Calibrated Forecasts and Game Dynamics

Sergiu Hart

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Sergiu Hart

Center for the Study of Rationality
Dept of Mathematics Dept of Economics
The Hebrew University of Jerusalem

`hart@huji.ac.il`
`http://www.ma.huji.ac.il/hart`

Joint work with

Dean P. Foster

University of Pennsylvania &

Amazon Research

Papers

Papers

- Dean P. Foster and Sergiu Hart
  Smooth Calibration, Leaky Forecasts, Finite Recall, and Nash Dynamics
  - 2012
  - *Games and Economic Behavior* 2018

www.ma.huji.ac.il/hart/abs/calib-eq.html

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- Dean P. Foster and Sergiu Hart
  Smooth Calibration, Leaky Forecasts, Finite Recall, and Nash Dynamics
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  - *Games and Economic Behavior* 2018

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- Dean P. Foster and Sergiu Hart
  An Integral Approach to Calibration
  - 2016 (in preparation)

Calibration

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- Forecaster says: "The chance of rain tomorrow is p"

Calibration

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- Forecaster is **CALIBRATED** if
  for every p: the proportion of rainy days among those days when the forecast was p equals p (or is close to p in the long run)

Calibration

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CALIBRATION ***can be guaranteed***

(no matter what the weather will be)

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NON-Bayesian, NO statistical assumptions !

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- Forecaster uses ***mixed*** forecasting
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  with probability $1/2$, forecast $= 60\%$)

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- Foster and Vohra 1994 [publ 1998]
- Hart 1995: proof using Minimax Theorem

The MINIMAX Theorem

The MINIMAX Theorem

THEOREM (von Neumann 1928)

IF

$X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ are compact convex sets,
and $f : X \times Y \to \mathbb{R}$ is a continuous function
that is convex-concave,
i.e., $f(\cdot, y) : X \to \mathbb{R}$ is convex for fixed y,
and $f(x, \cdot) : Y \to \mathbb{R}$ is concave for fixed x,

THEN

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y).$$

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  I have a strategy
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- finite game; probabilistic (mixed) strategies

Calibration Proof: Minimax

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  $\Rightarrow$ FINITE 2-person 0-sum game

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- FINITE δ-GRID, FINITE HORIZON
  $\Longrightarrow$ FINITE 2-person 0-sum game
- IF the strategy of the rainmaker IS KNOWN
  THEN the forecaster can get δ-calibrated forecasts

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BACK-casting (not fore-casting!)

("Politicians' Lemma")

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No Calibration

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  - Forecast is known before the rain/no-rain decision is made ("**LEAKY FORECASTS**")

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  - Forecaster uses a ***deterministic*** forecasting procedure

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Oakes 1985

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- **Smooth Calibration**: combine together the days when the forecast was ***close to*** p (*smooth* out the calibration score)
- **Main Result**:

> There exists a ***deterministic*** procedure that is **smoothly calibrated**.

Smooth Calibration

- **SMOOTH CALIBRATION**: combine together the days when the forecast was ***close to*** p (*smooth* out the calibration score)
- **Main Result**:

> There exists a ***deterministic*** procedure that is **SMOOTHLY CALIBRATED**.

Deterministic $\Longrightarrow$ result holds also when the forecasts are ***leaked***

Calibration

Calibration

- Set of **ACTIONS**: $A \subset \mathbb{R}^m$ (finite set)
- Set of **FORECASTS**: $C = \Delta(A)$
  - Example: $A = \{0, 1\}$, $C = [0, 1]$

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- **CALIBRATION SCORE** at time T for a sequence $(a_t, c_t)_{t=1,2,\ldots}$ in $A \times C$:

$$K_T = \frac{1}{T} \sum_{t=1}^{T} ||\bar{a}_t - c_t||$$

where

$$\bar{a}_t = \frac{\sum_{s=1}^{T} \mathbf{1}_{c_s = c_t}\, a_s}{\sum_{s=1}^{T} \mathbf{1}_{c_s = c_t}}$$

Smooth Calibration

Smooth Calibration

- A ***"smoothing function"*** is a Lipschitz function $\boldsymbol{\Lambda} : \boldsymbol{C} \times \boldsymbol{C} \to [\mathbf{0}, \mathbf{1}]$ with $\boldsymbol{\Lambda}(\boldsymbol{c}, \boldsymbol{c}) = \mathbf{1}$ for every $\boldsymbol{c}$.

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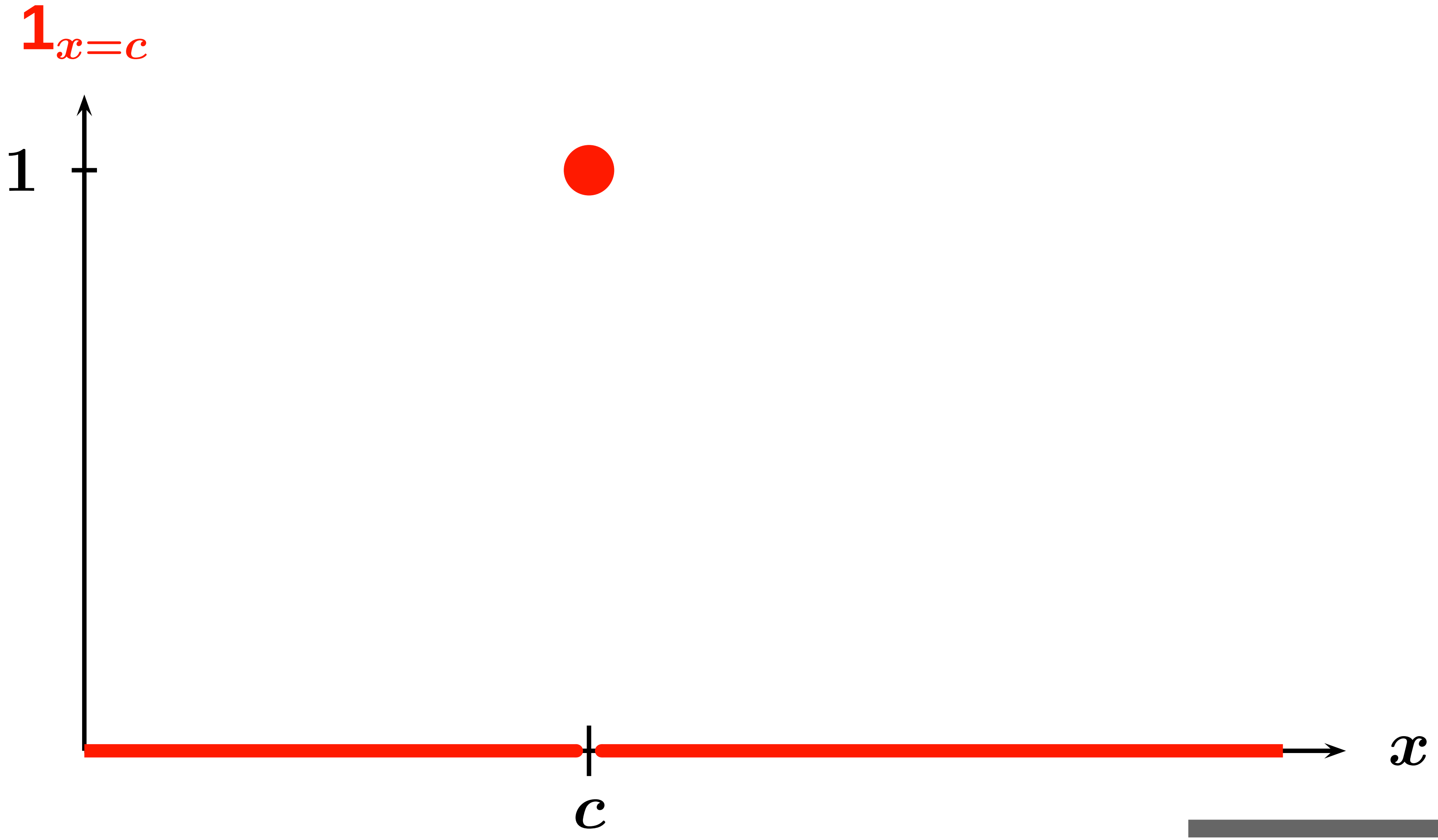
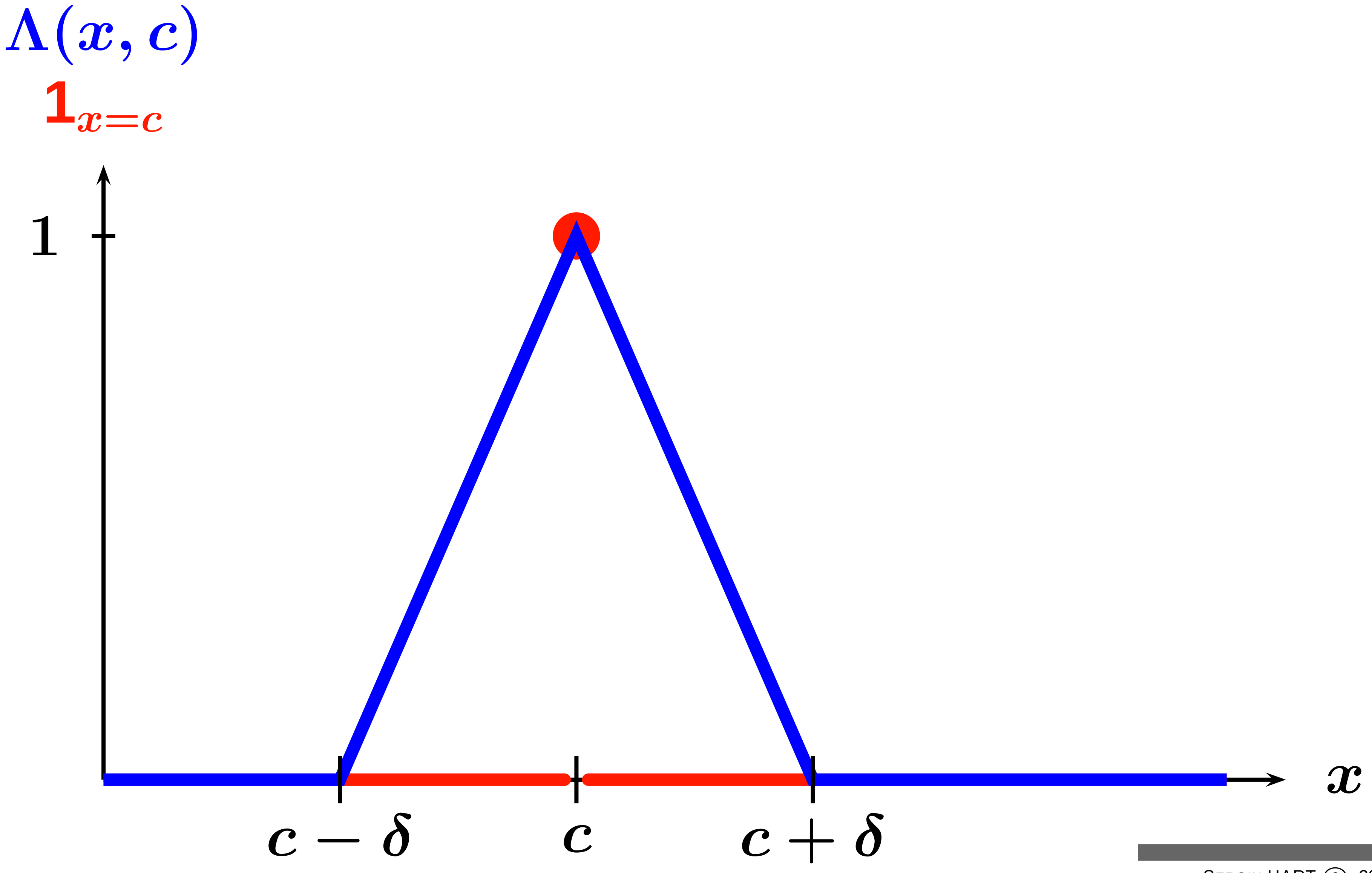
  - Example: $\Lambda(x,c) = [\delta - ||x - c||]_+/\delta$

Indicator Function

Indicator Function



Indicator and Λ Functions



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$$\bar{a}_t^\Lambda = \frac{\sum_{s=1}^{T} \Lambda(c_s, c_t)\, a_s}{\sum_{s=1}^{T} \Lambda(c_s, c_t)}, \quad c_t^\Lambda = \frac{\sum_{s=1}^{T} \Lambda(c_s, c_t)\, c_s}{\sum_{s=1}^{T} \Lambda(c_s, c_t)}$$

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    - a_t chosen ***after*** c_t is disclosed: **LEAKY** setup

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- Full monitoring, perfect recall

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(ε, L)-**SMOOTHLY CALIBRATED**

Smooth Calibration

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if there is T_0 such that $K_T^{\Lambda} \leq \varepsilon$ holds for:

- every $T \geq T_0$,
- every strategy of Player A, and
- every smoothing function Λ with Lipschitz constant $\leq L$

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For every $\varepsilon > 0$ and $L < \infty$

there exists a procedure

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Moreover:

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- it uses a fixed ***finite grid***, and
- forecasts may be ***leaked***

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  - nothing good ... (easier to pass the test)

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- For ***game dynamics***:

Smooth Calibration: Implications

- For ***forecasting***:
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- For ***game dynamics***:
  - Nash dynamics

Calibrated Learning

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Foster and Vohra 1997

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- every player uses a ***calibrated forecast*** on the play of the other players
- every player ***best replies*** to his forecast

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Calibrated Learning

CALIBRATED LEARNING:

- every player uses a ***calibrated forecast*** on the play of the other players
- every player ***best replies*** to his forecast

$\Rightarrow$ **time average of play**
(= empirical distribution of play)
is an approximate **CORRELATED EQUILIBRIUM**

Foster and Vohra 1997

Smooth Calibrated Learning

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(F) A ***smoothly calibrated deterministic*** procedure, which gives in each period t a "forecast" of play c_t in $\Pi_{i \in N} \Delta(A^i)$

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(P) A Lipschitz ***approximate best-reply*** mapping $g^i : \Pi_{i \in N} \Delta(A^i) \to \Delta(A^i)$ for each player i

In each period t, each player i:

1. runs the procedure **(F)** to get c_t
2. plays $g^i(c_t)$ given by **(P)**

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- has ***finite*** **MEMORY** and is ***stationary***

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SMOOTH CALIBRATED LEARNING

(with appropriate parameters):

- is a stochastic ***uncoupled*** dynamic
- has ***finite*** MEMORY and is ***stationary***
- ***Nash ε-equilibria are played at least $1 - \varepsilon$ of the time*** in the long run (a.s.)

Smooth Calibrated Learning: Proof

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$$\textbf{play}_t = g(c_t)$$

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- ***smooth calibration***
  $\Rightarrow$ $\textbf{play}_t = g(c_t) \approx c_t$

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  - use: g is Lipschitz
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  $\Rightarrow$ $\textbf{play}_t$ is an approximate Nash equilibrium
  - $g(\textbf{play}_t) = g(g(c_t)) \approx g(c_t) = \textbf{play}_t$

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  $\Rightarrow$ **FIXED POINT**

- **SMOOTH BEST REPLY**
  $\Rightarrow$ fixed point = **NASH EQUILIBRIUM**

Game Dynamics

Game Dynamics

- Best reply to **CALIBRATED** forecasts:

  $\longrightarrow$ **CORRELATED EQUILIBRIA**

Game Dynamics

- Best reply to **CALIBRATED** forecasts:
  $\longrightarrow$ **CORRELATED EQUILIBRIA**

- Best reply to **SMOOTHLY CALIBRATED** forecasts: $\longrightarrow$ **NASH EQUILIBRIA**

Dynamics and Equilibrium

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"LAW OF CONSERVATION OF COORDINATION":

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There must be some COORDINATION **—**

Dynamics and Equilibrium

"LAW OF CONSERVATION OF COORDINATION":

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(Hart and Mas-Colell 2003)

Integral Approach to Calibration

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- **INTEGRAL CALIBRATION** score:

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$$G_t^{\Lambda}(z) = \frac{1}{t}\sum_{s=1}^{t} \Lambda(c_s, z)(a_s - c_s)$$

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$$||G_t^{\Lambda}||_2 = \left(\int_C ||G_t^{\Lambda}(z)||^2 \, d\zeta(z)\right)^{1/2}$$

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- **INTEGRAL CALIBRATION**: Guarantee that

$$||G_t^{\Lambda}||_2 \le \varepsilon$$

for all t large enough, uniformly

Integral Calibration

Integral Calibration

- Choose the forecast c_t such that

$$\int \Lambda(c_t, z)\, G^{\Lambda}_{t-1}(z)\, d\zeta(z) \cdot (a - c_t) \leq 0$$

for every $a \in A$

Integral Calibration

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- Existence of such c_t is guaranteed by a **Fixed Point Theorem**

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  - $\Rightarrow$ **Deterministic Smooth** Calibration
  - $\Rightarrow$ **Deterministic Weak** Calibration

Integral Calibration

- Choose the forecast c_t such that

$$\int \Lambda(c_t, z)\; G^{\Lambda}_{t-1}(z)\, d\zeta(z) \cdot (a - c_t) \leq 0$$

  for every $a \in A$
- Existence of such c_t is guaranteed by a **Fixed Point Theorem**
- $\Rightarrow$ **Deterministic Integral** Calibration
  - $\Rightarrow$ **Deterministic Smooth** Calibration
  - $\Rightarrow$ **Deterministic Weak** Calibration
  - $\Rightarrow$ **Almost Deterministic** Calibration

Deterministic Nonlinear Calibration

- Choose the forecast c_t such that

$$\int \Lambda(c_t, z)\; G^{\Lambda}_{t-1}(z)\, d\zeta(z) \cdot (a - c_t) \le 0$$

  for every $a \in A$

- Existence of such c_t is guaranteed by a **Fixed Point Theorem**
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Integral Calibration

Integral Calibration

- Choose the **distribution** of the forecast c_t s.t.

$$\boldsymbol{E}\left[\int \boldsymbol{\Lambda}(\boldsymbol{c}_t, \boldsymbol{z})\, \boldsymbol{G}^{\boldsymbol{\Lambda}}_{t-1}(\boldsymbol{z})\, d\boldsymbol{\zeta}(\boldsymbol{z}) \cdot (\boldsymbol{a} - \boldsymbol{c}_t)\right] \leq \mathbf{0}$$

for every $\boldsymbol{a} \in \boldsymbol{A}$

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Integral Calibration

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  for every $a \in A$

- Existence of such distribution is guaranteed by a **Separation / Minimax Theorem**
- $\Rightarrow$ **Probabilistic** Calibration

Stochastic Linear Calibration

- Choose the **distribution** of the forecast c_t s.t.

$$E\left[\int \Lambda(c_t, z)\; G_{t-1}^{\Lambda}(z)\, d\zeta(z) \cdot (a - c_t)\right] \le 0$$

  for every $a \in A$
- Existence of such distribution is guaranteed by a **Separation / Minimax Theorem**
- $\Rightarrow$ **Probabilistic** Calibration

Integral Approach to Calibration

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fixed point $\longmapsto$ **deterministic** calibration

Integral Approach to Calibration

fixed point $\longmapsto$ **deterministic** calibration

separation / minimax $\longmapsto$ **stochastic** calibration

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- $\Leftrightarrow$ Brouwer's fixed-point theorem

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If f is bounded

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- $\Leftrightarrow$ Brouwer's fixed-point theorem
- "variational inequalities"

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BRIER SCORE = CALIBRATION + REFINEMENT

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$$= \mathbb{E}\,[\,\mathbb{VAR}\,[\,\mathbf{a}\,|\,\mathbf{c}\,]\,]$$

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Refinement / Discrimination

$$\textbf{REFINEMENT} = \frac{1}{T}\sum_{t=1}^{T} ||a_t - \bar{a}_t||^2$$

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$$= \mathbb{E}\,[\,\mathbb{VAR}\,[\,\mathbf{a} \,|\, \mathbf{c}\,]\,]$$

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Previous Work

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- Calibration and Correlated Equilibria:

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  - Foster and Vohra 1998
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  - Hart and Mas-Colell 2000, ...
- Weak Calibration (deterministic):
  - Kakade and Foster 2004 / 2008
  - Foster and Kakade 2006

Previous Work

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- Nash Dynamics:

Previous Work

- Nash Dynamics:
  - Foster and Young 2003
  - Kakade and Foster 2004 / 2008
  - Foster and Young 2006
  - Hart and Mas-Colell 2006
  - Germano and Lugosi 2007
  - Young 2009
  - Babichenko 2012

Previous Work

Previous Work

- Online Regression Problem:

Previous Work

- Online Regression Problem:
  - Foster 1991
  - J. Foster 1999
  - Vovk 2001
  - Azoury and Warmuth 2001
  - Cesa-Bianchi and Lugosi 2006

Successful Economic Forecasting

Successful Economic Forecasting

correctly forecasting

Successful Economic Forecasting ...

correctly forecasting

8 of the last 5 recessions