A Monotonicity Property of Binomial Probabilities

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Let $n \geq 1$ be an integer and $0 < p < 1$. Put $q := 1 - p$ and

$$a_k := \binom{n}{k} p^k q^{n-k}, \quad A_k := \sum_{i=0}^{k-1} a_i,$$
$$r_k := \frac{A_k}{k} \Bigg/ \frac{1 - A_k}{n + 1 - k}$$

for integer $0 \leq k \leq n$, with $r_0 := 0$ and $r_n := \infty$. Note that A_k/k is the average of the first k binomial terms, and $(1 - A_k)/(n + 1 - k)$ is the average of the remaining $n + 1 - k$ terms.

We prove the following result, settling a question of Ron Kaniel (communicated to us by Shmuel Kaniel and Micha Perles):

Proposition 1 *The sequence r_k is monotonically increasing in k.*

Denote
$$c_k := \frac{k(n - k)}{n + 1}.$$

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Lemma 2 *If $k \geq 1$ then: $r_k < r_{k+1}$ if and only if*

$$\frac{A_k(1-A_k)}{a_k} < A_k + c_k. \tag{1}$$

Proof. Noting that $A_{k+1} = A_k + a_k$, we have: $r_k < r_{k+1}$ if and only if

$$A_k(1-A_k-a_k)(k+1)(n+1-k) < (A_k+a_k)(1-A_k)k(n-k),$$

or

$$\begin{aligned} &A_k(1-A_k)[(k+1)(n+1-k)-k(n-k)] \\ &\qquad < a_k[(1-A_k)k(n-k)+A_k(k+1)(n+1-k)]. \end{aligned}$$

Simplifying yields

$$A_k(1-A_k)(n+1) < a_k[A_k(n+1)+k(n-k)],$$

which is (1). $\square$

Lemma 3 *If $k \leq (n-1)p-1$ then*

$$1 + c_k < \frac{a_{k+1}}{a_k}\, c_{k+1}.$$

Proof. We have

$$\begin{aligned} \frac{a_{k+1}}{a_k}\, c_{k+1} - c_k &= \frac{(n-k)p}{(k+1)q} \cdot \frac{(k+1)(n-k-1)}{n+1} - \frac{k(n-k)}{n+1} \\ &= \frac{n-k}{(n+1)q}\,[(n-k-1)p - kq] = \frac{(n-k)[(n-1)p-k]}{(n+1)q} > 1 \end{aligned}$$

since the assumption on k implies that $n-k \geq (n-1)q+2 > (n+1)q$ and $(n-1)p-k \geq 1$. $\square$

Lemma 4 *If $k \leq (n-1)p-1$ then $r_k < r_{k+1}$ implies $r_{k+1} < r_{k+2}$.*

Proof. Assume that $r_k < r_{k+1}$; by Lemma 2 for $k+1$ we have to show that

$$\frac{(A_k + a_k)(1 - A_k - a_k)}{a_{k+1}} < A_k + a_k + c_{k+1}.$$

(recall that $A_{k+1} = A_k + a_k$). Multiplying by a_{k+1}/a_k, this is equivalent to

$$\frac{A_k(1 - A_k)}{a_k} + 1 - A_k - A_k - a_k < \frac{a_{k+1}}{a_k}\left(A_k + a_k + c_{k+1}\right).$$

Now the left-hand side is $\leq (A_k + c_k) + 1 - 2A_k - a_k < 1 + c_k$ (for $k \geq 1$ by Lemma 2, and for $k = 0$ since $A_0 = c_0 = 0$ and $a_0 > 0$), whereas the right-hand side is $> (a_{k+1}/a_k)c_{k+1}$. Apply Lemma 3. $\square$

Corollary 5

$$r_0 < r_1 < \ldots < r_K$$

where $K := \lfloor (n-1)p \rfloor + 1$.

Proof. Start with $r_0 = 0 < r_1$ and apply Lemma 4 inductively. $\square$

Corollary 6

$$r_n > r_{n-1} > \ldots > r_{n-M}$$

where $M := \lfloor (n-1)q \rfloor + 1$.

Proof. Let $\tilde{a}_k := \binom{n}{k} q^k p^{n-k} = a_{n-k}$ and $\tilde{A}_k := \sum_{i=0}^{k-1} \tilde{a}_i = 1 - A_{n-k}$, with corresponding $\tilde{r}_k = 1/r_{n-k}$, and apply Corollary 5 to the sequence $\tilde{r}_k$. $\square$

Lemma 7 $K \geq n - M$.

Proof. $(n-1)p + (n-1)q = n - 1$ is an integer, therefore $\lfloor (n-1)p \rfloor + \lfloor (n-1)q \rfloor \geq n - 2$ (it equals either $n-1$ or $n-2$), and so $K + M \geq n$. $\square$

Proof of Proposition 1. Combine Corollaries 5 and 6 with Lemma 7. $\square$