Classical Cooperative Theory I: Core-Like Concepts

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1. Introduction

Pure bargaining games discussed in the previous two lectures are a special case of n-person cooperative games. In the general setup coalitions other than the grand coalition matter as well. The primitive is the *coalitional form* (or, "coalitional function", or "characteristic form"). The primitive can represent many different things, e.g., a simple voting game where we associate to a winning coalition the worth 1 and to a losing coalition the worth 0, or an economic market that generates a cooperative game. Von Neumann and Morgenstern (1944) suggested that one should look at what a coalition can guarantee (a kind of a constant-sum game between a coalition and its complement); however, that might not always be appropriate. Shapley and Shubik introduced the notion of a *C-game* (see Shubik (1982)): it is a game where there is no doubt on how to define the worth of a coalition. This happens, for example, in exchange economies where a coalition can reallocate its own resources, *independent* of what the complement does.

We assume we are given a coalitional function. Let N denote the set of players; a subset $S \subset N$ is called a *coalition*; $V(S)$ is the set of feasible outcomes for S.

How is an outcome defined? Assuming that some underlying utility functions for the players are specified, one can represent outcomes by the players' utilities. We thus use a payoff vector $a=(a^i)_{i \in S}$ in $\Re^S$ to represent an outcome, where a^i is player i'th utility of the outcome. So $V(S) \subset \Re^S$. Usually there are some assumptions made on the set $V(S)$; e.g., comprehensive, closed, convex, etc.

There are two special classes of games:

1) *Pure bargaining* games (PB): In these games only the grand coalition matters. Here $V(S)=\{x \in \Re^S$ such that $x^i \leq 0$ for all $i \in S\}$ for all[2] $S \neq N$.

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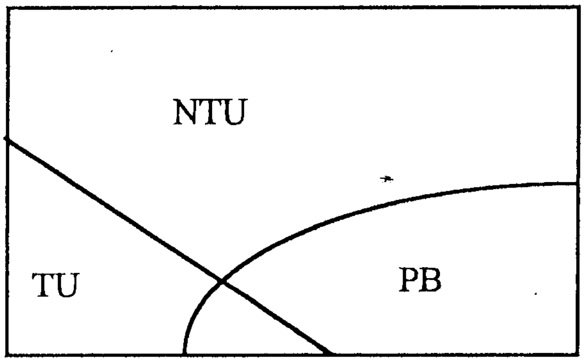
[1] Lecture notes written by Yossi Feinberg.

[2] Sometimes this is relaxed to: $(0,\ldots,0) \in \mathrm{bd}V(S)$ for all $S \neq N$, where "bd" stands for boundary.

2) *Transferable utility* games (TU): (used to be called "games with side payments"). Here one number represents what a coalition can get, and the members of the coalition can arbitrarily divide this amount among themselves. So a TU game is of the form $V(S)=\{x\in\Re^S$ such that $\sum_{i\in S} x^i\leq v(S)\}$ where $v(S)$ is a number for all $S\subset N$. Geometrically these sets are half-spaces with normal vector (1,...,1).

The general games are usually referred as games with *non-transferable utility,* or *NTU-games* for short. The following diagram shows the relationship between the different classes of games.



2. Solution Concepts

We will distinguish between two approaches to solution concepts (though the distinction is not always clear cut):

D - definition, description, discussion.

A - axiomatization.

Obviously there are other approaches, e.g., noncooperative, evolutionary, etc.

The D-approach stands for various formal or informal arguments, on how the solution has to look like. In the A-approach, one puts down a set of axioms and gets as a result that these axioms uniquely characterize the solution concept. Most solution concepts started out with the D-approach and only later where axiomatized; the Shapley Value started out with the A-approach.

A p.v. (*payoff vector*) is a vector $x\in\Re^N$. It is *feasible* if $x\in V(N)$, *efficient* (or Pareto optimal) if $x\in \mathrm{bd}V(N)$, and *individually rational* if $x^i\notin \mathrm{int}V(\{i\})$. The set $X:=\{x|\ x$ is an efficient and individually rational p.v.$\}$ is called the set of *imputations*. For simplicity we assume that the set of imputations is always non-empty. Thus in the TU case we consider only games which satisfy $v(N)\geq \sum_{i\in N} v(i)$. A solution concept associates payoff vectors (outcomes) with each game.

3. The D-Approach

3.1 Core

The idea of the Core is to look at those payoff vectors which no coalition can improve upon. Let x be a feasible p.v.; a coalition $\emptyset \neq S \subset N$ *can improve upon* x if there is a feasible outcome y for S ($y \in V(S)$) which is better for S, that is $y^i > x^i$ for all i in S (everyone in S must agree that y is better than x); we write $y \succ_S x$. We say that y *dominates* x if there exists a coalition S such that $y \succ_S x$. The *Core* is thus defined as the set of imputations that are not dominated by any p.v. .

In the NTU case a feasible p.v. x satisfies: $x \in \text{Core} \Leftrightarrow x^S \notin \text{int } V(S)$ for all[3] S.

In the TU case a feasible p.v. x satisfies: $x \in \text{Core} \Leftrightarrow x(S) \geq v(S)$ for all[4] S.

A first question one poses is the non-emptiness of the Core. This is connected with superadditivity which states that joining two coalitions may only increase their possibilities. The non-emptiness of the core is implied by *balancedness*, which is a generalization of superadditivity. Certain classes of games such as market games turn out to have a non-empty core (under general conditions). In the case of market games, the competitive (Walrasian) equilibrium is always in the Core. On the other hand, unless there is a veto player, voting games have an empty Core.

3.2 von Neumann and Morgenstern Stable Set

Recall that the Core is the set of all feasible p.v. that are not dominated by any p.v. .

Consider now the following definition of a solution:

The "Shmore"[5] is the set of all feasible p.v. that are not dominated by any p.v. in the Shmore.

It turns out this concept is indeed well defined. The idea behind it is that the set of "good" p.v. is to be compared against "good" p.v. . The definition of the Shmore can be rewritten by $x \in \text{Shmore} \Leftrightarrow y \not\succ x$ for all $y \in \text{Shmore}$. This can be further restated as follows: Let K=Shmore, then

1) $x, y \in K \Rightarrow x \not\succ y$;

2) $y \notin K \Rightarrow$ there exists $x \in K$ such that $x \succ y$.

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[3] We write x^S for the projection of x on $\Re^S$, i.e., $x^S = (x^i)_{i \in S}$.

[4] Here we define $x(S) = \sum_{i \in S} x^i$.

[5] This (temporary) name is due to R. J. Aumann.

A set K of efficient p.v. that satisfies these two conditions is called a *von Neumann and Morgenstern Stable Set*. Note that the basic concept here is a set concept, unlike the Core where the payoff vector's properties alone determine if it is in the solution. The Stable Set becomes a "standard of behavior" in the sense that if everyone believes that the solution is in K it will indeed be in K. Note that there may be more than one stable set in a game. Trivially, all von Neumann and Morgenstern Stable Sets contain the core.

There are games for which there are no Stable Sets. The first example was found by Lucas (1968) and was a 13 player game. Unfortunately even in simple cases it is difficult to calculate all (or even one of) the Stable Sets of a game. But finding this solution is very rewarding, it gives a lot of insight. For example, in voting games, where minimal winning coalitions seem important, it turns out that Stable Sets predict actually the formation of minimal blocking coalitions.

3.3 Bargaining Sets

The previous solution concepts are based on the idea that, given a p.v. , some coalitions or players may object to it (using other feasible p.v.). Along this line one can define a *counter objection* by objecting to the p.v. used for the original objection. Then follows the notion of *justified objections*, defined as objections that have no counter objection. Using these definitions one can define the Bargaining Set as the set of efficient p.v. for which there is no justified objection. This solution has many variants and was first conceived by Aumann and Maschler (1964); see Davis and Maschler (1963,1967), and also Mas-Colell (1989) for a new approach. The work on Bargaining Sets led to the following solution concept.

3.4 Nucleolus

This is a one point solution defined for the TU case. There are various suggestions for the generalization of the Nucleolus to the NTU case, but this is not yet settled.

Behind the notion of the Nucleolus is the following interpretation. Given a p.v. x each coalition S looks at $v(S)$-$x(S)$; this number represents the "complaint" of the coalition (it could be positive or negative). The higher the complaint the more loudly the coalition protests against x. Thus we want to minimize complaints under the "budget constraint" (the feasibility of x). We do so starting with the maximal complaint, i.e., we look at $Min_{p.v.\,x}\{Max_{S\subset N}(v(S)-x(S))\}$. Then we minimize the next highest complaint when considering only p.v. which minimized the highest complaint, and so on. What we get is the lexicographic minimum of all complaints. It turns out that we are left with a unique p.v. which is the Nucleolus. This solution concept is due to Schmeidler

(1969). The Nucleolus was applied to various problems, such as an airport landing fees problem in which airlines needed to share the cost of using runways (each coalition of airlines needing its distinct minimal runway length). We remark that when the Core is non-empty there is a feasible p.v. for which all complaints are non positive. Thus, in this case the complaints for the Nucleolus are non positive as well and we have that the Nucleolus is in the Core (it is moreover a special point in the Core, a kind of a symmetry center).

We have presented three kinds of solution concepts: one is a one-point solution (the Nucleolus); the second is a set of points (the Core), and the third is several sets of points (the Stable Sets).

4. The A-Approach

We move now to the second point of view on solution concepts, i.e., the axiomatic approach. One can always use the definition of a concept as its axiomatization, but obviously we would like to have more basic axioms with an intuitive meaning that characterize our concept. It should be noted that these characterizations are comparatively new.

All these axiomatizations have in common the *Consistency axiom* (also called the *Reduced Game Property*). Consistency is based on the following idea: Assume we have a game and its solution. Suppose that a certain set of players agree to the solution. The *reduced game* is the game played by the remaining players, on the remaining payoffs. Consistency requires that the solution of the reduced game be identical to the solution of the original game.

Formally, let (N,V) be an NTU game. Let x be a p.v. and $T\subset N$ be a coalition. We define the game (T,V^*) (where V^* depends on both x and T) by[6] $V^*(T)=\{y^T|(y^T,x^{T^c})\in V(N)\}$; i.e., we give x^{T^c} to the players in T^c and consider what we can give to the players in T so that the whole vector is feasible in the original game. For strict sub-coalitions $S\subset T$ $(S\neq T)$ we define $V^*(S)=\cup_{Q\subset T^c}\{y^S|(y^S,x^Q)\in V(S\cup Q)\}$; that is, we consider all sub-coalitions Q of T^c as those which can complement the members of S and create a feasible p.v. for S through a feasible p.v. for $S\cup Q$ in the original game.

The consistency or reduced game property states:

CONS: If x is a solution of (N,V) then x^T is a solution of (T,V^*) for all T.

It turns out that this definition of consistency yields many results.

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6 $T^c=N\setminus T$ is the complement of T.

4.1 The TU Case

A solution associates a set of feasible p.v. for each game, i.e., it is a mapping $(N,v)\to\sigma(N,v)$ where $\sigma(N,v)$ is a set of feasible p.v. . We shall consider the set of games with non-empty Core. The axioms are:

NE (non-emptiness): $\sigma(N,v)\neq\emptyset$.

IR (individual rationality): In any p.v. in the solution every player gets at least what the game guarantees him, i.e., $x\in\sigma(N,v)$ implies $x^i\geq v(i)$ for all i.

CONS : (as above).

SUPA (superadditivity): $\sigma(N,v)+\sigma(N,w)\subset\sigma(N,v+w)$ where the summation here is a set summation. Note that the set of players is always the same.

SIVA (single valuedness): $|\sigma(N,v)|=1$.

AN (anonymity): If the games (N,v) and (N',v') are isomorphic, i.e., there exists a one-to-one mapping $\Pi:N\to N'$ such that $v(S)=v'(\Pi S)$ for all S, then $\Pi\sigma(N,v)=\sigma(N',v')$ (Π is a "relabeling of the players").

INV (TU invariance): For all $a>0$ and $b\in\Re^N$, if $w(S)=av(S)+\sum_{i\in S}b^i$ for all S then $\sigma(N,w)=a\sigma(N,v)+b$.

Theorem (Peleg (1986)): The Core is the unique solution concept satisfying NE,IR,CONS,SUPA.

(Note that this result requires a world with 3 players at least.)

Theorem (Sobolev (1975)): The PreNucleolus (defined in the same way as the Nucleolus, with respect to all efficient but not necessarily individually rational p.v. x) is the unique solution concept satisfying SIVA,AN,INV,CONS.

(This result requires a world with an infinite number of players.)

The axiomatization of the Stable Sets is an open problem.

4.2 The NTU Case

Theorem (Peleg (1985)): The Core is the unique solution concept satisfying NE,IR,CONS (under some regularity conditions on the games considered). (Note that this result requires a world with an infinite number of players.)

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