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Formation of Cartels in Large Markets*

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The von Neumann–Morgenstern concept of solution is applied to nonatomic pure exchange markets consisting of a finite number of different types of trader, initially owning disjoint sets of goods. It is proved that if all traders of each type form a cartel and behave like a single (atomic) trader (their "representative"), then one gets solutions of the original market from those of the finite market of the representatives, and *all* solutions are obtained in this manner—i.e., the market *must* organize itself in cartels, and this will indeed be stable.

1. INTRODUCTION

Von Neumann and Morgenstern solutions [14] are meant to describe stable forms of social organization. In specific instances, they can often be interpreted to indicate which coalitions in a society will "form" (e.g., von Neumann–Morgenstern [14 especially Chaps V and X], Shapley [15], Bott [3], Hart [10]).

In this paper we apply the notion of von Neumann–Morgenstern solution to a Walrasian pure exchange economy[1] (or market), with a nonatomic continuum of traders, i.e., with "many" traders each of whom is initially endowed with a negligible proportion of the total resources in the market. In this market we assume that there is a finite number of "types" of trader, where the type of a trader is defined by his initial resources and his preferences. Moreover, we assume that each of the commodities is initially held by *only one* type of trader.

A solution $\mathscr{A}$ of such a market is called "symmetric" if for each allocation $\mathbf{x}$ in $\mathscr{A}$, any allocation obtained from $\mathbf{x}$ by permuting (in an arbitrary manner) the players *within* each type, is still in $\mathscr{A}$.

With the given nonatomic market one can associate a finite market, obtained by considering each type as a single player.

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[1] Without transferable utility; cf. Aumann–Peleg [2].

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Given a solution $\mathscr{B}$ of the finite market, one can obtain a set $\mathscr{B}_{NA}$ of allocations $\mathbf{x}$ in the original nonatomic market by associating with each allocation ξ in $\mathscr{B}$ the allocation $\mathbf{x}$ obtained by dividing the amount that ξ assigns to each type i equally between all traders of type i. Our main result asserts that $\mathscr{B}_{NA}$ is a solution of the nonatomic market, and that *all* symmetric solutions of the nonatomic market are obtained in this way.

This result can be interpreted as follows. Each of the traders in the finite market is thought of as consisting of the set of *all* traders of a given type in the original nonatomic market. Each such set appoints a representative to bargain for it; what is obtained by each of the representatives is then divided equally between all the traders he represents. Conceptually, therefore, the finite market is obtained from the nonatomic one by restricting all the traders of a given type to act in concert.

The main result makes two assertions: first, that this kind of restriction constitutes a stable form of organization for society as a whole; second, that if we look at forms of organization of society that do not discriminate between traders of the same type, then society *must* organize itself in this way. This means that if we look at society from a purely economic viewpoint—i.e., exluding discrimination that is practiced on noneconomic grounds—and if the various types initially have corners on different (and nonoverlapping) sets of goods, then the types *must necessarily* organize themselves into cartels. Moreover, these cartels will then act exactly as if they were in a (relatively) small market, whose traders are the cartels themselves rather than their individual members.

It is interesting to compare this behavior of the von Neumann–Morgenstern solutions with that of the core. For example, in a market with two commodities and two types of traders, the core in the nonatomic case consists of the competitive equilibrium only (see [1]), whereas in the finite case (i.e., two traders), it is exactly Edgeworth's [7] "contract curve"; the symmetric solution, on the other hand, is the contract curve in both cases.

It might be remarked that the core approach to the study of cartels (or "syndicates," "oligopolists") is fundamentally different from that of this paper. In the core approach, one assumes *a priori* that society has organized itself in a certain way (i.e., that certain cartels or syndicates have formed), and then studies the properties of the allocations in the core given this form of organization—see Shitovitz [17], Jaskold-Gabszewicz and Mertens [12], Hansen and Jaskold–Gabszewicz [9], Jaskold-Gabszewicz and Drèze [11], Drèze, Jaskold-Gabszewicz, Schmeidler, and Vind [6]. Our approach, on the other hand, makes no *a priori* assumption on the way society organizes itself: what we are studying are the solutions of the *nonatomic* (i.e., unorganized) market. The mode of organization

of society (i.e., which cartels "form") is an output of our approach, rather than being an input as in the core approach.

In Section 2 we present the mathematical model and state exactly our results, which are proved in Section 3. Section 4 is devoted to some remarks concerning our assumptions, and includes a number of counterexamples whenever these are not satisfied.

2. Mathematical Model and Statement of the Results

We shall be working in the Euclidean space R^m (m is the number of the commodities in the market). Superscripts will be used exclusively to denote coordinates for vectors in R^m. Following standard practice, for vectors x and y in R^m we write $x > y$ to mean $x^k > y^k$ for all k; $x \geqq y$ to mean $x^k \geqq y^k$ for all k; and $x \geqslant y$ to mean $x \geqq y$ but $x \neq y$.[2] The integral of a vector function is to be taken as the vector of the integrals of the components. The symbol $\setminus$ will be used for set-theoretic subtraction, whereas $-$ will be reserved for ordinary algebraic subtraction. Ω will denote the nonnegative octant of R^m.

A *commodity bundle* x is a point in Ω.

The *nonatomic market* (NA-market) consists of a measure space $(T, \mathscr{C}, \mu)$, where T is the set of *traders*, $\mathscr{C}$ is a σ field of subsets of T (a member of $\mathscr{C}$ being called a *coalition*), and μ is a nonatomic measure on $\mathscr{C}$. From now on, the terms "measure," "measurable," "integral," "integrable," are to be understood in the sense of the measure μ. A *null* set is a set of measure zero. *Null sets of traders are always ignored throughout this paper*; thus statements asserted for "all traders" are to be understood to hold for all traders except, possibly, for a null set of traders.

An *assignment* (of commodity bundles to traders) is an integrable function $\mathbf{x}$ from T to Ω. There is a fixed *initial assignment* $\mathbf{w}$. An *allocation* is an assignment $\mathbf{x}$ such that[3] $\int_T \mathbf{x} = \int_T \mathbf{w}$.

For each trader t there is defined on Ω a relation $\succ_t$, called *t preference*. This relation is assumed to be irreflexive, transitive, and complete. We write $x \sim_t y$ (*t indifference*) when neither $x \succ_t y$ nor $y \succ_t x$ holds.

The following assumptions on the preference relations are made:

(2.1) *Desirability* (of the commodities): $x \geqslant y$ implies $x \succ_t y$.

(2.2) *Continuity* (in the commodities): For each $y \in \Omega$, the sets $\{x \mid x \succ_t y\}$ and $\{x \mid y \succ_t x\}$ are open (relative to Ω).

[2] When x and y are scalars, we use both $x \geqslant y$ and $x \geqq y$ to mean $x > y$ or $x = y$.

[3] $\int_T \mathbf{x}$ is to be understood as $\int_T \mathbf{x}(t)\mu(dt)$, and the same for $\int_S \mathbf{x}$ where $S \in \mathscr{C}$.

(2.3) *Measurability*: If $\mathbf{x}$ and $\mathbf{y}$ are assignments, then the set $\{t \mid \mathbf{x}(t) \succ_t \mathbf{y}(t)\}$ is measurable.

(2.4) *Strong convexity*: $x \succcurlyeq_t y$ and $x \neq y$ imply $\alpha x + (1 - \alpha) y \succ_t x$ for all $0 < \alpha < 1$.

This is the usual model of a nonatomic market (see, e.g., Aumann [1]). The first three assumptions (desirability, continuity, and measurability) are almost always made on the preference relations. Convexity is also often assumed, but in forms weaker than this "strong convexity." (For a discussion on the convexity assumptions on preferences, see Debreu [4, Section 4.7]; the Debreu–Scarf [5] limit theorem on the cores was obtained also under this assumption, and the same is true for other results, e.g. [11].)

Our particular nonatomic game is characterized by some more assumptions. Two traders are of the same *type* if they have identical initial assignments and identical preference relations. We assume that

(2.5) the number of different types of traders in T is finite.

Let n be this number, and let T_i (for $1 \leqslant i \leqslant n$) be the coalition of all traders of type i. Without loss of generality we can make the following normalization[4]:

$$\mu(T_i) = 1 \qquad \text{for all } 1 \leqslant i \leqslant n. \tag{2.6}$$

The preference relation of the traders of type i will also be denoted $\succ_i$, and their initial assignment ω_i [i.e., $\mathbf{w}(t) = \omega_i$ for all $t \in T_i$].

One last but essential assumption is made on the initial assignment:

(2.7) The sets $M_i = \{k \mid \omega_i^k > 0\}$ for all $1 \leqslant i \leqslant n$ are disjoint.

The meaning of this assumption is that *each commodity is initially owned by only one type of trader.*[5]

Now we define the *finite market* (F-market) derived from the above nonatomic market. It will consist of n *traders*, their set $\{1, 2,..., n\}$ being denoted by N. An *assignment* ξ is a function from N to Ω; $\xi(i)$ will be denoted by ξ_i. The *initial assignment* is $\boldsymbol{\omega} = (\omega_i)_{i \in N}$. An *allocation* ξ is an assignment such that $\sum_{i=1}^n \xi_i = \sum_{i=1}^n \omega_i$. For each trader i, his preference relation is $\succ_i$.

An allocation $\mathbf{x}$ (or ξ) is *individually rational* (i.r.) if $\mathbf{x}(t) \succcurlyeq_t \mathbf{w}(t)$ for all $t \in T$ (or $\xi_i \succcurlyeq_i \omega_i$ for all $i \in N$).

[4] For example, by taking $[1/\mu(T_i)]\,\mathbf{x}(t)$ instead of $\mathbf{x}(t)$ (for $t \in T_i$) for any assignment (recall that assignments are *density* functions).

[5] A similar assumption is made in Ref. [9]: ω_i is the ith unit vector.

An NA allocation[6] is called *symmetric* if it assigns identical bundles to traders of the same type; i.e., for each i there is ξ_i in Ω such that

$$\mathbf{x}(t) = \xi_i \qquad \text{for all } t \in T_i\,. \tag{2.8}$$

It is obvious that $\boldsymbol{\xi} = (\xi_i)_{i \in N}$ defined by (2.8) for the symmetric NA allocation $\mathbf{x}$ is an F allocation, and the converse is also true: to every F allocation $\boldsymbol{\xi}$ corresponds a symmetric NA allocation $\mathbf{x}$ defined by (2.8).

For each set $\mathscr{A}$ of symmetric NA allocations, let $\mathscr{A}_{\mathrm{F}}$ denote the set of corresponding F allocations. If $\mathscr{B}$ is a set of F allocations, we will denote by $\mathscr{B}_{\mathrm{NA}}$ the set of corresponding symmetric NA allocations. That is,

$$\mathscr{B} = \mathscr{A}_{\mathrm{F}} \qquad \text{iff} \quad \mathscr{A} = \mathscr{B}_{\mathrm{NA}}\,.$$

We will now define the concepts of "domination" and "solution."

Let $\mathbf{x}$ and $\mathbf{y}$ be two given NA allocations. Then $\mathbf{x}$ *dominates* $\mathbf{y}$ *via* the nonempty coalition S (written $\mathbf{x} \succ_S \mathbf{y}$) if

$$\mathbf{x}(t) \underset{t}{\succ} \mathbf{y}(t) \qquad \text{for all } t \in S \tag{2.9}$$

and

$$\int_S \mathbf{x} \leqq \int_S \mathbf{w} \qquad \text{(effectiveness).} \tag{2.10}$$

Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be two given F allocations. Then $\boldsymbol{\xi}$ *dominates* $\boldsymbol{\eta}$ *via* the nonempty set $I \subset N$ (written $\boldsymbol{\xi} \succ_I \boldsymbol{\eta}$) if

$$\xi_i \underset{i}{\succ} \eta_i \qquad \text{for all } i \in I \tag{2.9'}$$

and

$$\sum_{i \in I} \xi_i \leqq \sum_{i \in I} \omega_i \qquad \text{(effectiveness).} \tag{2.10'}$$

If there is some S (or I) such that $\mathbf{x} \succ_S \mathbf{y}$ (or $\boldsymbol{\xi} \succ_I \boldsymbol{\eta}$), then $\mathbf{x}$ (or $\boldsymbol{\xi}$) dominates $\mathbf{y}$ (or $\boldsymbol{\eta}$), and we write $\mathbf{x} \succ \mathbf{y}$ (or $\boldsymbol{\xi} \succ \boldsymbol{\eta}$).

A von Neumann–Morgenstern *solution* [14] is a set $\mathscr{A}$ of i.r. allocations which is internally consistent (i.e., no two members of $\mathscr{A}$ dominate each other), and extradominative (i.e., each i.r. allocation not in $\mathscr{A}$ is dominated by some member of $\mathscr{A}$).

A *permutation* of the traders in the NA market is a one–one μ-measure preserving function π from T onto T, measurable in both directions, such that for all t in T, πt and t are of the same type (i.e., belong to T_i for the same i).

[6] That is, an allocation for the NA market, to distinguish from the allocations for the F market, which will be called F allocations.

A set $\mathscr{A}$ of NA allocations is *symmetric* if for each permutation π and for each $\mathbf{x} \in \mathscr{A}$, the allocation $\pi\mathbf{x}$ (defined by $(\pi\mathbf{x})(t) = \mathbf{x}(\pi t)$, for all t) is also in[7] $\mathscr{A}$.

A *symmetric solution* to the NA market is a solution that is a symmetric set.

We can now state our main result: *The operations $\mathscr{A} \to \mathscr{A}_F$ and $\mathscr{B} \to \mathscr{B}_{NA}$ provide a one-to-one correspondence between the symmetric solutions of the nonatomic market and the solutions of the derived finite market*, where the operation $\mathscr{A} \to \mathscr{A}_F$ makes sense since it turns out that all allocations in a symmetric solution are themselves symmetric.

3. Proof of the Results

The main result will be divided into two parts, each proved as an independent theorem.

THEOREM (3.1) *Let the nonatomic market satisfy* (2.1)–(2.7). *Let $\mathscr{B}$ be any solution of the derived finite market. Then $\mathscr{B}_{NA}$ is a symmetric solution of the nonatomic market.*

THEOREM (3.2) *Let the nonatomic market satisfy* (2.1)–(2.7). *Let $\mathscr{A}$ be any symmetric solution of it. Then every allocation in $\mathscr{A}$ is symmetric, and $\mathscr{A}_F$ is a solution of the derived finite market.*

Before proving these theorems, we will bring three lemmas.

LEMMA (3.3) *Let the nonatomic market satisfy* (2.5)–(2.7). *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two symmetric allocations, and let* $\boldsymbol{\xi}$ *and* $\boldsymbol{\eta}$ *be the corresponding finite allocations. Then* $\mathbf{x} \succ \mathbf{y}$ *implies* $\boldsymbol{\xi} \succ \boldsymbol{\eta}$.

Proof. Denote by S the nonnull coalition such that $\mathbf{x} \succ \mathbf{y}$ via S, and let $\alpha_i = \mu(S \cap T_i)$ for each i. Then the set

$$I = \{i \in N \mid \alpha_i > 0\}$$

is nonempty. Since $\mathbf{x} \succ \mathbf{y}$ via S, and $\mathbf{x}, \mathbf{y}$ are symmetric allocations, it follows that $\boldsymbol{\xi}_i \succ_i \boldsymbol{\eta}_i$ for each i in I. It remains only to prove that I is effective for $\boldsymbol{\xi}$, i.e., we have to show that

$$\sum_{i \in I} \xi_i^k \leqq \sum_{i \in I} \omega_i^k \qquad \text{for all } 1 \leqslant k \leqslant m. \tag{3.4}$$

[7] Note that an allocation in a symmetric set need not be symmetric.

From the effectiveness of S for $\mathbf{x}$, it follows that

$$\int_S \mathbf{x} \leqq \int_S \mathbf{w},$$

hence

$$\sum_{i \in I} \alpha_i \xi_i^k \leqq \sum_{i \in I} \alpha_i \omega_i^k \qquad \text{for all } 1 \leqslant k \leqslant m. \tag{3.5}$$

We distinguish two cases. First, when the kth commodity is not initially owned by any trader of type i, for all $i \in I$; i.e., when

$$k \notin \bigcup_{i \in I} M_i .$$

Then $\omega_i^k = 0$ for all $i \in I$, and the right side of (3.5) vanishes. Hence the left side vanishes too, and $\xi_i^k = 0$ for all $i \in I$, which proves (3.4) for this k.

The second case is when the kth commodity is initially owned by some type in I. Then, by Assumption (2.7),

$$\sum_{i \in I} \omega_i^k = \sum_{i=1}^{n} \omega_i^k$$

and (3.4) holds because ξ is an allocation.

This completes the proof of the lemma. Q.E.D.

LEMMA (3.6) *Let the nonatomic market satisfy* (2.1)–(2.6). *Let* $\mathbf{x}$ *be an i.r. allocation, and let* $1 \leqslant i \leqslant n$ *be such that the set*

$$S = \{t \in T_i \mid \mathbf{x}(t) \neq \xi\}$$

is nonnull, where

$$\xi = \int_{T_i} \mathbf{x}.$$

Then there is $\eta \leqslant \xi$ *such that the set*

$$U = \{t \in T_i \mid \mathbf{x}(t) \underset{i}{\prec} \eta\}$$

is also nonnull, and $\eta \succcurlyeq_i \omega_i$.

Proof. First we will show that the set

$$V = \{t \in T_i \mid \mathbf{x}(t) \underset{i}{\prec} \xi\}$$

is nonnull. Suppose the contrary, i.e., $\mathbf{x}(t) \succcurlyeq_i \xi$ for all $t \in T_i$.

Let

$$C = \{x \in \Omega \mid x \underset{i}{\succcurlyeq} \xi, x \neq \xi\},$$

then C is a convex set because of Assumption (2.4) of *strong* convexity (ξ cannot be obtained as a convex combination of vectors x_j such that $x_j \succcurlyeq_i \xi$ and $x_j \neq \xi$).

Then $\mathbf{x}(t) \in C$ for all $t \in S$, and

$$\frac{1}{\mu(S)} \int_S \mathbf{x}$$

is also in C (this is well known, e.g., Shitovitz [17, Lemma 5.6.1]). We got a contradiction, since obviously

$$\frac{1}{\mu(S)} \int_S \mathbf{x} = \xi.$$

Therefore V is a nonnull set.

Since $\xi \geqslant 0$ [otherwise $\mathbf{x}(t) = 0$ for all t and S is null], there is some k with $\xi^k > 0$. Let e_k denote the unit vector in R^m with the kth coordinate equal to 1 and all others 0. The continuity assumption (2.2) implies the existence of $\epsilon > 0$ small enough such that $\eta = \xi - \epsilon e_k$ is still in Ω, the set

$$U = \{t \in T_i \mid \mathbf{x}(t) \underset{i}{\prec} \eta\}$$

is nonnull, and $\eta \succcurlyeq_i \omega_i$ [the latter, since $\mathbf{x}(t) \succcurlyeq_i \omega_i$ and $\xi \succ_i \mathbf{x}(t)$ for all $t \in V$, hence $\xi \succ_i \omega_i$]. Q.E.D.

LEMMA (3.7) *Let* $\mathbf{x}$ *be an* NA *allocation, let* $S_1 \subset T$ *and let* $0 \leqslant \alpha \leqslant 1$. *Then there is* $S_\alpha \subset S_1$ *such that*

$$\mu(S_\alpha) = \alpha\mu(S_1)$$

and

$$\int_{S_\alpha} \mathbf{x} = \alpha \int_{S_1} \mathbf{x}.$$

Proof. This is a direct consequence of Lyapunov's theorem [13], used for the vector of nonatomic measures

$$(\mu, \nu^1, \nu^2, \ldots, \nu^m),$$

where ν^k is defined by

$$\nu^k(S) = \int_S \mathbf{x}^k \, d\mu \qquad \text{for all } S \in \mathscr{C}. \qquad \text{Q.E.D.}$$

We can now prove the two theorems.

Proof of Theorem (3.1) The internal consistency of $\mathscr{B}_{\mathrm{NA}}$ follows immediately from the internal consistency of $\mathscr{B}$ and Lemma (3.3).

Next, let $\mathbf{x}$ be any i.r. NA allocation not in $\mathscr{B}_{\mathrm{NA}}$; we will prove that there is some member of $\mathscr{B}_{\mathrm{NA}}$ dominating it.

First, suppose $\mathbf{x}$ is symmetric, and let $\boldsymbol{\xi}$ be the corresponding F allocation. $\boldsymbol{\xi}$ is not in $\mathscr{B}$ (otherwise $\mathbf{x}$ would be in $\mathscr{B}_{\mathrm{NA}}$), hence there is some $\boldsymbol{\eta}$ in $\mathscr{B}$ such that $\boldsymbol{\eta} \succ_I \boldsymbol{\xi}$. Let

$$S = \bigcup_{i \in I} T_i ,$$

then obviously $\mathbf{y} \succ_S \mathbf{x}$, where $\mathbf{y}$ is the symmetric NA allocation corresponding to $\boldsymbol{\eta}$ (hence in $\mathscr{B}_{\mathrm{NA}}$), and $\mathbf{x}$ is dominated by a member of $\mathscr{B}_{\mathrm{NA}}$.

Second, suppose $\mathbf{x}$ is not symmetric. Let

$$\xi_i = \int_{T_i} \mathbf{x}$$

for all $1 \leqslant i \leqslant n$. Since $\mathbf{x}$ is not symmetric, there are some i such that the set

$$S_i = \{t \in T_i \mid \mathbf{x}(t) \neq \xi_i\}$$

is nonnull. Let I be the set of all such i, then I is nonempty. For each such i, we use Lemma (3.6) to get a vector $\eta_i \leqslant \xi_i$, such that[8] $\eta_i \succcurlyeq_i \omega_i$ and the set

$$U_i = \{t \in T_i \mid \mathbf{x}(t) \underset{i}{\prec} \eta_i\} \tag{3.8}$$

is also nonnull. Let

$$\alpha_i = \mu(U_i) > 0 \tag{3.9}$$

and let

$$\delta = \sum_{i \in I} (\xi_i - \eta_i) \geqslant 0.$$

For each i not in I, we define[9]

$$\eta_i = \xi_i + [1/(n - |I|)]\, \delta,$$

where $|I|$ is the number of elements of I. It follows immediately from its definition that $\boldsymbol{\eta} = (\eta_i)_{i \in N}$ is an F allocation. For i not in I, let U_i and α_i be defined also by (3.8) and (3.9); for all such i, obviously $U_i = T_i$ and $\alpha_i = 1$.

[8] To ensure that $\boldsymbol{\eta}$ is i.r.

[9] In case $I = N$, we take $\eta_i = \xi_i$ for all i.

Let

$$\alpha = \min_{1 \leqslant i \leqslant n} \alpha_i > 0,$$

and choose for each i any coalition V_i satisfying $V_i \subset U_i$ and $\mu(V_i) = \alpha$ (the measure μ being nonatomic, this is always possible).

If $\eta \in \mathscr{B}$, let $\mathbf{y}$ be the corresponding NA allocation in $\mathscr{B}_{\mathrm{NA}}$. Then obviously $\mathbf{y} \succ \mathbf{x}$ via the coalition

$$\bigcup_{i=1}^{n} V_i .$$

If $\eta \notin \mathscr{B}$, then there is an F allocation ζ in $\mathscr{B}$ such that $\zeta \succ \eta$ via some nonempty set J. Let $\mathbf{z} \in \mathscr{B}_{\mathrm{NA}}$ correspond to ζ, then $\mathbf{z} \succ \mathbf{x}$ via the coalition

$$\bigcup_{i \in J} V_i .$$

In any case, we have found a member of $\mathscr{B}_{\mathrm{NA}}$ dominating $\mathbf{x}$. This proves that $\mathscr{B}_{\mathrm{NA}}$ is extradominative, and hence an NA solution.

Last, $\mathscr{B}_{\mathrm{NA}}$ is by definition a set of symmetric allocations, hence a symmetric set ($\pi\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathscr{B}_{\mathrm{NA}}$ and all permutations π). This completes the proof that $\mathscr{B}_{\mathrm{NA}}$ is a symmetric solution. Q.E.D.

The next lemma is of independent interest. Note that it does not make use of Assumption (2.7), but only of the usual assumptions of the nonatomic market.

LEMMA (3.10) *Let the nonatomic market satisfy* (2.1)–(2.6), *and let* $\mathscr{A}$ *be any symmetric solution. Then every allocation* $\mathbf{x}$ *in* $\mathscr{A}$ *is symmetric.*

Proof. Suppose $\mathbf{x}$ is not symmetric, then there is some i_0 such that the set

$$\{t \in T_{i_0} \mid \mathbf{x}(t) \neq \xi_{i_0}\}$$

is nonnull, where $\xi_{i_0} = \int_{T_{i_0}} \mathbf{x}$. Using Lemma (3.6) for this i_0, we get $\eta_{i_0} \leqslant \xi_{i_0}$ such that the set

$$S_{i_0} = \{t \in T_{i_0} \mid \mathbf{x}(t) \underset{i_0}{\prec} \eta_{i_0}\}$$

is also nonnull, and[10] $\eta_{i_0} \succcurlyeq_{i_0} \omega_{i_0}$. Let $\delta = \xi_{i_0} - \eta_{i_0} \geqslant 0$, and let $\mathbf{y}$ be the NA allocation

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{x}(t) + [1/(n-1)]\delta && \text{for} \quad t \in T_i,\ i \neq i_0, \\ &= \eta_{i_0} && \text{for} \quad t \in T_{i_0}. \end{aligned}$$

[10] Once more, to ensure that $\mathbf{y}$ is i.r.

For all $i \neq i_0$, we use Lemma (3.7) for $\mathbf{y}$, T_i and $\alpha = \mu(S_{i_0}) > 0$, to get a set $S_i \subset T_i$ such that $\mu(S_i) = \alpha$ and $\int_{S_i} \mathbf{y} = \alpha \int_{T_i} \mathbf{y}$. Then obviously the set

$$S = \bigcup_{i=1}^{n} S_i$$

is effective for $\mathbf{y}$, and $\mathbf{y} \succ \mathbf{x}$ via S.

Hence $\mathbf{y}$ is not a member of $\mathscr{A}$, and there is some $\mathbf{z}$ in $\mathscr{A}$ such that $\mathbf{z} \succ \mathbf{y}$ via some nonnull coalition U. Without loss of generality, we assume that

$$\mu(U \cap T_{i_0}) \leqslant \mu(S_{i_0}) = \alpha.$$

[Indeed, because of Lemma (3.7), we may choose U so that $\mu(U)$—and *a fortiori* $\mu(U \cap T_{i_0})$—is arbitrarily small.]

If $\mu(U \cap T_{i_0}) = 0$, then by desirability (2.1), we have $\mathbf{z} \succ \mathbf{x}$ via U. If $\mu(U \cap T_{i_0}) > 0$, let $V_{i_0} \subset S_{i_0}$ such that $\mu(U \cap T_{i_0}) = \mu(V_{i_0})$, and define π to be a permutation interchanging V_{i_0} with $U \cap T_{i_0}$, and being otherwise the identity. Then $\pi\mathbf{x}$ is also in $\mathscr{A}$ ($\mathscr{A}$ is symmetric), and obviously $\mathbf{z} \succ \pi\mathbf{x}$ via U. In any case we get domination between two members of $\mathscr{A}$, which is impossible. Hence every allocation in $\mathscr{A}$ is symmetric. Q.E.D.

Proof of Theorem (3.2) The set $\mathscr{A}_F$ is well defined by Lemma (3.10). Its internal consistency and extradominance follow easily from the same properties of $\mathscr{A}$ [for the latter, we use also Lemma (3.3)]. Q.E.D.

4. Discussion

In this section we will make some remarks concerning our assumptions and their relation to the results.

First, the assumption (2.4) of strong convexity. It was used only in Lemma (3.6), and it ensures the "equal treatment" character of all the allocations in the solutions of the nonatomic market.

An instructive example of the problems arising when we assume that the preferences are convex, but not strongly convex, is the following: there are three commodities and three types of traders, initially owning (1, 0, 0), (0, 1, 0), and (0, 0, 1), respectively. The utility functions are

$$u_1(x) = u_2(x) = \min\{\sqrt{x^1} + \sqrt{x^2} + \sqrt{x^3}, (\sqrt{3})(x^1 + x^2 + x^3)\},$$

$$u_3(x) = \sqrt{x^1} + \sqrt{x^2} + \sqrt{x^3}.$$

Let D denote the diagonal

$$D = \{x \in \Omega \mid x^1 = x^2 = x^3\},$$

and let F be the set

$$F = \{x \in \Omega \mid \sqrt{x^1} + \sqrt{x^2} + \sqrt{x^3} \geqq (\sqrt{3})(x^1 + x^2 + x^3)\},$$

i.e., F is the part of the commodity space where the preference relations of types 1 and 2 are convex but *not* strong convex. It is easy to check that the set $\mathscr{B}$ of all finite allocations ξ such that $u_i(\xi_i) \geqslant 1$ for all $i = 1, 2, 3$, $\xi_1, \xi_2 \in D \cup F$ and $\xi_3 \in D$, is a solution of the derived finite market. But $\mathscr{B}_{\mathrm{NA}}$ is not a solution of the nonatomic market, since it does not dominate all those allocations we get from members ξ of $\mathscr{B}$ by "dividing" ξ_1 and ξ_2 in a nonsymmetric but utility-preserving form. For example, let $S_1 \subset T_1$ and $S_2 \subset T_2$ be coalitions of measure 1/2, and let

$$\begin{aligned} \mathbf{x}(t) &= (1/3, 1/3, 0), && t \in S_1 \cup S_2, \\ &= (1/9, 1/9, 4/9), && t \in (T_1 \backslash S_1) \cup (T_2 \backslash S_2), \\ &= (5/9, 5/9, 5/9), && t \in T_3. \end{aligned}$$

Then $\mathbf{x}$ is not symmetric, hence is not in $\mathscr{B}_{\mathrm{NA}}$, but is not dominated by any member of $\mathscr{B}_{\mathrm{NA}}$, since there is in $\mathscr{B}_{\mathrm{NA}}$ an allocation $\mathbf{y}$ with the same utilities:

$$\begin{aligned} \mathbf{y}(t) &= (2/9, 2/9, 2/9), && t \in T_1 \cup T_2, \\ &= (5/9, 5/9, 5/9), && t \in T_3. \end{aligned}$$

However, $\mathbf{x}$ cannot be added to $\mathscr{B}_{\mathrm{NA}}$, since it dominates allocations there: let $\mathbf{z}$ be

$$\begin{aligned} \mathbf{z}(t) &= (1/5, 1/5, 1/5), && t \in T_1 \cup T_2, \\ &= (3/5, 3/5, 3/5), && t \in T_3, \end{aligned}$$

then $\mathbf{x} \succ \mathbf{z}$ via $S_1 \cup S_2$, and $\mathbf{z}$ is in $\mathscr{B}_{\mathrm{NA}}$.

Second, some remarks concerning Assumption (2.7). As indicated, this property of the market is the direct cause of the formation of the cartels, and without it the market will no longer behave this way. For example, suppose there are two commodities and three types of players, with

$$\begin{aligned} &\omega_1 = (1, 0), \qquad \omega_2 = (1, 0), \qquad \omega_3 = (0, 2), \\ &u_1(x) = u_3(x) = \sqrt{x^1} + \sqrt{x^2}, \\ &u_2(x) = \sqrt[3]{x^1} + \sqrt[3]{x^2}. \end{aligned}$$

Let $\mathscr{B}$ be the set of all finite allocations

$$[(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma)]$$

such that $\alpha + \beta + \gamma = 2$, $1/4 \leqq \alpha \leqq 1$, $1/8 \leqq \beta \leqq 1$ and $1/2 \leqq \gamma$. Then $\mathscr{B}$ is a solution of the derived finite market, but $\mathscr{B}_{\mathrm{NA}}$ is *not* a solution of the nonatomic one: let $\mathbf{x}$ and $\mathbf{y}$ be the symmetric allocations obtained from

$$\xi = [(2/3, 2/3), (2/3, 2/3), (2/3, 2/3)]$$

and

$$\eta = [(1, 1), (1/2, 1/2), (1/2, 1/2)],$$

respectively, then $\mathbf{x}$ and $\mathbf{y}$ are both in $\mathscr{B}_{\mathrm{NA}}$, but $\mathbf{x} \succ \mathbf{y}$ [via $T_2 \cup S_3$, where $S_3 \subset T_3$ and $\mu(S_3) = 1/2$].

To understand this phenomenon, note that Assumption (2.7) is not satisfied, and Lemma (3.3) is no longer true—although $\mathbf{x} \succ \mathbf{y}$ via $T_2 \cup S_3$, ξ cannot dominate η via $\{2, 3\}$ (because it is not effective); therefore Theorem (3.1) does not hold any more.

The same happens to Theorem (3.2). Let $\mathscr{A}$ be the set of all allocations

$$\begin{aligned} \mathbf{x}(t) &= (\alpha, \alpha), & t \in T_1, \\ &= (\beta, \beta), & t \in T_2, \\ &= (\gamma, \gamma), & t \in T_3, \end{aligned}$$

such that $\alpha + \beta + \gamma = 2$ and either (i) $\alpha = \beta$, $1/4 \leqq \alpha$, and $1/2 \leqq \gamma$, or (ii) $\alpha = 1/4$ and $1/8 \leqq \beta < 1/4$. This is a symmetric solution of the nonatomic market, but $\mathscr{A}_{\mathrm{F}}$ is *not* a solution to the finite market, since it is not extradominative; e.g., the allocation

$$[(1/4, 1/4), (3/4, 3/4), (1, 1)]$$

is not dominated by any member of $\mathscr{A}_{\mathrm{F}}$.

Note that Assumption (2.7) is not only a sufficient condition in Lemma (3.3), but it is also necessary. Whenever it is not satisfied, one can find symmetric allocations $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x} \succ \mathbf{y}$ but there is no dominance between the corresponding finite allocations ξ and η. And it is obvious that without Lemma (3.3), both Theorems (3.1) and (3.2) can no longer be true.

A last remark concerns the proof of Theorem (3.1). As can be checked, the proof can be adjusted[11] to fit the case in which the market is not nonatomic, but rather is any r *replication* of the finite market (i.e., when there are r traders of each type, instead of only one). This means that any solution of a finite market satisfying our assumptions gives a solution of the r-replicated market, for *any* r (and not only in the limit, as for other concepts—see, e.g., Debreu–Scarf [5] for the core, Shapley [16] for the

[11] We take there $\alpha = 1/r$, and $V_i \subset U_i$ consisting of exactly one trader.

value[12]). However, in the proof of Theorem (3.2), we do make essential use of the nonatomic character [in Lemma (3.7)—Lyapunov's theorem].

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[12] For another theorem that holds for *all r* replications, see Galil [8].