} dynamics leading to Nash equilibrium

"natural":

- adaptive
- simple and efficient:
  - computation (performed at each step)
  - time (how long to reach equilibrium)
  - information (of each player)
FACT

There are no general, natural dynamics leading to Nash equilibrium

"natural":

- adaptive
- simple and efficient:
  - computation (performed at each step)
  - time (how long to reach equilibrium)
  - information (of each player)
FACT

There are no general, natural dynamics leading to Nash equilibrium

"natural":
- adaptive
- simple and efficient:
  - computation (performed at each step)
  - time (how long to reach equilibrium)
  - information (of each player)
Each player knows *only* his own payoff (utility) function.
Each player knows *only* his own payoff (utility) function

(does *not* know the payoff functions of the other players)
UNCOPLED DYNAMICS:

Each player knows \textit{only} his own payoff (utility) function

(does \textit{not} know the payoff functions of the other players)

\textit{Hart and Mas-Colell, AER 2003}
Each player knows *only* his own payoff (utility) function

(does *not* know the payoff functions of the other players)

(privacy-preserving, decentralized, distributed ...)

**Hart and Mas-Colell, AER 2003**
$N$-person game in strategic (normal) form:

- Players

\[ i = 1, 2, \ldots, N \]
$N$-person game in strategic (normal) form:

- **Players**
  
  \[ i = 1, 2, \ldots, N \]

- For each player $i$: **Actions**
  
  \[ \alpha^i \text{ in } A^i \]
$N$-person game in strategic (normal) form:

- **Players**
  
  $i = 1, 2, ..., N$

- **Actions**
  
  $a^i$ in $A^i$

- **Payoffs (utilities)**
  
  $u^i(a) \equiv u^i(a^1, a^2, ..., a^N)$
Time

\[ t = 1, 2, \ldots \]
Dynamics

- **Time**

\[ t = 1, 2, \ldots \]

- At period \( t \) each player \( i \) chooses an action \( a^i_t \) in \( A^i \)
Dynamics

- Time

\[ t = 1, 2, \ldots \]

- At period \( t \) each player \( i \) chooses an action \( a^i_t \) in \( A^i \)

  according to a probability distribution \( \sigma^i_t \) in \( \Delta(A^i) \)
Fix the set of players 1, 2, ..., $N$ and their action spaces $A^1, A^2, ..., A^N$
Fix the set of players 1, 2, ..., N and their action spaces $A^1, A^2, ..., A^N$

A general dynamic:
Fix the set of players 1, 2, ..., $N$ and their action spaces $A^1, A^2, ..., A^N$.

A general dynamic:

$$\sigma^i_t \equiv \sigma^i_t (\text{HISTORY} ; \text{GAME})$$
Fix the set of players $1, 2, \ldots, N$ and their action spaces $A^1, A^2, \ldots, A^N$.

A general dynamic:

$$\sigma_t^i \equiv \sigma_t^i (\text{HISTORY}; \text{GAME})$$

$$\equiv \sigma_t^i (\text{HISTORY}; u^1, \ldots, u^i, \ldots, u^N)$$
Uncoupled Dynamics

Fix the set of players 1, 2, ..., $N$ and their action spaces $A^1, A^2, ..., A^N$

- A general dynamic:

$$\sigma^i_t \equiv \sigma^i_t ( \text{HISTORY} ; \text{GAME} )$$

$$\equiv \sigma^i_t ( \text{HISTORY} ; u^1, ..., u^i, ..., u^N )$$

- An **UNCOPLED** dynamic:
Uncoupled Dynamics

Fix the set of players 1, 2, ..., $N$ and their action spaces $A^1, A^2, ..., A^N$

- A general dynamic:

$$\sigma^i_t \equiv \sigma^i_t \ ( \text{HISTORY} ; \ \text{GAME} )$$

$$\equiv \sigma^i_t \ ( \text{HISTORY} ; \ u^1, ..., u^i, ..., u^N )$$

- An UNCOUPLED dynamic:

$$\sigma^i_t \equiv \sigma^i_t \ ( \text{HISTORY} ; \ u^i )$$
Simplest **uncoupled dynamics**
Simplest **uncoupled dynamics**:

\[ \sigma^i_t \equiv f^i(a_{t-1}; u^i) \]

where \( a_{t-1} = (a^1_{t-1}, a^2_{t-1}, \ldots, a^N_{t-1}) \in A \) are the actions of all the players in the previous period.
Simplest **uncoupled dynamics**:

\[ \sigma_t^i \equiv f^i(a_{t-1}; u^i) \]

where \( a_{t-1} = (a_{t-1}^1, a_{t-1}^2, \ldots, a_{t-1}^N) \in A \)

are the actions of all the players in the previous period

*Only last period matters ("1-recall")*
Simplest **uncoupled dynamics**: 

\[ \sigma_t^i \equiv f^i(a_{t-1}; u^i) \]

where \( a_{t-1} = (a_{t-1}^1, a_{t-1}^2, \ldots, a_{t-1}^N) \in A \) are the actions of all the players in the previous period.

- Only last period matters (**1-recall**)
- Time \( t \) does not matter (**stationary**)

**Uncoupled Dynamics**
Theorem. There are **no** uncoupled dynamics with 1-recall

\[ \sigma^i_t \equiv f^i(a_{t-1}; u^i) \]

that yield almost sure convergence of play to pure Nash equilibria of the stage game in all games where such equilibria exist.
Theorem. There are NO uncoupled dynamics with 1-recall

$$\sigma_t^i \equiv f^i(a_{t-1}; u^i)$$

that yield almost sure convergence of play to pure Nash equilibria of the stage game in all games where such equilibria exist.

Hart and Mas-Colell, GEB 2006
Consider the following two-person game, which has a unique pure Nash equilibrium

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>1,0</td>
<td>0,1</td>
<td>1,0</td>
</tr>
<tr>
<td>R2</td>
<td>0,1</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td>R3</td>
<td>0,1</td>
<td>0,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>
Consider the following two-person game, which has a unique pure Nash equilibrium \((R3, C3)\)

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>1,0</td>
<td>0,1</td>
<td>1,0</td>
</tr>
<tr>
<td>R2</td>
<td>0,1</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td>R3</td>
<td>0,1</td>
<td>0,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>
Consider the following two-person game, which has a unique pure Nash equilibrium (R3,C3)

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>1,0</td>
<td>0,1</td>
<td>1,0</td>
</tr>
<tr>
<td>R2</td>
<td>0,1</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td>R3</td>
<td>0,1</td>
<td>0,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Assume *by way of contradiction* that we are given an uncoupled, 1-recall, stationary dynamic that yields almost sure convergence to pure Nash equilibria when these exist.
Proof

Suppose the play at time $t - 1$ is $(R_1, C_1)$
Proof

- Suppose the play at time $t - 1$ is $(R_1, C_1)$
- **ROWENA** is best replying at $(R_1, C_1)$

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>R1</strong></td>
<td>1,0</td>
<td>0,1</td>
<td>1,0</td>
</tr>
<tr>
<td><strong>R2</strong></td>
<td>0,1</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td><strong>R3</strong></td>
<td>0,1</td>
<td>0,1</td>
<td><strong>1,1</strong></td>
</tr>
</tbody>
</table>
Proof

Suppose the play at time $t-1$ is $(R1, C1)$

ROWENA is best replying at $(R1, C1)$

$\Rightarrow$ ROWENA will play $R1$ also at $t$
Proof

Suppose the play at time $t-1$ is $(R1,C1)$

ROWENA is best replying at $(R1,C1)$

$\implies$ ROWENA will play R1 also at $t$

Proof:

Change the payoff function of COLIN so that $(R1,C1)$ is the unique pure Nash eq.

```
    C1   C2   C3
R1  1,0  0,1  1,0
R2  0,1  1,0  1,0
R3  0,1  0,1  1,1
```
Suppose the play at time $t - 1$ is $(R_1, C_1)$

**Rowena** is best replying at $(R_1, C_1)$

$\Rightarrow$ **Rowena** will play $R_1$ also at $t$

**Proof:**

Change the payoff function of Colin so that $(R_1, C_1)$ is the unique pure Nash eq.

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>1,1</td>
<td>0,1</td>
<td>1,0</td>
</tr>
<tr>
<td>$R_2$</td>
<td>0,1</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td>$R_3$</td>
<td>0,1</td>
<td>0,1</td>
<td>1,0</td>
</tr>
</tbody>
</table>
Proof

- Suppose the play at time $t-1$ is $(R1,C1)$
- **ROWENA is best replying at** $(R1,C1)$
- $\implies$ **ROWENA will play** $R1$ **also at** $t$

*Proof:*

- Change the payoff function of COLIN so that $(R1,C1)$ is the unique pure Nash eq.
- In the new game, ROWENA **must** play $R1$ after $(R1,C1)$ (by 1-recall, stationarity, and a.s. convergence to the pure Nash eq.)
Proof

Suppose the play at time $t-1$ is $(R1,C1)$

**ROWENA is best replying at** $(R1,C1)$

$\implies$ **ROWENA will play R1 also at** $t$

*Proof:*

- Change the payoff function of **COLIN** so that $(R1,C1)$ is the unique pure Nash eq.

- In the new game, **ROWENA** must play R1 after $(R1,C1)$ (by 1-recall, stationarity, and a.s. convergence to the pure Nash eq.)

- By **uncoupledness**, the same holds in the original game
Proof

Suppose the play at time $t - 1$ is $(R_1, C_1)$

Rowena is best replying at $(R_1, C_1)$

$\Rightarrow$ Rowena will play $R_1$ also at $t$
Proof

• ROWENA is best replying at $t - 1$
• $\implies$ ROWENA will play the same action at $t$
Similarly for COLIN:

A player who is best replying cannot switch
Similarly for COLIN:

**A player who is best replying cannot switch**

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>R1</strong></td>
<td>1,0</td>
<td>0,1</td>
<td>1,0</td>
</tr>
<tr>
<td><strong>R2</strong></td>
<td>0,1</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td><strong>R3</strong></td>
<td>0,1</td>
<td>0,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>
Similarly for **COLIN**:

**A player who is best replying cannot switch**

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>1,0</td>
<td>0,1</td>
<td>1,0</td>
</tr>
<tr>
<td>R2</td>
<td>0,1</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td>R3</td>
<td>0,1</td>
<td>0,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>
Similarly for COLIN:

A player who is best replying cannot switch

<table>
<thead>
<tr>
<th></th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>1,0 ↔</td>
<td>0,1</td>
<td>1,0 ↔</td>
</tr>
<tr>
<td>R2</td>
<td>0,1</td>
<td>1,0 ↔</td>
<td>1,0 ↔</td>
</tr>
<tr>
<td>R3</td>
<td>0,1</td>
<td>0,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>

⇒ (R3,C3) cannot be reached
Similarly for COLIN:

A player who is best replying cannot switch

\[
\begin{array}{ccc}
\text{C1} & \text{C2} & \text{C3} \\
\hline
\text{R1} & 1,0 & 0,1 & 1,0 \\
\text{R2} & 0,1 & 1,0 & 1,0 \\
\text{R3} & 0,1 & 0,1 & 1,1 \\
\end{array}
\]

⇒ (R3, C3) cannot be reached
(unless we start there)
Theorem. **THERE EXIST** *uncoupled dynamics with 2-RECALL*

\[ \sigma_i^t \equiv f^i(a_{t-2}, a_{t-1}; u^i) \]

*that yield almost sure convergence of play to pure Nash equilibria of the stage game in every game where such equilibria exist.*
Define the strategy of each player $i$ as follows:
Possibility

Define the strategy of each player $i$ as follows:

**IF:**
- Everyone played the same in the previous two periods: $a_{t-2} = a_{t-1} = a$; and
- Player $i$ best replied: $a^i \in BR^i(a^{-i}; u^i)$

**THEN:** At $t$ player $i$ *plays $a^i$ again*: $a^i_t = a^i$
Define the strategy of each player $i$ as follows:

**IF:**

- Everyone played the same in the previous two periods: $a_{t-2} = a_{t-1} = a$; and
- Player $i$ best replied: $a^i \in BR^i(a^{-i}; u^i)$

**THEN:** At $t$ player $i$ *plays $a^i$ again*: $a^i_t = a^i$

**ELSE:** At $t$ player $i$ *randomizes uniformly over* $A^i$
"Good": 
"Good":

- simple
"Good":
  - simple

"Bad":

Possibility
"Good":
- simple

"Bad":
- exhaustive search
"Good":
  - simple

"Bad":
  - exhaustive search
  - all players must use it
"Good":

- simple

"Bad":

- exhaustive search
- all players must use it
- takes a long time
Dynamics

FACT

There are no general, natural dynamics leading to Nash equilibrium

"natural":
- adaptive
- simple and efficient:
  - computation
  - time
  - information
FACT

There are no general, natural dynamics leading to Nash equilibrium

"natural":
- adaptive
- simple and efficient:
  - computation
  - time
- information: uncoupledness ✓
FACT

There are no general, natural dynamics leading to Nash equilibrium

"natural":
- adaptive
- simple and efficient:
  - computation: finite recall ✓
  - time
  - information: uncoupledness ✓
FACT

There are no general, natural dynamics leading to Nash equilibrium

"natural":
- adaptive
- simple and efficient:
  - computation: finite recall ✓
  - time to reach equilibrium ?
  - information: uncoupledness ✓
HOW LONG TO EQUILIBRIUM?
HOW LONG TO EQUILIBRIUM?

Estimate the number of time periods it takes until a Nash equilibrium is reached
HOW LONG TO EQUILIBRIUM?

Estimate the number of time periods it takes until a Nash equilibrium is reached.

How?
HOW LONG TO EQUILIBRIUM?

Estimate the number of time periods it takes until a Nash equilibrium is reached

- How?
- An uncoupled dynamic
  \[ \approx \]
  A distributed computational procedure
HOW LONG TO EQUILIBRIUM?

Estimate the number of time periods it takes until a Nash equilibrium is reached

- How?
- An uncoupled dynamic
  \[ \approx \]
  A distributed computational procedure

⇒ COMMUNICATION COMPLEXITY
Distributed computational procedure
Communication Complexity

- **Distributed computational procedure**

  - **START**: Each participant has some private information
Communication Complexity

- **Distributed computational procedure**
  - **START**: Each participant has some private information
  - **COMMUNICATION**: Messages are transmitted between the participants
Distributed computational procedure

START: Each participant has some private information

COMMUNICATION: Messages are transmitted between the participants

END: All participants reach agreement on the result
**Distributed computational procedure**

**START:** Each participant has some private information

**COMMUNICATION:** Messages are transmitted between the participants

**END:** All participants reach agreement on the result
**Distributed computational procedure**

- **START:** Each participant has some private information

- **COMMUNICATION:** Messages are transmitted between the participants

- **END:** All participants reach agreement on the result
Distributed computational procedure

**START**: Each participant has some private information

**COMMUNICATION**: Messages are transmitted between the participants

**END**: All participants reach agreement on the result

**COMMUNICATION COMPLEXITY** = the minimal number of rounds needed
**Communication Complexity**

- **Distributed computational procedure**
  - **START**: Each participant has some private information
  - **COMMUNICATION**: Messages are transmitted between the participants
  - **END**: All participants reach agreement on the result

- **COMMUNICATION COMPLEXITY** = the minimal number of rounds needed

---

Yao 1979, Kushilevitz and Nisan 1997
How Long to Equilibrium
How Long to Equilibrium

Uncoupled dynamic leading to Nash equilibria
How Long to Equilibrium

- Uncoupled dynamic leading to Nash equilibria

- **START:** Each player knows his own payoff function

```plaintext
INPUTS
```
How Long to Equilibrium

- **Uncoupled dynamic leading to Nash equilibria**

- **START:** Each player knows his own payoff function

- **COMMUNICATION:** The actions played are commonly observed
**How Long to Equilibrium**

- **Uncoupled dynamic leading to Nash equilibria**

  - **START:** Each player knows his own payoff function
  - **COMMUNICATION:** The actions played are commonly observed
  - **END:** All players play a Nash equilibrium
How Long to Equilibrium

- **Uncoupled dynamic leading to Nash equilibria**

- **START:** Each player knows his own payoff function

- **COMMUNICATION:** The actions played are commonly observed

- **END:** All players play a Nash equilibrium

- **COMMUNICATION COMPLEXITY** = the minimal number of rounds needed
How Long to Equilibrium

Uncoupled dynamic leading to Nash equilibria

START: Each player knows his own payoff function

COMMUNICATION: The actions played are commonly observed

END: All players play a Nash equilibrium

COMMUNICATION COMPLEXITY = the minimal number of rounds needed

Conitzer and Sandholm 2004
An uncoupled dynamic leading to Nash equilibria is \textsc{Time-Efficient} if
How Long to Equilibrium

An uncoupled dynamic leading to Nash equilibria is **TIME-EFFICIENT** if the **TIME IT TAKES**
An uncoupled dynamic leading to Nash equilibria is **TIME-EFFICIENT** if the **TIME IT TAKES** is **POLYNOMIAL** in the number of players.
An uncoupled dynamic leading to Nash equilibria is **TIME-EFFICIENT** if the **TIME IT TAKES** is **POLYNOMIAL** in the number of players (rather than: *exponential*)
How Long to Equilibrium

- An uncoupled dynamic leading to Nash equilibria is **TIME-EFFICIENT** if the **TIME IT TAKES** is **POLYNOMIAL** in the number of players (rather than: exponential)

**Theorem.** There are **NO TIME-EFFICIENT** uncoupled dynamics that reach a pure Nash equilibrium in all games where such equilibria exist.
An uncoupled dynamic leading to Nash equilibria is **TIME-EFFICIENT** if the **TIME IT TAKES** is **POLYNOMIAL** in the number of players (rather than: *exponential*).

**Theorem.** *There are NO TIME-EFFICIENT uncoupled dynamics that reach a pure Nash equilibrium in all games where such equilibria exist.*

*Hart and Mansour, GEB 2009*
An uncoupled dynamic leading to Nash equilibria is **TIME-EFFICIENT** if the **TIME IT TAKES** is **POLYNOMIAL** in the number of players (rather than: exponential)

**Theorem.** *There are NO TIME-EFFICIENT uncoupled dynamics that reach a pure Nash equilibrium in all games where such equilibria exist.*

In fact: exponential, like exhaustive search

---

Hart and Mansour, GEB 2009
How Long to Equilibrium

Intuition:
How Long to Equilibrium

Intuition:

- different games have different equilibria
How Long to Equilibrium

Intuition:

- different games have different equilibria
- the dynamic procedure must distinguish between them
Intuition:

- different games have different equilibria
- the dynamic procedure must distinguish between them
- no single player can do so by himself
FACT

There are NO general, natural dynamics leading to Nash equilibrium
FACT

There are NO general, natural dynamics leading to Nash equilibrium

RESULT

There CANNOT BE general, natural dynamics leading to Nash equilibrium
RESULT

There CANNOT BE general, natural dynamics leading to Nash equilibrium
RESULT

There CANNOT BE general, natural dynamics leading to Nash equilibrium

Perhaps we are asking too much?
RESULT

There CANNOT BE general, natural dynamics leading to Nash equilibrium

Perhaps we are asking too much?
For instance, the size of the data (the payoff functions) is exponential rather than polynomial in the number of players.
CORRELATED EQUILIBRIUM:

Nash equilibrium when players receive payoff-irrelevant information before playing the game

Aumann, JME 1974
A Correlated Equilibrium is a Nash equilibrium when players receive payoff-irrelevant signals before playing the game.
A **Correlated Equilibrium** is a Nash equilibrium when players receive payoff-irrelevant signals before playing the game.

- **Examples:**
A **Correlated Equilibrium** is a Nash equilibrium when players receive payoff-irrelevant signals before playing the game.

- **Examples:**
  - Independent signals
A **Correlated Equilibrium** is a Nash equilibrium when players receive payoff-irrelevant signals before playing the game.

**Examples:**

- Independent signals $\iff$ Nash equilibrium
A **Correlated Equilibrium** is a Nash equilibrium when players receive payoff-irrelevant signals before playing the game.

- **Examples:**
  - Independent signals $\iff$ Nash equilibrium
  - Public signals ("sunspots")
A **Correlated Equilibrium** is a Nash equilibrium when players receive payoff-irrelevant signals before playing the game.

**Examples:**
- Independent signals $\iff$ Nash equilibrium
- Public signals ("sunspots") $\iff$ convex combinations of Nash equilibria
A Correlated Equilibrium is a Nash equilibrium when players receive payoff-irrelevant signals before playing the game.

- **Examples:**
  - Independent signals ⇔ Nash equilibrium
  - Public signals (“sunspots”) ⇔ convex combinations of Nash equilibria
  - Butterflies play the Chicken Game (“Speckled Wood” *Pararge aegeria*)
"Chicken" game

<table>
<thead>
<tr>
<th></th>
<th>LEAVE</th>
<th>STAY</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEAVE</td>
<td>5, 5</td>
<td>3, 6</td>
</tr>
<tr>
<td>STAY</td>
<td>6, 3</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
"Chicken" game

<table>
<thead>
<tr>
<th></th>
<th>LEAVE</th>
<th>STAY</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEAVE</td>
<td>5, 5</td>
<td>3, 6</td>
</tr>
<tr>
<td>STAY</td>
<td>6, 3</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

a Nash equilibrium
"Chicken" game

<table>
<thead>
<tr>
<th></th>
<th>LEAVE</th>
<th>STAY</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEAVE</td>
<td>5, 5</td>
<td>3, 6</td>
</tr>
<tr>
<td>STAY</td>
<td>6, 3</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

another Nash equilibrium
"Chicken" game

<table>
<thead>
<tr>
<th></th>
<th>LEAVE</th>
<th>STAY</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEAVE</td>
<td>5, 5</td>
<td>3, 6</td>
</tr>
<tr>
<td>STAY</td>
<td>6, 3</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

a (publicly) correlated equilibrium

\[ \begin{array}{cccc}
\text{LEAVE} & \text{STAY} \\
\hline
5, 5 & 3, 6 \\
6, 3 & 0, 0 \\
\end{array} \]
"Chicken" game

<table>
<thead>
<tr>
<th></th>
<th>LEAVE</th>
<th>STAY</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEAVE</td>
<td>5, 5</td>
<td>3, 6</td>
</tr>
<tr>
<td>STAY</td>
<td>6, 3</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>S</td>
<td>1/3</td>
<td>0</td>
</tr>
</tbody>
</table>

another correlated equilibrium

- after signal L play LEAVE
- after signal S play STAY
A Correlated Equilibrium is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974)

**Examples:**
- Independent signals $\iff$ Nash equilibrium
- Public signals (“sunspots”) $\iff$ convex combinations of Nash equilibria
- Butterflies play the Chicken Game (“Speckled Wood” *Pararge aegeria*)
A **Correlated Equilibrium** is a Nash equilibrium when the players receive payoff-irrelevant signals before playing the game (Aumann 1974)

**Examples:**
- Independent signals ⇔ Nash equilibrium
- Public signals ("sunspots") ⇔ convex combinations of Nash equilibria
- Butterflies play the Chicken Game ("Speckled Wood" *Pararge aegeria*)
- Boston Celtics’ front line
Correlated Equilibrium

Signals (public, correlated) are **unavoidable**
Correlated Equilibrium

- Signals (public, correlated) are **unavoidable**
- **Common Knowledge** of **Rationality** ⇔ **Correlated Equilibrium** (Aumann 1987)
Correlated Equilibrium

- Signals (public, correlated) are **unavoidable**
- **Common Knowledge of Rationality** $\iff$ **Correlated Equilibrium** (Aumann 1987)

A joint distribution $\pi$ is a correlated equilibrium $\iff$

$$\sum_{s^{-i}} u(j, s^{-i}) \pi(j, s^{-i}) \geq \sum_{s^{-i}} u(k, s^{-i}) \pi(j, s^{-i})$$

for all $i \in N$ and all $j, k \in S^i$
RESULT

THERE EXIST *general, natural dynamics* leading to *CORRELATED EQUILIBRIA*
RESULT

THERE EXIST general, natural dynamics leading to CORRELATED EQUILIBRIA

Regret Matching

Hart and Mas-Colell, Ec’ca 2000
RESULT

THERE EXIST *general, natural dynamics* *leading to* *CORRELATED EQUILIBRIA*

- *Regret Matching*
- General regret-based dynamics

_Hart and Mas-Colell, Ec’ca 2000, JET 2001_
"REGRET": the increase in past payoff, if any, if a different action would have been used
Regret Matching

"REGRET": the increase in past payoff, if any, if a different action would have been used.

"MATCHING": switching to a different action with a probability that is proportional to the regret for that action.
THERE EXIST general, natural dynamics leading to CORRELATED EQUILIBRIA
THERE EXIST \textit{general}, \textit{natural dynamics} leading to \textbf{CORRELATED EQUILIBRIA}

- "\textit{general}": in all games
THERE EXIST general, natural dynamics leading to CORRELATED EQUILIBRIA

- "general": in all games
- "natural": 
THERE EXIST general, natural dynamics leading to CORRELATED EQUILIBRIA

- "general": in all games
- "natural": adaptive (also: close to "behavioral")
THERE EXIST *general, natural dynamics* leading to **CORRELATED EQUILIBRIA**

- "*general*": in all games
- "*natural*":
  - adaptive (also: close to "behavioral")
  - simple and efficient:
    - computation, time, information
THERE EXIST general, natural dynamics leading to CORRELATED EQUILIBRIA

- "general": in all games
- "natural":
  - adaptive (also: close to "behavioral")
  - simple and efficient: computation, time, information
- "leading to correlated equilibria": statistics of play become close to CORRELATED EQUILIBRIA
Dynamics and Equilibrium

Nash equilibrium: a fixed-point of a non-linear map
**Dynamics and Equilibrium**

- **Nash equilibrium**: a *fixed-point* of a non-linear map

- **Correlated equilibrium**: a solution of finitely many *linear inequalities*
Nash equilibrium: a fixed-point of a non-linear map

Correlated equilibrium: a solution of finitely many linear inequalities

Set-valued fixed-point (curb sets)?
"LAW OF CONSERVATION OF COORDINATION":
"Law of Conservation of Coordination":

There must be some Coordination —
"LAW OF CONSERVATION OF COORDINATION":

There must be some COORDINATION —

either in the EQUILIBRIUM notion,
"Law of Conservation of Coordination":

There must be some coordination —
either in the equilibrium notion,
or in the dynamic
A. Demarcate the BORDER between
A. **Demarcate the BORDER** between classes of dynamics where convergence to equilibria **CAN** be obtained
A. Demarcate the **BORDER** between

- classes of dynamics where convergence to equilibria **CAN** be obtained, and

- classes of dynamics where convergence to equilibria **CANNOT** be obtained
The "Program"

A. Demarcate the BORDER between

- classes of dynamics where convergence to equilibria CAN be obtained, and
- classes of dynamics where convergence to equilibria CANNOT be obtained

B. Find NATURAL dynamics for the various equilibrium concepts
Dynamics and Equilibrium
Dynamics and Equilibrium
Dynamics and Equilibrium
Dynamics and Equilibrium
My Game Theory

INSIGHTS, IDEAS, CONCEPTS
My Game Theory

INSIGHTS, IDEAS, CONCEPTS

FORMAL MODELS
My Game Theory

INSIGHTS, IDEAS, CONCEPTS

FORMAL MODELS
My Game Theory

INSIGHTS,
IDEAS,
CONCEPTS

FORMAL
MODELS
My Game Theory

- INSIGHTS,
- IDEAS,
- CONCEPTS

- FORMAL MODELS
My Game Theory

INSIGHTS,
IDEAS,
CONCEPTS

FORMAL
MODELS
My Game Theory

INSIGHTS, IDEAS, CONCEPTS

FORMAL MODELS