A Monotonicity Property of Binomial Probabilities

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May 11, 2008

Let \( n \geq 1 \) be an integer and \( 0 < p < 1 \). Put \( q := 1 - p \) and

\[
\begin{align*}
    a_k &:= \binom{n}{k} p^k q^{n-k}, \quad A_k := \sum_{i=0}^{k-1} a_i, \\
    r_k &:= \frac{A_k}{k} / \left(\frac{1 - A_k}{n + 1 - k}\right)
\end{align*}
\]

for integer \( 0 \leq k \leq n \), with \( r_0 := 0 \) and \( r_n := \infty \). Note that \( A_k/k \) is the average of the first \( k \) binomial terms, and \( (1 - A_k)/(n + 1 - k) \) is the average of the remaining \( n + 1 - k \) terms.

We prove the following result, settling a question of Ron Kaniel (communicated to us by Shmuel Kaniel and Micha Perles):

**Proposition 1** The sequence \( r_k \) is monotonically increasing in \( k \).

Denote

\[
    c_k := \frac{k(n - k)}{n + 1}.
\]
Lemma 2  If $k \geq 1$ then: $r_k < r_{k+1}$ if and only if

\[ \frac{A_k(1 - A_k)}{a_k} < A_k + c_k. \]  

Proof. Noting that $A_{k+1} = A_k + a_k$, we have: $r_k < r_{k+1}$ if and only if

\[ A_k(1 - A_k - a_k)(n + 1 - k) < (A_k + a_k)(1 - A_k)k(n - k), \]

or

\[ A_k(1 - A_k)[(k + 1)(n + 1 - k) - k(n - k)] < a_k[(1 - A_k)k(n - k) + A_k(k + 1)(n + 1 - k)]. \]

Simplifying yields

\[ A_k(1 - A_k)(n + 1) < a_k[A_k(n + 1) + k(n - k)], \]

which is (1). \hfill \Box

Lemma 3  If $k \leq (n - 1)p - 1$ then

\[ 1 + c_k < \frac{a_{k+1}}{a_k} c_{k+1}. \]

Proof. We have

\[ \frac{a_{k+1}}{a_k} c_{k+1} - c_k = \frac{(n - k)p}{(k + 1)q} \cdot \frac{(k + 1)(n - k - 1)}{n + 1} \cdot \frac{A_k(n - k)}{n + 1} = \frac{n - k}{(n + 1)q} [(n - k - 1)p - kq] = \frac{(n - k)[(n - 1)p - k]}{(n + 1)q} > 1 \]

since the assumption on $k$ implies that $n - k \geq (n - 1)q + 2 > (n + 1)q$ and $(n - 1)p - k \geq 1$. \hfill \Box

Lemma 4  If $k \leq (n - 1)p - 1$ then $r_k < r_{k+1}$ implies $r_{k+1} < r_{k+2}$.
Proof. Assume that $r_k < r_{k+1}$; by Lemma 2 for $k + 1$ we have to show that

$$\frac{(A_k + a_k)(1 - A_k - a_k)}{a_{k+1}} < A_k + a_k + c_{k+1}. $$

(recall that $A_{k+1} = A_k + a_k$). Multiplying by $a_{k+1}/a_k$, this is equivalent to

$$\frac{A_k(1 - A_k)}{a_k} + 1 - A_k - A_k - a_k < \frac{a_{k+1}}{a_k} (A_k + a_k + c_{k+1}).$$

Now the left-hand side is $\leq (A_k + c_k) + 1 - 2A_k - a_k < 1 + c_k$ (for $k \geq 1$ by Lemma 2, and for $k = 0$ since $A_0 = c_0 = 0$ and $a_0 > 0$), whereas the right-hand side is $> (a_{k+1}/a_k)c_{k+1}$, and so $\frac{a_{k+1}}{a_k} (A_k + a_k + c_{k+1})$. Apply Lemma 3. □

Corollary 5

$$r_0 < r_1 < ... < r_K$$

where $K := \lfloor (n - 1)p \rfloor + 1$.

Proof. Start with $r_0 = 0 < r_1$ and apply Lemma 4 inductively. □

Corollary 6

$$r_n > r_{n-1} > ... > r_{n-M}$$

where $M := \lfloor (n - 1)q \rfloor + 1$.

Proof. Let $\tilde{a}_k := \binom{n}{k}q^kp^{n-k} - a_{n-k}$ and $\tilde{A}_k := \sum_{i=0}^{k-1} \tilde{a}_i = 1 - A_{n-k}$, with corresponding $\tilde{r}_k = 1/r_{n-k}$, and apply Corollary 5 to the sequence $\tilde{r}_k$. □

Lemma 7 $K \geq n - M$.

Proof. $(n - 1)p + (n - 1)q = n - 1$ is an integer, therefore $\lfloor (n - 1)p \rfloor + \lfloor (n - 1)q \rfloor \geq n - 2$ (it equals either $n - 1$ or $n - 2$), and so $K + M \geq n$. □

Proof of Proposition 1. Combine Corollaries 5 and 6 with Lemma 7. □