Two(!) Good To Be True

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July 2013
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Joint work with

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Sergiu Hart and Phil Reny
“Revenue Maximization in Two Dimensions”
(2010, in preparation)
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“Revenue Maximization in Two Dimensions”
(2010, in preparation)

Sergiu Hart and Phil Reny
“Maximizing Revenue with Multiple Goods: Nonmonotonicity and Other Observations”
(2011)
www.ma.huji.ac.il/hart/abs/monot-m.html
Sergiu Hart and Phil Reny
“Implementation of Reduced Form Mechanisms: A Simple Approach and a New Characterization” (2011)
www.ma.huji.ac.il/hart/abs/q-mech.html
Sergiu Hart and Noam Nisan
“Approximate Revenue Maximization with Multiple Items”
(2012)
www.ma.huji.ac.il/hart/abs/m-approx.html
Sergiu Hart and Noam Nisan
“Approximate Revenue Maximization with Multiple Items”
(2012)
www.ma.huji.ac.il/hart/abs/m-approx.html

Sergiu Hart and Noam Nisan
“The Menu-Size Complexity of Auctions”
(2013)
www.ma.huji.ac.il/hart/abs/m-corr.html
1 SELLER
A Simple Problem

1 SELLER

1 BUYER
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
A Simple Problem

1 SELLER
1 BUYER
k GOODS (ITEMS)

OBJECTIVE:
MAXIMIZE the REVENUE of the SELLER
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
A Simple Problem

1 SELLER

1 BUYER

$k$ GOODS (ITEMS)

values of GOODS to BUYER:

\[ x = (x_1, x_2, \ldots, x_k) \]
A Simple Problem

1 SELLER

1 BUYER

k GOODS (ITEMS)

values of GOODS to BUYER:
\[ x = (x_1, x_2, \ldots, x_k) \]

additive valuation
(good 1 and good 2 = \( x_1 + x_2 \))
A Simple Problem

1 SELLER

1 BUYER

$k$ GOODS (ITEMS)

values of GOODS to BUYER:
\[ x = (x_1, x_2, \ldots, x_k) \]

additive valuation
(good 1 and good 2 = \[ x_1 + x_2 \])

BUYER knows the value \( x \)
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
- values of GOODS to BUYER: $x = (x_1, x_2, \ldots, x_k)$
- additive valuation
  (good 1 and good 2 = $x_1 + x_2$)
- BUYER knows the value $x$
- SELLER does not know the value $x$
A Simple Problem

1 SELLER

1 BUYER

k GOODS (ITEMS)

values of GOODS to BUYER:
\[ x = (x_1, x_2, \ldots, x_k) \]

additive valuation
(good 1 and good 2 = \( x_1 + x_2 \))

BUYER knows the value \( x \)

SELLER does not know the value \( x \)

\( x \) distributed according to \( \mathcal{F} \) (c.d.f. on \( \mathbb{R}_+^k \))
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
  - values of GOODS to BUYER: $x = (x_1, x_2, \ldots, x_k)$
  - additive valuation
    (good 1 and good 2 = $x_1 + x_2$)
  - BUYER knows the value $x$
  - SELLER does not know the value $x$
  - $x$ distributed according to $\mathcal{F}$ (c.d.f. on $\mathbb{R}_+^k$)
  - SELLER knows the distribution $\mathcal{F}$ of $x$
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
A Simple Problem

1 SELLER

1 BUYER

k GOODS (ITEMS)

SELLER and BUYER:

quasi-linear utilities (i.e., additive in monetary payments)
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)

SELLER and BUYER:
- quasi-linear utilities (i.e., additive in monetary payments)
- risk-neutral (i.e., linear in probabilities)
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)

SELLER and BUYER:

- quasi-linear utilities (i.e., additive in monetary payments)
- risk-neutral (i.e., linear in probabilities)
  (or: linear in quantities)
A Simple Problem

1 SELLER

1 BUYER

$k$ GOODS (ITEMS)

SELLER and BUYER:

- quasi-linear utilities (i.e., additive in monetary payments)
- risk-neutral (i.e., linear in probabilities)
  (or: linear in quantities)

SELLER:

- no value and no cost for the GOODS
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)

**OBJECTIVE:**

MAXIMIZE the REVENUE of the SELLER
ONE GOOD ($k = 1$):
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ONE GOOD ($k = 1$):

- **SELLER** posts a **PRICE** $p$
ONE GOOD ($k = 1$):

- **SELLER posts** a **PRICE** $p$
- **BUYER chooses** between:
  - get the good and pay $p$, or
  - get nothing and pay nothing

Myerson 1981
ONE GOOD \((k = 1)\): 

- **SELLER** posts a **PRICE** \(p\) 
- **BUYER** chooses between: 
  - get the good and pay \(p\), or 
  - get nothing and pay nothing 

\(p\) such that \(\text{REVENUE } R = p \cdot \Pr[X > p] = p \cdot (1 - F(p))\) is **MAXIMAL**

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Myerson 1981
One Good: Solution

One Good ($k = 1$):

- **Seller posts** a price $p$
- **Buyer chooses** between:
  - get the good and pay $p$, or
  - get nothing and pay nothing

$p$ such that Revenue $R = p \cdot \Pr[X > p]$

$= p \cdot (1 - F(p))$ is maximal

$$\text{Rev}(F) = \max_p p \cdot (1 - F(p))$$

Myerson 1981
$X \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
22 & \text{with probability } \frac{1}{2}
\end{cases}$
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10 & \text{with probability } \frac{1}{2} \\
22 & \text{with probability } \frac{1}{2} 
\end{cases}$

$p = 10 \rightarrow R = 10 \cdot 1 = 10$
One Good: Example

\[ X \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2
\end{cases} \]

\[ \begin{align*}
p = 10 & \implies R = 10 \cdot 1 = 10 \\
p = 22 & \implies R = 22 \cdot 1/2 = 11
\end{align*} \]
One Good: Example

\[ X \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- \( p = 10 \rightarrow R = 10 \cdot 1 = 10 \)
- \( p = 22 \rightarrow R = 22 \cdot 1/2 = 11 \)
One Good: Example

\[ X \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
22 & \text{with probability } \frac{1}{2}
\end{cases} \]

\[ p = 10 \rightarrow R = 10 \cdot 1 = 10 \]

\[ p = 22 \rightarrow R = 22 \cdot \frac{1}{2} = 11 \]

\[ \text{Rev}(F) = 11 \quad p = 22 \]
Two Goods ($k = 2$)
Two Goods ($k = 2$), Independent
Two Goods ($k = 2$), Independent

sell separately:
Two Goods $(k = 2)$, Independent

- sell separately:

  \[
  \text{PRICE} = p_1 \quad \text{for good 1} \\
  \text{PRICE} = p_2 \quad \text{for good 2}
  \]
Two Goods

Two Goods ($k = 2$), Independent
Two Goods: Example

Two Goods \((k = 2)\), Independent
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[ X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2
\end{cases} \]
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[ X_1, X_2 \sim \begin{cases} 10 & \text{with probability } \frac{1}{2} \\ 22 & \text{with probability } \frac{1}{2} \end{cases} \]

\[ \text{REV}(X_1) = \text{REV}(X_2) = 11 \]
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[ X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

\[ \text{REV}(X_1) = \text{REV}(X_2) = 11 \]

\[ \max(10 \cdot 1, 22 \cdot 1/2) = 11 \]
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[ X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
22 & \text{with probability } \frac{1}{2}
\end{cases} \]

\[ \text{REV}(X_1) = \text{REV}(X_2) = 11 \]

\[ \max(10 \cdot 1, 22 \cdot 1/2) = 11 \]

\[ \text{REV}(X_1) + \text{REV}(X_2) = 11 + 11 = 22 \]
Two Goods: Example

Two Goods ($k = 2$), Independent

\[ X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- $\text{REV}(X_1) = \text{REV}(X_2) = 11$
  \[ \max(10 \cdot 1, 22 \cdot 1/2) = 11 \]
- $\text{REV}(X_1) + \text{REV}(X_2) = 11 + 11 = 22$
- sell the two goods together ("bundle") for the price $p_{12} = 32$:
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[ X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- \( \text{Rev}(X_1) = \text{Rev}(X_2) = 11 \)
- \( \max(10 \cdot 1, 22 \cdot 1/2) = 11 \)
- \( \text{Rev}(X_1) + \text{Rev}(X_2) = 11 + 11 = 22 \)

sell the two goods together ("bundle") for the price \( p_{12} = 32 \):

\[ R = 32 \cdot 3/4 = 24 > 22 \]
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[ X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- \(\text{Rev}(X_1) = \text{Rev}(X_2) = 11\)
  \[\max(10 \cdot 1, \ 22 \cdot 1/2) = 11\]
- \(\text{Rev}(X_1) + \text{Rev}(X_2) = 11 + 11 = 22\)
- sell the two goods together ("bundle") for the price \(p_{12} = 32\):
  \[R = 32 \cdot 3/4 = 24 > 22\]
OUTCOME:
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$q_i = \text{probability that } \text{BUYER gets good } i$
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$$q = (q_1, \ldots, q_k) \in [0, 1]^k$$
OUTCOME:

- $q_i$ = probability that BUYER gets good $i$
  
  $q = (q_1, \ldots, q_k) \in [0, 1]^k$

- $s$ = payment from BUYER to SELLER ("REVENUE")
General Mechanism

OUTCOME:

- \( q_i \) = probability that BUYER gets good \( i \)
  \[ q = (q_1, \ldots, q_k) \in [0, 1]^k \]

- \( s \) = payment from BUYER to SELLER ("REVENUE")

PAYOFF (utility) of BUYER when his valuation is

\[ x = (x_1, \ldots, x_k) \]
General Mechanism

OUTCOME:

- \( q_i \) = probability that \textbf{BUYER} gets good \( i \)
- \( q = (q_1, \ldots, q_k) \in [0, 1]^k \)
- \( s \) = payment from \textbf{BUYER} to \textbf{SELLER} ("REVENUE")

PAYOFF (utility) of \textbf{BUYER} when his valuation is \( x = (x_1, \ldots, x_k) \):

- \( b = q_1 \cdot x_1 + \ldots + q_k \cdot x_k - s \)
General Mechanism

OUTCOME:

- $q_i$ = probability that BUYER gets good $i$
  
  $q = (q_1, ..., q_k) \in [0, 1]^k$

- $s$ = payment from BUYER to SELLER
  ("REVENUE")

PAYOFF (utility) of BUYER when his valuation is $x = (x_1, ..., x_k)$:

- $b = q_1 \cdot x_1 + ... + q_k \cdot x_k - s = q \cdot x - s$
Simple Mechanism

MENU $\mathcal{M}$: a SET of possible OUTCOMES
Simple Mechanism

**MENU** $\mathcal{M}$: a set of possible outcomes

$$\mathcal{M} = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$
**Simple Mechanism**

**MENU** $\mathcal{M}$: a set of possible outcomes

$\mathcal{M} = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$

- **SELLER posts a MENU** $\mathcal{M}$
Simple Mechanism

**MENU $\mathcal{M}$**: a **SET** of possible **OUTCOMES**

$$\mathcal{M} = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $\mathcal{M}$
- **BUYER chooses** one **OUTCOME** in **MENU** $\mathcal{M}$:
Simple Mechanism

**MENU** $\mathcal{M}$: a set of possible outcomes

$$\mathcal{M} = \{ (q, s) \} \subset [0, 1]^k \times \mathbb{R}$$

- **seller** posts a **MENU** $\mathcal{M}$
- **buyer** chooses one outcome in **MENU** $\mathcal{M}$:
  - outcome chosen by buyer when his valuation is $x$: $(q(x), s(x)) \in \mathcal{M}$
Simple Mechanism

MENU $\mathcal{M}$: a SET of possible OUTCOMES

$$\mathcal{M} = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$

- SELLER posts a MENU $\mathcal{M}$
- BUYER chooses one OUTCOME in MENU $\mathcal{M}$:
  - OUTCOME chosen by BUYER when his valuation is $x$: $(q(x), s(x)) \in \mathcal{M}$
  - payoff of SELLER: $s(x)$
**Simple Mechanism**

**MENU** \( \mathcal{M} \): a SET of possible OUTCOMES

\[ \mathcal{M} = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R} \]

- **SELLER** posts a MENU \( \mathcal{M} \)
- **BUYER** chooses one OUTCOME in MENU \( \mathcal{M} \):
  - OUTCOME chosen by **BUYER** when his valuation is \( x \):
    \[ (q(x), s(x)) \in \mathcal{M} \]
  - payoff of **SELLER**: \( s(x) \)
  - payoff of **BUYER**: \( b(x) = q(x) \cdot x - s(x) \)
"Menu" Mechanism

**MENU** \( \mathcal{M} \): a **SET** of possible **OUTCOMES**

\[
\mathcal{M} = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}
\]

- **SELLER** posts a **MENU** \( \mathcal{M} \)
- **BUYER** chooses one **OUTCOME** in **MENU** \( \mathcal{M} \):
  - OUTCOME chosen by **BUYER** when his valuation is \( x \):
    \[
    (q(x), s(x)) \in \mathcal{M}
    \]
  - payoff of **SELLER**: \( s(x) \)
  - payoff of **BUYER**: \( b(x) = q(x) \cdot x - s(x) \)
"Menu" Mechanism

**MENU \( \mathcal{M} \):** a set of possible outcomes

\[ \mathcal{M} = \{ (q, s) \} \subset [0, 1]^k \times \mathbb{R} \]

- **Seller posts a menu** \( \mathcal{M} \)
- **Buyer chooses** one outcome in menu \( \mathcal{M} \):
  - outcome chosen by buyer when his valuation is \( x \):
    \[ (q(x), s(x)) \in \mathcal{M} \]
  - payoff of seller:
    \[ s(x) \]
  - payoff of buyer:
    \[ b(x) = q(x) \cdot x - s(x) \]

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**The Revelation Principle:**
"Menu" Mechanism

**MENU** \( \mathcal{M} \): a set of possible outcomes
\[
\mathcal{M} = \{ (q, s) \} \subset [0, 1]^k \times \mathbb{R}
\]

- **SELLER** posts a **MENU** \( \mathcal{M} \)

- **BUYER** chooses one **OUTCOME** in **MENU** \( \mathcal{M} \):
  - **OUTCOME** chosen by **BUYER** when his valuation is \( x \):
    \[
    (q(x), s(x)) \in \mathcal{M}
    \]
  - payoff of **SELLER**: \( s(x) \)
  - payoff of **BUYER**: \( b(x) = q(x) \cdot x - s(x) \)

**The Revelation Principle**: Every mechanism is equivalent to a **MENU MECHANISM**
"Menu" Mechanism

**MENU \( \mathcal{M} \):** a set of possible outcomes
\[
\mathcal{M} = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}
\]

- **Seller posts** a **MENU** \( \mathcal{M} \)
- **Buyer chooses** one outcome in **MENU** \( \mathcal{M} \):
  - **Outcome** chosen by **Buyer** when his valuation is \( x \):
    \[
    (q(x), s(x)) \in \mathcal{M}
    \]
  - Payoff of **Seller**: \( s(x) \)
  - Payoff of **Buyer**: \( b(x) = q(x) \cdot x - s(x) \)

**The Revelation Principle:** Every mechanism is equivalent to a **MENU MECHANISM** ("direct mechanism")
Incentive Compatibility (IC)
Incentive Compatibility (IC)

\[ q_1(x) \cdot x_1 + \cdots + q_k(x) \cdot x_k - s(x) \geq 0 \]

\[ q_1(y) \cdot x_1 + \cdots + q_k(y) \cdot x_k - s(y) \]

(for all \( x \) and \( y \))
- **Incentive Compatibility (IC)**

\[ q_1(x) \cdot x_1 + \ldots + q_k(x) \cdot x_k - s(x) \geq 0 \]

\[ q_1(y) \cdot x_1 + \ldots + q_k(y) \cdot x_k - s(y) \]

(for all \( x \) and \( y \))

- **Individual Rationality (IR) / Participation**
Incentive Compatibility (IC)

\[ q_1(x) \cdot x_1 + \ldots + q_k(x) \cdot x_k - s(x) \geq 0 \]

\[ q_1(y) \cdot x_1 + \ldots + q_k(y) \cdot x_k - s(y) \]

(for all \( x \) and \( y \))

Individual Rationality (IR) / Participation

\[ q_1(x) \cdot x_1 + \ldots + q_k(x) \cdot x_k - s(x) \geq 0 \]

(for all \( x \))
Maximize Revenue:
Maximize Revenue:

\[
R = \mathbb{E}_{x \sim F}[s(x)] = \int s(x)d\mathcal{F}(x)
\]
Maximize Revenue:

maximize

\[ R = \mathbb{E}_{x \sim F}[s(x)] = \int s(x) dF(x) \]

subject to

\((q, s)\) satisfies IC & IR
Incentive Compatibility (IC)
Incentive Compatibility (IC)

\( b(x) \) is a convex function of \( x \)
Incentive Compatibility (IC)

- \( b(x) \) is a convex function of \( x \)

Proof
Incentive Compatibility (IC)

- \( b(x) \) is a convex function of \( x \)

Proof

- for every \( y \): \( q(y) \cdot x - s(y) \) affine in \( x \)
Incentive Compatibility (IC)

- $b(x)$ is a convex function of $x$

**Proof**

- for every $y$: $q(y) \cdot x - s(y)$ affine in $x$
- $b(x) = \max_y \{ q(y) \cdot x - s(y) \}$ convex in $x$
Incentive Compatibility (IC)

- $b(x)$ is a convex function of $x$

**Proof**

- for every $y$: $q(y) \cdot x - s(y)$ affine in $x$
- $b(x) = \max_y \{q(y) \cdot x - s(y)\}$ convex in $x$
- $b(x) = q(x) \cdot x - s(x)$
Incentive Compatibility

Incentive Compatibility (IC)

- \( b(x) \) is a convex function of \( x \)

Proof

- for every \( y \): \( q(y) \cdot x - s(y) \) affine in \( x \)
- \( b(x) = \max_y \{ q(y) \cdot x - s(y) \} \) convex in \( x \)
- \( b(x) = q(x) \cdot x - s(x) = \nabla b(x) \cdot x - s(x) \)
Incentive Compatibility

Incentive Compatibility (IC) \iff

\begin{itemize}
  \item \( b(x) \) is a convex function of \( x \) and
  \item \( q_i(x) = \partial b(x)/\partial x_i \) for a.e. \( x \) and all \( i \)
  \item \( q(x) = \nabla b(x) \) for a.e. \( x \)
\end{itemize}

Proof

\begin{itemize}
  \item for every \( y \): \( q(y) \cdot x - s(y) \) affine in \( x \)
  \item \( b(x) = \max_y \{q(y) \cdot x - s(y)\} \) convex in \( x \)
  \item \( b(x) = q(x) \cdot x - s(x) = \nabla b(x) \cdot x - s(x) \)
\end{itemize}
Incentive Compatibility (IC) \iff

- \( b(x) \) is a convex function of \( x \) and
- \( q_i(x) = \partial b(x) / \partial x_i \) for a.e. \( x \) and all \( i \)
- \( q(x) = \nabla b(x) \) for a.e. \( x \)

Proof

- for every \( y \): \( q(y) \cdot x - s(y) \) affine in \( x \)
- \( b(x) = \max_y \{ q(y) \cdot x - s(y) \} \) convex in \( x \)
- \( b(x) = q(x) \cdot x - s(x) = \nabla b(x) \cdot x - s(x) \)
- \( s(x) = \nabla b(x) \cdot x - b(x) \)
Incentive Compatibility (IC) ⇔

- $b(x)$ is a convex function of $x$ and
- $q_i(x) = \frac{\partial b(x)}{\partial x_i}$ for a.e. $x$ and all $i$
- $q(x) = \nabla b(x)$ for a.e. $x$

$$s(x) = \nabla b(x) \cdot x - b(x)$$
Incentive Compatibility (IC) \iff

- \( b(x) \) is a convex function of \( x \) and
- \( q_i(x) = \frac{\partial b(x)}{\partial x_i} \) for a.e. \( x \) and all \( i \)
- \( q(x) = \nabla b(x) \) for a.e. \( x \)
- \( s(x) = \nabla b(x) \cdot x - b(x) \)
\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \text{ s.t.} \]
\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_+^k \to \mathbb{R}_+ \text{ s.t.} \]

\[ b \text{ is a convex function} \]
Maximal Revenue

\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_+^k \to \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)
$\mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \rightarrow \mathbb{R}_+ \text{ s.t.}

- b \text{ is a convex function, } b(0) = 0
- 0 \leq \partial b(x)/\partial x_i \leq 1 \text{ for a.e. } x$
Maximal Revenue

\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)

\[ R(b, \mathcal{F}) = \mathbb{E}_\mathcal{F}[\nabla b(x) \cdot x - b(x)] \]
$B^k = \text{ set of all functions } b : \mathbb{R}^k_+ \rightarrow \mathbb{R}_+ \text{ s.t.} $

- $b$ is a convex function, $b(0) = 0$
- $0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1$ for a.e. $x$

$R(b, \mathcal{F}) = \mathbb{E}_{\mathcal{F}}[\nabla b(x) \cdot x - b(x)]$

$= \mathbb{E}_{\mathcal{F}}[b'(x; x) - b(x)]$
Maximal Revenue

- \( \mathcal{B}^k \) = set of all functions \( b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \) s.t.
  - \( b \) is a convex function, \( b(0) = 0 \)
  - \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)
  
- \( R(b, \mathcal{F}) = E_\mathcal{F}[\nabla b(x) \cdot x - b(x)] \)
  = \( E_\mathcal{F}[b'(x; x) - b(x)] \)

- \( \text{REV}(\mathcal{F}) = \max_{b \in \mathcal{B}^k} R(b, \mathcal{F}) \)
Maximal Revenue

\[ \mathcal{B}_k = \text{set of all functions } b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)

\[ R(b, \mathcal{F}) = \mathbb{E}_\mathcal{F}[\nabla b(x) \cdot x - b(x)] = \mathbb{E}_\mathcal{F}[b'(x; x) - b(x)] \]

\[ \text{REV}(\mathcal{F}) = \max_{b \in \mathcal{B}_k} R(b, \mathcal{F}) \]

- \( \mathcal{B}_k \) is a closed convex set
Maximal Revenue

\( \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^+_k \rightarrow \mathbb{R}_+ \text{ s.t.} \)

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)

\[ R(b, \mathcal{F}) = \mathbb{E}_\mathcal{F}[\nabla b(x) \cdot x - b(x)] = \mathbb{E}_\mathcal{F}[b'(x; x) - b(x)] \]

\[ \text{REV}(\mathcal{F}) = \max_{b \in \mathcal{B}^k} R(b, \mathcal{F}) \]

- \( \mathcal{B}^k \) is a closed \textbf{convex} set
- \( R(b, \mathcal{F}) \) is \textbf{linear} in \( b \)
Maximal Revenue

- $\mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \text{ s.t.}$
- $b$ is a convex function, $b(0) = 0$
- $0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \text{ for a.e. } x$

\[
R(b, \mathcal{F}) = \mathbb{E}_\mathcal{F}[\nabla b(x) \cdot x - b(x)] = \mathbb{E}_\mathcal{F}[b'(x; x) - b(x)]
\]

$\text{REV}(\mathcal{F}) = \max_{b \in \mathcal{B}^k} R(b, \mathcal{F})$

- $\mathcal{B}^k$ is a closed convex set
- $R(b, \mathcal{F})$ is linear in $b$

$\text{REV}(\mathcal{F}) = \max_{b \in \text{EXT}(\mathcal{B}^k)} R(b, \mathcal{F})$

(EXT = set of extreme points)
Maximal Revenue: One Good

\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)

\[ R(b, \mathcal{F}) = \mathbb{E}_\mathcal{F}[\nabla b(x) \cdot x - b(x)] \]

\[ \text{REV}(\mathcal{F}) = \max_{b \in \mathcal{B}^k} R(b, \mathcal{F}) \]
Maximal Revenue: One Good

- \( B^1 \) = set of all functions \( b : \mathbb{R}_+^1 \to \mathbb{R}_+ \) s.t.
  - \( b \) is a convex function, \( b(0) = 0 \)
  - \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)
- \( R(b, \mathcal{F}) = \mathbb{E}_\mathcal{F}[\nabla b(x) \cdot x - b(x)] \)
- \( \text{REV}(\mathcal{F}) = \max_{b \in B^1} R(b, \mathcal{F}) \)
Maximal Revenue: One Good

\[ \mathcal{B}^1 = \text{set of all functions } b : \mathbb{R}^1_+ \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq b'(x) \leq 1 \) for a.e. \( x \)

\[ R(b, \mathcal{F}) = \mathbb{E}_{\mathcal{F}}[\nabla b(x) \cdot x - b(x)] \]

\[ \text{REV}(\mathcal{F}) = \max_{b \in \mathcal{B}^1} R(b, \mathcal{F}) \]
Maximal Revenue: One Good

- $\mathcal{B}^1 = \text{set of all functions } b : \mathbb{R}_+^1 \rightarrow \mathbb{R}_+ \text{ s.t.}$
- $b$ is a convex function, $b(0) = 0$
- $0 \leq b'(x) \leq 1 \text{ for a.e. } x$

- $R(b, \mathcal{F}) = E_{\mathcal{F}}[b'(x) \cdot x - b(x)]$
- $\text{REV}(\mathcal{F}) = \max_{b \in \mathcal{B}^1} R(b, \mathcal{F})$
Maximal Revenue: One Good

- $\mathcal{B}^1 = \text{set of all functions } b : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ s.t.}
  - b \text{ is a convex function, } b(0) = 0
  - 0 \leq b'(x) \leq 1 \text{ for a.e. } x$

- $\text{REV}(F) = \max_{b \in \mathcal{B}^1} \mathbb{E}_F[b'(x) \cdot x - b(x)]$
Maximal Revenue: One Good

\[ \mathcal{B}^1 = \text{set of all functions } b : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ s.t.} \]

\[ b \text{ is a convex function, } b(0) = 0 \]

\[ 0 \leq b'(x) \leq 1 \text{ for a.e. } x \]

\[ \text{REV}(F) = \max_{b \in \mathcal{B}^1} \mathbb{E}_F[b'(x) \cdot x - b(x)] \]
Maximal Revenue: One Good

- \( \mathcal{B}^1 \) = set of all functions \( b : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) s.t.
  - \( b \) is a convex function, \( b(0) = 0 \)
  - \( 0 \leq b'(x) \leq 1 \) for a.e. \( x \)

- \( \text{REV}(F) = \max_{b \in \mathcal{B}^1} \mathbb{E}_F[b'(x) \cdot x - b(x)] \)

- \( \mathcal{B}^1 = \) closed convex hull of \( \{b_p\}_{p \geq 0} \) where \( b_p(x) = \max\{0, x - p\} \)
Maximal Revenue: One Good

- $\mathcal{B}^1 = \text{set of all functions } b : \mathbb{R}_+ \to \mathbb{R}_+ \text{ s.t.}$
  - $b$ is a convex function, $b(0) = 0$
  - $0 \leq b'(x) \leq 1$ for a.e. $x$

- $\text{REV}(F) = \max_{b \in \mathcal{B}^1} E_F[b'(x) \cdot x - b(x)]$

- $\mathcal{B}^1 = \text{closed convex hull of } \{b_p\}_{p \geq 0}$ where
  - $b_p(x) = \max\{0, x - p\}$

- $\text{REV}(F) = \max_{p \geq 0} E_F[b_p'(x) \cdot x - b_p(x)]$
Maximal Revenue: One Good

- \( \mathcal{B}^1 \) = set of all functions \( b : \mathbb{R}_+ \to \mathbb{R}_+ \) s.t.
  - \( b \) is a convex function, \( b(0) = 0 \)
  - \( 0 \leq b'(x) \leq 1 \) for a.e. \( x \)

- \( \text{REV}(F) = \max_{b \in \mathcal{B}^1} \mathbb{E}_F[b'(x) \cdot x - b(x)] \)

- \( \mathcal{B}^1 = \) closed convex hull of \( \{b_p\}_{p \geq 0} \) where
  \( b_p(x) = \max\{0, x - p\} \)

- \( \text{REV}(F) = \max_{p \geq 0} \mathbb{E}_F[b'_p(x) \cdot x - b_p(x)] \)
  \( = \max_{p \geq 0} \mathbb{E}_F[(x - (x - p))1_{x \geq p}] \)
Maximal Revenue: One Good

- \( \mathcal{B}^1 \) = set of all functions \( b : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) s.t.
  - \( b \) is a convex function, \( b(0) = 0 \)
  - \( 0 \leq b'(x) \leq 1 \) for a.e. \( x \)

- \( \text{REV}(F) = \max_{b \in \mathcal{B}^1} \mathbb{E}_F[b'(x) \cdot x - b(x)] \)

- \( \mathcal{B}^1 = \) closed convex hull of \( \{b_p\}_{p \geq 0} \) where \( b_p(x) = \max\{0, x - p\} \)

- \( \text{REV}(F) = \max_{p \geq 0} \mathbb{E}_F[b'_p(x) \cdot x - b_p(x)] \)
  \[= \max_{p \geq 0} \mathbb{E}_F[(x - (x - p))1_{x \geq p}] \]
  \[= \max_{p \geq 0} p \cdot (1 - F(p)) \]
Maximal Revenue: \( k \geq 2 \) Goods
Maximal Revenue: $k \geq 2$ Goods

\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- $b$ is a convex function, $b(0) = 0$
- $0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1$ for a.e. $x$
Maximal Revenue: $k \geq 2$ Goods

- $\mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \rightarrow \mathbb{R}_+$ s.t.
  - $b$ is a convex function, $b(0) = 0$
  - $0 \leq \partial b(x)/\partial x_i \leq 1$ for a.e. $x$

- $\text{REV}(\mathcal{F}) = \max_{b \in \text{EXT}(\mathcal{B}^k)} R(b, \mathcal{F})$
Maximal Revenue: $k \geq 2$ Goods

\[ \mathcal{B}_k = \text{set of all functions } b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- $b$ is a convex function, $b(0) = 0$
- $0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1$ for a.e. $x$

\[ \text{REV}(\mathcal{F}) = \max_{b \in \text{EXT}(\mathcal{B}_k)} R(b, \mathcal{F}) \]

\[ \text{EXTREME points of } \mathcal{B}_k = ? \]
Maximal Revenue: $k \geq 2$ Goods

- $\mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \to \mathbb{R}_+ \text{ s.t.} \$
  - $b$ is a convex function, $b(0) = 0$
  - $0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \text{ for a.e. } x$

- $\text{REV}(\mathcal{F}) = \max_{b \in \text{EXT}(\mathcal{B}^k)} R(b, \mathcal{F})$

- EXTREME points of $\mathcal{B}^k = ?$

EXTREMELY COMPLEX!
Two Goods: Example 1
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
22 & \text{with probability } \frac{1}{2}
\end{cases} \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

Separate:

\[ \text{Rev}(X) + \text{Rev}(Y) \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2
\end{cases} \]

Separate:

\[ \text{Rev}(X) + \text{Rev}(Y) \]
\[ \max(10 \cdot 1, 22 \cdot 1/2) = 11 \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

Separate:
\[
REV(X) + REV(Y) = 11 + 11 = 22 \\
\max(10 \cdot 1, 22 \cdot 1/2) = 11
\]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

Separate:

\[ \text{REV}(X) + \text{REV}(Y) = 11 + 11 = 22 \]
Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
22 & \text{with probability } \frac{1}{2}
\end{cases} \]

- Separate:
  \[ \text{REV}(X) + \text{REV}(Y) = 11 + 11 = 22 \]

- Bundled:
  \[ \text{REV}(X + Y) \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- **Separate:**
  \[ \text{REV}(X) + \text{REV}(Y) = 11 + 11 = 22 \]

- **Bundled:**
  \[ \text{REV}(X + Y) \]
  \[ \max(20 \cdot 1, 32 \cdot 3/4, 44 \cdot 1/4) = 24 \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- Separate:
  \[ \text{REV}(X) + \text{REV}(Y) = 11 + 11 = 22 \]

- Bundled:
  \[ \text{REV}(X + Y) = 32 \cdot 3/4 = 24 \]
  \[ \max(20 \cdot 1, 32 \cdot 3/4, 44 \cdot 1/4) = 24 \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- **Separate:**
  \[ \text{REV}(X) + \text{REV}(Y) = 11 + 11 = 22 \]

- **Bundled:**
  \[ \text{REV}(X + Y) = 32 \cdot 3/4 = 24 \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases} \]

- **Separate:**
  \[ \text{REV}(X) + \text{REV}(Y) = 11 + 11 = 22 \]

- **Bundled:**
  \[ \text{REV}(X + Y) = 32 \cdot 3/4 = 24 \]  
  \( \text{PRICE FOR THE BUNDLE} \)
Two Goods: Example 2
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

Separate:
\[ \text{REV}(X) + \text{REV}(Y) \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 50 & \text{with probability } 1/2 \end{cases} \]

Separate:

\[ \text{REV}(X) + \text{REV}(Y) \]
\[ \max(10 \cdot 1, 50 \cdot 1/2) = 25 \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 10 & \text{with probability } \frac{1}{2} \\ 50 & \text{with probability } \frac{1}{2} \end{cases} \]

Separate:

\[ \text{REV}(X) + \text{REV}(Y) = 25 + 25 = 50 \]
\[ \max(10 \cdot 1, 50 \cdot \frac{1}{2}) = 25 \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

Separate:

\[ \text{REV}(X) + \text{REV}(Y) = 25 + 25 = 50 \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

- **Separate:**
  \[ \text{REV}(X) + \text{REV}(Y) = 25 + 25 = 50 \]

- **Bundled:**
  \[ \text{REV}(X + Y) \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

- **Separate:**
  \[ \text{Rev}(X) + \text{Rev}(Y) = 25 + 25 = 50 \]

- **Bundled:**
  \[ \text{Rev}(X + Y) = \max(20 \cdot 1, 60 \cdot 3/4, 100 \cdot 1/4) = 45 \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

Separate:

\[ \text{REV}(X) + \text{REV}(Y) = 25 + 25 = 50 \]

Bundled:

\[ \text{REV}(X + Y) = 60 \cdot 3/4 = 45 \]
\[ \max(20 \cdot 1, 60 \cdot 3/4, 100 \cdot 1/4) = 45 \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

- **Separate:**
  \[ \text{REV}(X) + \text{REV}(Y) = 25 + 25 = 50 \]

- **Bundled:**
  \[ \text{REV}(X + Y) = 60 \cdot \frac{3}{4} = 45 \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ X, Y \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

Separate:

\[ \text{REV}(X) + \text{REV}(Y) = 25 + 25 = 50 \]

Bundled:

\[ \text{REV}(X + Y) = 60 \cdot 3/4 = 45 \]

PRICE FOR EACH GOOD
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } \frac{1}{3} \\
1 & \text{w/probability } \frac{1}{3} \\
2 & \text{w/probability } \frac{1}{3}
\end{cases} \quad \text{(IID)} \]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } \frac{1}{3} \\
1 & \text{w/probability } \frac{1}{3} \\
2 & \text{w/probability } \frac{1}{3}
\end{cases} \quad \text{(IID)} \]

Separate:
\[ \max(0 \cdot 1, 1 \cdot \frac{2}{3}, 2 \cdot \frac{1}{3}) = \frac{2}{3} \]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)}
\]

- **Separate:** \[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]
\[ \max(0 \cdot 1, 1 \cdot \frac{2}{3}, 2 \cdot \frac{1}{3}) = \frac{2}{3} \]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)} \]

Separate: \[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } \frac{1}{3} \\
1 & \text{w/probability } \frac{1}{3} \\
2 & \text{w/probability } \frac{1}{3} 
\end{cases} \quad \text{(IID)}
\]

- **Separate:** \[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]
- **Bundled:**
  \[ \max(0 \cdot 1, 1 \cdot \frac{8}{9}, 2 \cdot \frac{6}{9}, 3 \cdot \frac{3}{9}, 4 \cdot \frac{1}{9}) \]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } \frac{1}{3} \\
1 & \text{w/probability } \frac{1}{3} \\
2 & \text{w/probability } \frac{1}{3} 
\end{cases} \quad \text{(IID)} \]

- **Separate:** \[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]
- **Bundled:** \[
\max(0 \cdot 1, 1 \cdot \frac{8}{9}, 2 \cdot \frac{6}{9}, 3 \cdot \frac{3}{9}, 4 \cdot \frac{1}{9}) = \frac{4}{3}
\]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3
\end{cases} \quad \text{(IID)}

- Separate: \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)

- Bundled: \( R = \frac{4}{3} \)
  \[
  \max(0 \cdot 1, 1 \cdot \frac{8}{9}, 2 \cdot \frac{6}{9}, 3 \cdot \frac{3}{9}, 4 \cdot \frac{1}{9}) = \frac{4}{3}
  \]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } \frac{1}{3} \\
1 & \text{w/probability } \frac{1}{3} \\
2 & \text{w/probability } \frac{1}{3}
\end{cases} \quad \text{(IID)}
\]

- **Separate:** \[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]
- **Bundled:** \[ R = \frac{4}{3} \]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \quad \text{(IID)} \\
2 & \text{w/probability } 1/3 
\end{cases} \]

- **Separate:** \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)
- **Bundled:** \( R = \frac{4}{3} \)
- \( b(x, y) = \max(0, x - 2, y - 2, x + y - 3) \)
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3 \\
\end{cases} \quad \text{(IID)} \]

- **Separate:** \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)
- **Bundled:** \( R = \frac{4}{3} \)

\( b(x, y) = \max(0, x - 2, y - 2, x + y - 3) \)

\[
\begin{align*}
  s(2, 0) &= s(0, 2) = 2 \\
  s(2, 1) &= s(1, 2) = s(2, 2) = 3
\end{align*}
\]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 \text{ w/probability } 1/3 \\
1 \text{ w/probability } 1/3 \\
2 \text{ w/probability } 1/3
\end{cases} \]  

(IID)

Separate: \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)

Bundled: \( R = \frac{4}{3} \)

\[ b(x, y) = \max(0, x - 2, y - 2, x + y - 3) \]

\[ s(2, 0) = s(0, 2) = 2 \]

\[ s(2, 1) = s(1, 2) = s(2, 2) = 3 \]

\[ R = 2 \cdot \frac{2}{9} + 3 \cdot \frac{3}{9} = \frac{13}{9} \]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3 
\end{cases} \] (IID)

- **Separate:** \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)
- **Bundled:** \( R = \frac{4}{3} \)
- \( b(x, y) = \max(0, x - 2, y - 2, x + y - 3) \)
  \[ R(b) = \frac{13}{9} \]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)}
\]

- Separate: \[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]
- Bundled: \[ R = \frac{4}{3} \]

\[ b(x, y) = \max(0, x - 2, y - 2, x + y - 3) \]

\[ R(b) = \frac{13}{9} \]
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 
0 & \text{w/probability } \frac{1}{3} \\
1 & \text{w/probability } \frac{1}{3} \\
2 & \text{w/probability } \frac{1}{3} 
\end{cases} \quad \text{(IID)} \]

- **Separate:** \[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]
- **Bundled:** \[ R = \frac{4}{3} \]

\[ b(x, y) = \max(0, x - 2, y - 2, x + y - 3) \]

\[ R(b) = \frac{13}{9} = \text{REV}(X, Y) \]

THE UNIQUE OPTIMAL MECHANISM
Two Goods: Example 3

\[ X, Y \sim \begin{cases} 0 & \text{w/probability } \frac{1}{3} \\ 1 & \text{w/probability } \frac{1}{3} \\ 2 & \text{w/probability } \frac{1}{3} \end{cases} \quad \text{(IID)} \]

\[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]

\[ b(x, y) = \max(0, x - 2, y - 2, x + y - 3) \]

\[ R(b) = \frac{13}{9} = \text{Rev}(X, Y) \]

The unique optimal mechanism

Price for each good and for bundle
Two Goods: Example 4

\( (X, Y) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases} \)
Two Goods: Example 4

\[(X, Y) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases} \]

\[b(x, y) = \max(0, \frac{1}{2}x - \frac{1}{2}, y - 2, x + y - 5)\]
Two Goods: Example 4

\[(X, Y) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3
\end{cases} \]

\[b(x, y) = \max(0, \frac{1}{2} x - \frac{1}{2}, y - 2, x + y - 5)\]

\[R(b) = 2.5\]
Two Goods: Example 4

\[(X, Y) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3
\end{cases}\]

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\end{cases} \]

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THE UNIQUE OPTIMAL MECHANISM
Two Goods: Example 4

\[(X, Y) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
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**THE UNIQUE OPTIMAL MECHANISM**

\[b_1(x, y) = \max(0, x - 1, y - 2, x + y - \_\_\_)\]
Two Goods: Example 4

\[(X, Y) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3
\end{cases}\]

\[b(x, y) = \max(0, \frac{1}{2}x - \frac{1}{2}, y - 2, x + y - 5)\]

\[R(b) = 2.5 = \text{REV}(X, Y)\]

**THE UNIQUE OPTIMAL MECHANISM**

\[b_1(x, y) = \max(0, x - 1, y - 2, x + y - \ldots)\]

\[b_0(x, y) = \max(0, y - 2, x + y - \ldots)\]
Two Goods: Example 4

\[(X, Y) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases}\]

\[b(x, y) = \max(0, \frac{1}{2}x - \frac{1}{2}, y - 2, x + y - 5)\]

\[R(b) = 2.5 = \text{Rev}(X, Y)\]

THE UNIQUE OPTIMAL MECHANISM

\[b_1(x, y) = \max(0, x - 1, y - 2, x + y - 4)\]

\[b_0(x, y) = \max(0, y - 2, x + y - \quad )\]
Two Goods: Example 4

\[(X, Y) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases} \]

\[b(x, y) = \max(0, \frac{1}{2}x - \frac{1}{2}, y - 2, x + y - 5)\]

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THE UNIQUE OPTIMAL MECHANISM

\[b_1(x, y) = \max(0, x - 1, y - 2, x + y - 4)\]

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Two Goods: Example 4

\[(X, Y) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
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(3, 3) & \text{w/probability } 1/3 
\end{cases} \]

\[b(x, y) = \max(0, \frac{1}{2} x - \frac{1}{2}, y - 2, x + y - 5)\]

\[R(b) = 2.5 = \text{Rev}(X, Y)\]

**THE UNIQUE OPTIMAL MECHANISM**

\[b_1(x, y) = \max(0, x - 1, y - 2, x + y - 4)\]

\[R(b_1) = 2.33...\]

\[b_0(x, y) = \max(0, y - 2, x + y - 5)\]
Two Goods: Example 4

\[(X, Y) \sim \begin{cases} (1, 0) & \text{w/probability } 1/3 \\ (0, 2) & \text{w/probability } 1/3 \\ (3, 3) & \text{w/probability } 1/3 \end{cases} \]

\[b(x, y) = \max(0, \frac{1}{2}x - \frac{1}{2}, y - 2, x + y - 5)\]

\[R(b) = 2.5 = \text{REV}(X, Y)\]

THE UNIQUE OPTIMAL MECHANISM

\[b_1(x, y) = \max(0, x - 1, y - 2, x + y - 4)\]

\[R(b_1) = 2.33\ldots\]

\[b_0(x, y) = \max(0, y - 2, x + y - 5)\]

\[R(b_0) = 2.33\ldots\]
Two Goods: Example 4

\[(X, Y) \sim \begin{cases} 
(1, 0) \text{ w/probability } 1/3 \\
(0, 2) \text{ w/probability } 1/3 \\
(3, 3) \text{ w/probability } 1/3 
\end{cases}\]

\[b(x, y) = \max(0, \frac{1}{2}x - \frac{1}{2}, y - 2, x + y - 5)\]

\[R(b) = 2.5 = \text{REV}(X, Y)\]

THE UNIQUE OPTIMAL MECHANISM

PRICE FOR LOTTERIES ON GOODS
Two Goods: Example 4’

\[ X, Y \sim \begin{cases} 
1 & \text{w/ probability } 1/6 \\
2 & \text{w/ probability } 1/2 \\
4 & \text{w/ probability } 1/3 
\end{cases} \quad \text{(IID)} \]
Two Goods: Example 4’

\[ X, Y \sim \begin{cases} 
1 & \text{w/probability } 1/6 \\
2 & \text{w/probability } 1/2 \\
4 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)}

**THE UNIQUE OPTIMAL MECHANISM:**

\[ b(x, y) = \max(0, \frac{1}{2}x - 1, \frac{1}{2}y - 1, x + y - 4) \]
Two Goods: Example 4’

\[ X, Y \sim \begin{cases} 
1 & \text{w/probability } 1/6 \\
2 & \text{w/probability } 1/2 \\
4 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)}

THE UNIQUE OPTIMAL MECHANISM:

\[ b(x, y) = \max(0, \frac{1}{2}x - 1, \frac{1}{2}y - 1, x + y - 4) \]

PRICE FOR LOTTERIES ON GOODS
Revenue maximizing mechanisms:
Multiple Goods

Revenue maximizing mechanisms:

1. post a price for each good separately
Revenue maximizing mechanisms:

1. post a price for each good separately
2. post a price for the bundle
Revenue maximizing mechanisms:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
Multiple Goods

Revenue maximizing mechanisms:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries
Revenue maximizing mechanisms:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries

1 – 3: deterministic mechanisms
4: stochastic mechanisms
Revenue maximizing mechanisms:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries
Revenue maximizing mechanisms:

1. post a price for each good separately
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4. post prices for various lotteries

Multiple Goods, I.I.D. Uniform

\[ X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \]
Multiple Goods, I.I.D. Uniform

\[ X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \]

- \( k = 1: b(x) = \max(0, x_1 - \frac{1}{2}) \)
Multiple Goods, I.I.D. Uniform

$X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.}$

- $k = 1$: $b(x) = \max(0, x_1 - \frac{1}{2})$
- $k = 2$: $b(x) = \max(0, x_i - \frac{2}{3}, x_1 + x_2 - \frac{4-\sqrt{2}}{3})$
Multiple Goods, I.I.D. Uniform

\[ X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \]

- \( k = 1 \): \( b(x) = \max(0, x_1 - \frac{1}{2}) \)
- \( k = 2 \):
  \[ b(x) = \max(0, x_i - \frac{2}{3}, x_1 + x_2 - \frac{4-\sqrt{2}}{3}) \]
- \( k = 3 \): \( b(x) = \max(0, x_i - \frac{3}{4}, x_i + x_j - \frac{6-\sqrt{2}}{4}, x_1 + x_2 + x_3 - s) \)
Multiple Goods, I.I.D. Uniform

\[ X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \]

- \( k = 1: b(x) = \max(0, x_1 - \frac{1}{2}) \)
- \( k = 2: b(x) = \max(0, x_i - \frac{2}{3}, x_1 + x_2 - \frac{4 - \sqrt{2}}{3}) \)
- \( k = 3: b(x) = \max(0, x_i - \frac{3}{4}, x_i + x_j - \frac{6 - \sqrt{2}}{4}, x_1 + x_2 + x_3 - s) \)

where \( s = \frac{9}{4} - \frac{\sqrt{6}}{4} \cos\left(\frac{1}{3} \arctan\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right) \)

\[ -\frac{3\sqrt{2}}{4} \sin\left(\frac{1}{3} \arctan\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right) \]
Multiple Goods, I.I.D. Uniform

\( X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \)

- \( k = 1: \ b(x) = \max(0, \ x_1 - \frac{1}{2}) \)
- \( k = 2: \)
  \[
  b(x) = \max(0, \ x_i - \frac{2}{3}, \ x_1 + x_2 - \frac{4-\sqrt{2}}{3})
  \]
- \( k = 3: \ b(x) = \max(0, \ x_i - \frac{3}{4}, \ x_i + x_j - \frac{6-\sqrt{2}}{4}, \ x_1 + x_2 + x_3 - s) \)

where \( s \approx 1.2257... \) = solution of 3rd degree equation with coefficients in \( \mathbb{Q}[\sqrt{2}] \)
Multiple Goods, I.I.D. Uniform

\[ X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \]

- \( k = 1: \quad b(x) = \max(0, x_1 - \frac{1}{2}) \)
- \( k = 2: \quad b(x) = \max(0, x_i - \frac{2}{3}, x_1 + x_2 - \frac{4-\sqrt{2}}{3}) \)
- \( k = 3: \quad b(x) = \max(0, x_i - \frac{3}{4}, x_i + x_j - \frac{6-\sqrt{2}}{4}, x_1 + x_2 + x_3 - s) \)
  
  \[ \ldots \]
Multiple Goods, I.I.D. Uniform

$X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.}$

- $k = 1$: $b(x) = \max(0, x_1 - \frac{1}{2})$
- $k = 2$:
  \[
  b(x) = \max(0, x_i - \frac{2}{3}, x_1 + x_2 - \frac{4-\sqrt{2}}{3})
  \]
- $k = 3$: $b(x) = \max(0, x_i - \frac{3}{4}, x_i + x_j - \frac{6-\sqrt{2}}{4}, x_1 + x_2 + x_3 - s) \ldots$

Manelli & Vincent 2006, Hart & Reny 2010
Monotonicity
If valuations of **BUYER** increase
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: 
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $y > x$. 
If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $y > x$.

$$q(x)x - s(x) \geq q(y)x - s(y) \quad (\text{IC: } x \rightarrow y)$$
Monotonicity

If valuations of BUYER increase then maximal revenue of SELLER increases (weakly)

Proof for $k = 1$: Let $y > x$.

\[ q(x)x - s(x) \geq q(y)x - s(y) \quad (\text{IC: } x \rightarrow y) \]
\[ q(y)y - s(y) \geq q(x)y - s(x) \quad (\text{IC: } y \rightarrow x) \]
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $y > x$.

\[
q(x)x - s(x) \geq q(y)x - s(y) \quad \text{(IC: } x \rightarrow y) \\
q(y)y - s(y) \geq q(x)y - s(x) \quad \text{(IC: } y \rightarrow x) \\
\Rightarrow (q(y) - q(x))(y - x) \geq 0 \quad \text{(add)}
\]
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** \( k = 1 \): Let \( y > x \).

\[
q(x)x - s(x) \geq q(y)x - s(y) \quad \text{(IC: } x \rightarrow y) \\
q(y)y - s(y) \geq q(x)y - s(x) \quad \text{(IC: } y \rightarrow x) \\
\Rightarrow (q(y) - q(x))(y - x) \geq 0 \quad \text{(add)} \\
\Rightarrow q(y) \geq q(x) \quad \text{(} y > x \text{)}
\]
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** \( k = 1 \): Let \( y > x \).

\[
\begin{align*}
q(x)x - s(x) & \geq q(y)x - s(y) \quad \text{(IC: } x \rightarrow y) \\
q(y)y - s(y) & \geq q(x)y - s(x) \quad \text{(IC: } y \rightarrow x) \\
\Rightarrow (q(y) - q(x))(y - x) & \geq 0 \quad \text{(add)} \\
\Rightarrow q(y) & \geq q(x) \quad \text{(} y > x \text{)} \\
s(y) - s(x) & \geq (q(y) - q(x))x \quad \text{(IC: } x \rightarrow y) 
\end{align*}
\]
Monotonicity

If valuations of **BUYER** *increase* then maximal revenue of **SELLER** *increases* (weakly)

**Proof for** $k = 1$: Let $y > x$.

\[ q(x)x - s(x) \geq q(y)x - s(y) \quad \text{(IC: } x \rightarrow y) \]
\[ q(y)y - s(y) \geq q(x)y - s(x) \quad \text{(IC: } y \rightarrow x) \]
\[ \Rightarrow (q(y) - q(x))(y - x) \geq 0 \quad \text{(add)} \]
\[ \Rightarrow q(y) \geq q(x) \quad (y > x) \]
\[ s(y) - s(x) \geq (q(y) - q(x))x \quad \text{(IC: } x \rightarrow y) \]
\[ \Rightarrow s(y) - s(x) \geq 0 \]
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $y > x$.

\[ \Rightarrow s(y) - s(x) \geq 0 \]
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$:

- $y > x \Rightarrow s(y) \geq s(x)$
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$:

- $y > x \Rightarrow s(y) \geq s(x)$
- Every **IC** mechanism has **monotonic** $s$
Monotonicity

If valuations of BUYER increase then maximal revenue of SELLER increases (weakly)

Proof for $k = 1$:

- $y > x \Rightarrow s(y) \geq s(x)$
- Every IC mechanism has monotonic $s$
- $\Rightarrow$ Revenue of every IC mechanism is monotonic w.r.t. to BUYER valuations
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

Proof for $k = 1$:

- $y > x \implies s(y) \geq s(x)$
- Every **IC** mechanism has **monotonic** $s$
- $\implies$ Revenue of every **IC** mechanism is **monotonic** w.r.t. **BUYER** valuations
- $\implies$ Maximal revenue is **monotonic** w.r.t. **BUYER** valuations
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$:

- $y > x \implies s(y) \geq s(x)$
- Every **IC** mechanism has monotonic $s$
- $\implies$ Revenue of every **IC** mechanism is monotonic w.r.t. to **BUYER** valuations
- $\implies$ Maximal revenue is monotonic w.r.t. **BUYER** valuations

**Proof for** $k > 1$ ?
$b(x, y) = \max(0, x - 10, y - 20, x + y - 40)$
Non-Monotonicity: Example

\[ b(x, y) = \max(0, x - 10, y - 20, x + y - 40) \]
Non-Monotonicity: Example

\[ b(x, y) = \max(0, x - 10, y - 20, x + y - 40) \]
Non-Monotonicity: Example

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Non-Monotonicity: Example

\[ b(x, y) = \max(0, x - 10, y - 20, x + y - 40) \]
Non-Monotonicity: Example

\[ b(x, y) = \max(0, x - 10, y - 20, x + y - 40) \]

(12, 24) : \( y - 20 \)

(18, 26) : \( x - 10 \)
Non-Monotonicity: Example

\[ b(x, y) = \max(0, x - 10, y - 20, x + y - 40) \]

- **(12, 24)**: \( y - 20 \)
- **(18, 26)**: \( x - 10 \)

- **x** increases
- **y** increases
- \( s \) **DECREASES**!
Non-Monotonicity: Example

\[ b(x, y) = \max(0, x - 10, y - 20, x + y - 40) \]
Non-Monotonicity: Example

\[ b(x, y) = \max(0, x - 10, y - 20, x + y - 40) \]

There exist distributions \( \mathcal{F} \) for which this \( b \) MAXIMIZES REVENUE.
Non-Monotonicity: Example

\[ b(x, y) = \max(0, x - 10, y - 20, x + y - 40) \]

There exist distributions \( F \) for which this \( b \) **MAXIMIZES REVENUE**
(moreover: unique maximizer; robust)
Non-Monotonicity

\[ b(x, y) = \max(0, x - 10, y - 20, x + y - 40) \]

- There exist distributions \( F \) for which this \( b \) **MAXIMIZES REVENUE** (moreover: unique maximizer; robust)

- **NON-MONOTONICITY** occurs also for **I.I.D.**
Summary: Multiple Goods
Maximizing revenue with multiple goods:
Maximizing revenue with multiple goods:

- many of the results for ONE GOOD are FALSE for MULTIPLE GOODS
Maximizing revenue with multiple goods:

- many of the results for ONE GOOD are FALSE for MULTIPLE GOODS
- is an extremely complex problem (even for simple distributions)
Maximizing revenue with multiple goods:

- many of the results for **ONE GOOD** are **FALSE** for **MULTIPLE GOODS**
- is an extremely complex problem (even for simple distributions)
- “**what we have learned from one good is too good to be true for two goods**”
Maximizing revenue with multiple goods:

- many of the results for ONE GOOD are FALSE for MULTIPLE GOODS
- is an extremely complex problem (even for simple distributions)
- “what we have learned from one good is too good to be true for two goods”
- ?
Maximizing revenue with multiple goods:

- many of the results for ONE GOOD are FALSE for MULTIPLE GOODS
- is an extremely complex problem (even for simple distributions)
- “what we have learned from one good is too good to be true for two goods”

APPROXIMATION using SIMPLE mechanisms?
Two Goods, Independent
MAXIMAL REVENUE of selling SEPARATELY:

\[ S \text{Rev}(F_1 \times F_2) = \text{Rev}(F_1) + \text{Rev}(F_2) \]
Two Goods, Independent

MAXIMAL REVENUE of selling SEPARATELY:

\[
\text{SRev}(F_1 \times F_2) = \text{Rev}(F_1) + \text{Rev}(F_2)
\]

Theorem 1. For any two independent goods:

\[
\text{SRev}(F_1 \times F_2) \geq \frac{1}{2} \text{Rev}(F_1 \times F_2)
\]
Two Goods, Independent

MAXIMAL REVENUE of selling SEPARATELY:

\[ \text{SRev}(F_1 \times F_2) = \text{Rev}(F_1) + \text{Rev}(F_2) \]

Theorem 1. For any two independent goods:

\[ \text{SRev}(F_1 \times F_2) \geq \frac{1}{2} \text{Rev}(F_1 \times F_2) \]

Theorem 2. For any two i.i.d. goods:

\[ \text{SRev}(F \times F) \geq \frac{e}{e+1} \text{Rev}(F \times F) \]

\[ \frac{e}{e+1} \approx 73\% \]
Theorem 2. For any two i.i.d. goods:

\[ \text{SREV}(F \times F) \geq \frac{e}{e+1} \text{REV}(F \times F) \]

\[ \frac{e}{e+1} \approx 73\% \]
Theorem 2. For any two i.i.d. goods:

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\[
\text{SRev}(F \times F) = 2 \text{Rev}(F) = 2p^* \cdot (1 - F(p^*))
\]
Two Goods, I.I.D.

Theorem 2. For any two i.i.d. goods:

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Posting the optimal one-good price per unit guarantees at least 73% of the optimal revenue.
Two Goods: Theorem 1
Two Goods: Theorem 1

\[ S_{\text{REV}}(F_1 \times F_2) \geq \frac{1}{2} R_{\text{REV}}(F_1 \times F_2) \]
Two Goods: Theorem 1

\[ \text{SR}_{\text{EV}}(F_1 \times F_2) \geq \frac{1}{2} \text{REV}(F_1 \times F_2) \]

Proof.
Two Goods: Theorem 1

$$\text{SREV}(F_1 \times F_2) \geq \frac{1}{2} \text{REV}(F_1 \times F_2)$$

Proof. Let $X \sim F_1$, $Y \sim F_2$, independent
Two Goods: Theorem 1

**SRev**(\(F_1 \times F_2\)) \(\geq\) \(\frac{1}{2}\) **Rev**(\(F_1 \times F_2\))

**Proof.** Let \(X \sim F_1, Y \sim F_2\), independent

\[
\text{Rev}(X, Y) \\
\leq \text{Rev}((X, Y)1_{x \geq y}) + \text{Rev}((X, Y)1_{y \geq x})
\]
Two Goods: Theorem 1

\[ \text{SERev}(F_1 \times F_2) \geq \frac{1}{2} \text{Rev}(F_1 \times F_2) \]

**Proof.** Let \( X \sim F_1, \ Y \sim F_2 \), independent

- \( \text{Rev}(X, Y) \)
  \[ \leq \text{Rev}((X, Y)1_{X \geq Y}) + \text{Rev}((X, Y)1_{Y \geq X}) \]

- **Claim.** \( \text{Rev}((X, Y)1_{X \geq Y}) \leq 2 \text{Rev}(X) \)
Two Goods: Theorem 1

\[ \text{SREV}(F_1 \times F_2) \geq \frac{1}{2} \text{REV}(F_1 \times F_2) \]

**Proof.** Let \( X \sim F_1, \ Y \sim F_2, \) independent

\[ \text{REV}(X, Y) \leq \text{REV}((X, Y)1_{X \geq Y}) + \text{REV}((X, Y)1_{Y \geq X}) \]

**Claim.** \[ \text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X) \]
\[ \text{REV}((X, Y)1_{Y \geq X}) \leq 2 \text{REV}(Y) \]
Two Goods: Theorem 1

\[ \text{SR}_{\text{REV}}(F_1 \times F_2) \geq \frac{1}{2} \text{REV}(F_1 \times F_2) \]

Proof. Let \( X \sim F_1, Y \sim F_2 \), independent

- \( \text{REV}(X, Y) \)
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Claim. \( \text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X) \)
\( \text{REV}((X, Y)1_{Y \geq X}) \leq 2 \text{REV}(Y) \)

\[ \Rightarrow \text{REV}(X, Y) \leq 2 \text{REV}(X) + 2 \text{REV}(Y) \]
Two Goods: Theorem 1

\[ \text{SRev}(F_1 \times F_2) \geq \frac{1}{2} \text{Rev}(F_1 \times F_2) \]

Proof. Let \( X \sim F_1, \ Y \sim F_2 \), independent

\[ \text{Rev}(X, Y) \leq \text{Rev}((X, Y)1_{X \geq Y}) + \text{Rev}((X, Y)1_{Y \geq X}) \]

Claim. \( \text{Rev}((X, Y)1_{X \geq Y}) \leq 2 \text{Rev}(X) \)
\( \text{Rev}((X, Y)1_{Y \geq X}) \leq 2 \text{Rev}(Y) \)

\[ \Rightarrow \text{Rev}(X, Y) \leq 2 \text{Rev}(X) + 2 \text{Rev}(Y) = 2 \text{SRev}(X, Y) \]
Two Goods: Theorem 1

Claim. \( \text{REV}( (X, Y) \mathbb{1}_{X \geq Y} ) \leq 2 \text{REV}(X) \)
Claim. $\text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X)$
Claim. $\text{REV}((X, Y)_{1_{X \geq Y}}) \leq 2 \text{REV}(X)$

Proof.
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Proof.

Let $(q, s)$ be IC&IR for $(X, Y)$.
Claim. $\text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X)$

Proof.

Let $(q, s)$ be IC&IR for $(X, Y)$. For every fixed $y$: 
Claim. \( \text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X) \)

Proof.

Let \((q, s)\) be IC&IR for \((X, Y)\).
For every fixed \(y\):
Instead of giving \(y\) with probability \(q_2\),
give a "monetary refund" = \(q_2y\):
Claim. $\text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X)$

Proof.

Let $(q, s)$ be IC&IR for $(X, Y)$. For every fixed $y$:

Instead of giving $y$ with probability $q_2$, give a "monetary refund" $= q_2 y$:

- $\tilde{q}(x) := q_1(x, y)$
- $\tilde{s}(x) := s(x, y) - q_2(x, y)y$
Claim. $\text{REV}((X, Y)_{1_{X \geq Y}}) \leq 2 \text{REV}(X)$

Proof.

Let $(q, s)$ be IC&IR for $(X, Y)$. For every fixed $y$:
- Instead of giving $y$ with probability $q_2$, give a "monetary refund" $= q_2y$:

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Then: $(\tilde{q}, \tilde{s})$ is IC&IR for $X$. 
Claim. $\text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X)$

Proof.

- Let $(q, s)$ be IC&IR for $(X, Y)$.
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    Instead of giving $y$ with probability $q_2$, give a "monetary refund" $= q_2 y$:

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Then: $(\tilde{q}, \tilde{s})$ is IC&IR for $X$.

$\text{REV}(X) \geq \mathbb{E}[\tilde{s}(X)]$
Claim. \( \text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X) \)

Proof.

\( \tilde{s}(x) = s(x, y) - q_2(x, y)y \)

\( \text{REV}(X) \geq E[\tilde{s}(X)] \)
Claim. $\text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X)$

Proof. For every $y$:

- $\tilde{s}(x) = s(x, y) - q_2(x, y)y$
- $\text{REV}(X) \geq \mathbb{E}[\tilde{s}(X)] \geq \mathbb{E}[\tilde{s}(X)1_{X \geq y}]$
Claim. $\text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X)$

Proof. For every $y$:

1. $\tilde{s}(x) = s(x, y) - q_2(x, y)y$
2. $\text{REV}(X) \geq E[\tilde{s}(X)] \geq^* E[\tilde{s}(X)1_{X \geq y}]$
Two Goods: Theorem 1

Claim. $\text{REV}((X, Y)^1_{X \geq Y}) \leq 2 \text{REV}(X)$

Proof. For every $y$:

1. $\tilde{s}(x) = s(x, y) - q_2(x, y)y \geq s(x, y) - y$

2. $\text{REV}(X) \geq E[\tilde{s}(X)] \geq E[\tilde{s}(X)^1_{X \geq y}]$
Two Goods: Theorem 1

Claim. \( \text{REV}( (X, Y)^1_{X \geq Y}) \leq 2 \text{REV}(X) \)

Proof. For every \( y \):

- \( \tilde{s}(x) = s(x, y) - q_2(x, y)y \geq s(x, y) - y \)
- \( \text{REV}(X) \geq E[\tilde{s}(X)] \geq E[\tilde{s}(X)^1_{X \geq y}] \geq E[s(X, y)^1_{X \geq y}] - y E[1_{X \geq y}] \)
Two Goods: Theorem 1

Claim. \( \text{REV}(\{(X, Y) \mid X \geq Y\}) \leq 2 \text{REV}(X) \)

Proof. For every \( y \):

\[ \tilde{s}(x) = s(x, y) - q_2(x, y)y \geq s(x, y) - y \]

\[ \text{REV}(X) \geq \mathbb{E}[\tilde{s}(X)] \geq \mathbb{E}[\tilde{s}(X)1_{X \geq y}] \]
\[ \geq \mathbb{E}[s(X, y)1_{X \geq y}] - y \mathbb{E}[1_{X \geq y}] \]
\[ = \mathbb{E}[s(X, y)1_{X \geq y}] - y \mathbb{P}[X \geq y] \]
Two Goods: Theorem 1

Claim. \( \text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X) \)

Proof. For every \( y \):

\[ \tilde{s}(x) = s(x, y) - q_2(x, y)y \geq s(x, y) - y \]

\[ \text{REV}(X) \geq E[\tilde{s}(X)] \geq^* E[\tilde{s}(X)1_{X \geq y}] \]

\[ \geq E[s(X, y)1_{X \geq y}] - y E[1_{X \geq y}] \]

\[ = E[s(X, y)1_{X \geq y}] - y \Pr[X \geq y] \]

\[ \geq E[s(X, y)1_{X \geq y}] - \text{REV}(X) \]
Claim. $\text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X)$

Proof. For every $y$:

- $\tilde{s}(x) = s(x, y) - q_2(x, y)y \geq s(x, y) - y$

- $\text{REV}(X) \geq E[\tilde{s}(X)] \geq E[\tilde{s}(X)1_{X \geq y}]$
  $\geq E[s(X, y)1_{X \geq y}] - yE[1_{X \geq y}]$
  $= E[s(X, y)1_{X \geq y}] - y\Pr[X \geq y]$
  $\geq E[s(X, y)1_{X \geq y}] - \text{REV}(X)$

- $E[s(X, y)1_{X \geq y}] \leq 2 \text{REV}(X)$
Two Goods: Theorem 1

Claim. \( \text{REV}((X, Y)1_{X \geq Y}) \leq 2 \text{REV}(X) \)

Proof. For every \( y \):

1. \( \tilde{s}(x) = s(x, y) - q_2(x, y)y \geq s(x, y) - y \)
2. \( \text{REV}(X) \geq E[\tilde{s}(X)] \geq* E[\tilde{s}(X)1_{X \geq y}] \)
   \( \geq E[s(X, y)1_{X \geq y}] - y E[1_{X \geq y}] \)
   \( = E[s(X, y)1_{X \geq y}] - y \Pr[X \geq y] \)
   \( \geq E[s(X, y)1_{X \geq y}] - \text{REV}(X) \)
3. \( E[s(X, y)1_{X \geq y}] \leq 2 \text{REV}(X) \)
4. Take expectation over \( y \sim Y \)
   \( (X \text{ and } Y \text{ are independent}) \)
Theorem 1. For every one-dimensional $F_1$, $F_2$:

$$\text{SRev}(F_1 \times F_2) \geq \frac{1}{2} \text{Rev}(F_1 \times F_2)$$
Theorem 1. For every one-dimensional $F_1, F_2$:

\[ \text{SREV}(F_1 \times F_2) \geq \frac{1}{2} \text{REV}(F_1 \times F_2) \]

Theorem 2. For every one-dimensional $F$:

\[ \text{SREV}(F \times F) \geq 73\% \text{REV}(F \times F) \]
Theorem 1. For every one-dimensional $F_1, F_2$:

$$\text{SRev}(F_1 \times F_2) \geq \frac{1}{2} \text{Rev}(F_1 \times F_2)$$

Theorem 2. For every one-dimensional $F$:

$$\text{SRev}(F \times F) \geq 73\% \text{Rev}(F \times F)$$

Proposition. There is a one-dimensional $F$:

$$\text{SRev}(F \times F) \approx 78\% \text{Rev}(F \times F)$$
Two Goods

Theorem 1. For every one-dimensional $F_1, F_2$:

$$SREV(F_1 \times F_2) \geq \frac{1}{2} REV(F_1 \times F_2)$$

Theorem 2. For every one-dimensional $F$:

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Proposition. There is a one-dimensional $F$:

$$SREV(F \times F) \approx 78\% \REV(F \times F)$$

Hart and Nisan (2012)
A class of IC&IR mechanisms
A class of IC&IR mechanisms $\mathbb{M}$
A family of distributions $\mathcal{F}$
A class of IC&IR mechanisms $M$

A family of distributions $F$

**Guaranteed Fraction of Optimal Revenue**
A class of IC&IR mechanisms $\mathbb{M}$

A family of distributions $\mathbb{F}$

**GUARANTEED FRACTION OF OPTIMAL REVENUE**
A class of IC&IR mechanisms $\mathcal{M}$

A family of distributions $\mathcal{F}$

**GUARANTEED FRACTION OF OPTIMAL REVENUE**

= maximal fraction $\alpha$ in $[0, 1]$ such that for every distribution $\mathcal{F}$ in $\mathcal{F}$ there is a mechanism $\mathcal{M}$ in $\mathcal{M}$ with

$$R(\mathcal{M}, \mathcal{F}) \geq \alpha \text{Rev}(\mathcal{F})$$
---

**GFOR**

- A class of IC&IR mechanisms $\mathcal{M}$
- A family of distributions $\mathcal{F}$

**Guaranteed Fraction of Optimal Revenue**

$= \text{maximal fraction } \alpha \text{ in } [0, 1] \text{ such that for every distribution } \mathcal{F} \text{ in } \mathcal{F} \text{ there is a mechanism } \mathcal{M} \text{ in } \mathcal{M}$

with

\[
R(\mathcal{M}, \mathcal{F}) \geq \alpha \text{ Rev}(\mathcal{F})
\]

\[
\text{GFOR} = \inf_{\mathcal{F} \in \mathcal{F}} \frac{\text{M-Rev}(\mathcal{F})}{\text{Rev}(\mathcal{F})}
\]
A class of IC&IR mechanisms $\mathcal{M}$

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**Guaranteed Fraction of Optimal Revenue**

$= \text{maximal fraction } \alpha \text{ in } [0, 1] \text{ such that for every distribution } \mathcal{F} \text{ in } \mathcal{F} \text{ there is a mechanism } \mathcal{M} \text{ in } \mathcal{M} \text{ with}$

$$R(\mathcal{M}, \mathcal{F}) \geq \alpha \text{ Rev}(\mathcal{F})$$

$$\text{GFOR} = \inf_{\mathcal{F} \in \mathcal{F}} \frac{\text{M-Rev}(\mathcal{F})}{\text{Rev}(\mathcal{F})} = \inf_{\mathcal{F} \in \mathcal{F}} \frac{\sup_{\mathcal{M} \in \mathcal{M}} R(\mathcal{M}, \mathcal{F})}{\sup_{\mathcal{M}} R(\mathcal{M}, \mathcal{F})}$$
GFOR: Two Goods
GFOR: Two Goods

- **SEPARATE** selling of **INDEPENDENT** goods:
GFOR: Two Goods

- **SEPARATE** selling of **INDEPENDENT** goods:

\[ 0.50 \leq \text{GFOR} \]
GFOR: Two Goods

- **SEPARATE** selling of **INDEPENDENT** goods:
  
  \[ 0.50 \leq GFOR \]

- **SEPARATE** selling of **IID** goods:
GFOR: Two Goods

- **SEPARATE** selling of **INDEPENDENT** goods:
  \[ 0.50 \leq \text{GFOR} \]

- **SEPARATE** selling of **IID** goods:
  \[ 0.73 \leq \text{GFOR} \]
GFOR: Two Goods

1 **BUYER**, 2 **GOODS**

- **SEPARATE** selling of **INDEPENDENT** goods:
  
  \[ 0.50 \leq \text{GFOR} \]

- **SEPARATE** selling of **IID** goods:
  
  \[ 0.73 \leq \text{GFOR} \]
1 BUYER, 2 GOODS

- SEPARATE selling of INDEPENDENT goods:
  
  \[ 0.50 \leq \text{GFOR} \]

- SEPARATE selling of IID goods:
  
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\( n \) BUYERS, 2 GOODS
GFOR: Two Goods

1 BUYER, 2 GOODS

- **SEPARATE** selling of INDEPENDENT goods:
  
  \[ 0.50 \leq \text{GFOR} \]

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\( n \) BUYERS, 2 GOODS

- **SEPARATE** selling of INDEPENDENT goods:
GFOR: Two Goods

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n BUYERS, 2 GOODS

- SEPARATE selling of INDEPENDENT goods:
  
  \[ 0.50 \leq \text{GFOR} \]
1 BUYER, 2 GOODS

- SEPARATE selling of INDEPENDENT goods:
  \[ 0.50 \leq \text{GFOR} \leq 0.78 \]

- SEPARATE selling of IID goods:
  \[ 0.73 \leq \text{GFOR} \leq 0.78 \]

n BUYERS, 2 GOODS

- SEPARATE selling of INDEPENDENT goods:
  \[ 0.50 \leq \text{GFOR} \leq 0.78 \]
GFOR: \( n \) Buyers, Two Goods

\[ n \text{ BUYERS, 2 GOODS} \]

- \( \text{SEPARATE selling of INDEPENDENT goods:} \)

\[ 0.50 \leq \text{GFOR} \]
$n$ Buyers, Two Goods

$\begin{align*}
\text{n \ BUYERS, 2 \ GOODS} \\
\text{\bullet SEPARATE selling of INDEPENDENT goods:} \\
0.50 \leq \text{GFOR}
\end{align*}$

Holds for:
GFOR: \( n \) Buyers, Two Goods

\[ \begin{align*}
\text{\( n \) BUYERS, 2 GOODS} \\
\text{\( \bullet \) SEPARATE selling of INDEPENDENT goods:} \\
\text{\( 0.50 \leq \text{GFOR} \)} \\
\text{Holds for:} \\
\text{\( \bullet \) BAYESIAN-NASH implementation}
\end{align*} \]
$n$ BUYERS, 2 GOODS

- SEPARATE selling of INDEPENDENT goods:

\[ 0.50 \leq \text{GFOR} \]

Holds for:

- BAYESIAN-NASH implementation
- DOMINANT-STRATEGY implementation
GFOR: $n$ Buyers, Two Goods

$n$ BUYERS, 2 GOODS

- SEPARATE selling of INDEPENDENT goods:

$$0.50 \leq \text{GFOR}$$

Holds for:

- BAYESIAN-NASH implementation
- DOMINANT-STRATEGY implementation

(in each case: use the same implementation for single goods and for the two goods)
Notations:
Notations:

$F$ : c.d.f. on $\mathbb{R}_+$
Notations:

- $F : \text{c.d.f. on } \mathbb{R}_+$
- $F^\times_k \equiv F \times \ldots \times F$

\[ k \]
Many Goods

Notations:

- $F$ : c.d.f. on $\mathbb{R}_+$
- $F^\times k \equiv F \times \ldots \times F$
- $\text{REV}(F^\times k)$ : MAXIMAL REVENUE from $k$ goods distributed *independently*, each one according to $F$
Many Goods

Notations:

- $F$ : c.d.f. on $\mathbb{R}_+$
- $F^{\times k} \equiv F \times \ldots \times F$

- $\text{Rev}(F^{\times k})$ : MAXIMAL REVENUE from $k$ goods distributed *independently*, each one according to $F$

- $\text{BRev}(F^{\times k})$ : MAXIMAL REVENUE selling the $k$ goods BUNDLED
Many Goods

Notations:

- $F$: c.d.f. on $\mathbb{R}_+$
- $F^{\times k} \equiv F \times \ldots \times F$

- $\text{Rev}(F^{\times k})$: MAXIMAL REVENUE from $k$ goods distributed \textit{independently}, each one according to $F$

- $\text{BRev}(F^{\times k})$: MAXIMAL REVENUE selling the $k$ goods \textbf{BUNDLED} $\rightarrow$ PRICE $p_k$ for all the $k$ goods together
Many Goods: $k \rightarrow \infty$
Theorem 3. For every one-dimensional $F$ with finite expectation ($\mathbb{E}(F) < \infty$):
Theorem 3. For every one-dimensional $F$ with finite expectation ($\mathbb{E}(F) < \infty$):

$$\lim_{k \to \infty} \frac{\text{BREV}(F \times k)}{\text{REV}(F \times k)} = 1$$
Theorem 3. For every one-dimensional $F$ with finite expectation ($\mathbb{E}(F) < \infty$):

$$\lim_{k \to \infty} \frac{\text{BREV}(F^\times k)}{\text{REV}(F^\times k)} = 1$$

Armstrong (1999), Bakos & Brynjolfsson (1999)
Theorem 3

\[
\lim_{k \to \infty} \frac{\text{BREV}(F \times^k)}{\text{REV}(F \times^k)} = 1
\]
Proof of Theorem 3

\[
\lim_{k \to \infty} \frac{\text{BR} \\text{EV}(F \times k)}{\text{REV}(F \times k)} = 1
\]
Proof of Theorem 3

\[ \lim_{k \to \infty} \frac{\text{BR}_{\text{EV}}(F \times^k)}{\text{REV}(F \times^k)} = 1 \]

Proof.
Proof of Theorem 3

\[ \lim_{k \to \infty} \frac{\text{BREV}(F \times k)}{\text{REV}(F \times k)} = 1 \]

Proof.

- \( \text{BREV}(F \times k) \leq \text{REV}(F \times k) \leq k \ E(F) \)
Proof of Theorem 3

\[
\lim_{k \to \infty} \frac{\text{B} \text{R} \text{E} \text{V}(F \times k)}{\text{R} \text{E} \text{V}(F \times k)} = 1
\]

Proof.

\[ \text{B} \text{R} \text{E} \text{V}(F \times k) \leq \text{R} \text{E} \text{V}(F \times k) \leq k \ \text{E}(F) \]

Proof:

\[
s(x) = q(x) \cdot x - b(x) \leq x_1 + \ldots + x_k \text{ (IR)}
\]
Proof of Theorem 3

\[
\lim_{k \to \infty} \frac{\text{BREV}(F^\times k)}{\text{REV}(F^\times k)} = 1
\]

Proof. Let \( X_i \) be i.i.d.-\( F \).

- \( \text{BREV}(F^\times k) \leq \text{REV}(F^\times k) \leq k \text{E}(F) \)

Proof:

\[
s(x) = q(x) \cdot x - b(x) \leq x_1 + \ldots + x_k \quad (\text{IR})
\]
Proof of Theorem 3

\[ \lim_{k \to \infty} \frac{\text{BREV}(F \times k)}{\text{REV}(F \times k)} = 1 \]

Proof. Let \( X_i \) be i.i.d. \( F \).

\[ \text{BREV}(F \times k) \leq \text{REV}(F \times k) \leq k \mathbb{E}(F) \]

Proof:

\[ s(x) = q(x) \cdot x - b(x) \leq x_1 + \ldots + x_k \text{ (IR)} \]

\[ \mathbb{E}(s(X)) \leq \mathbb{E}(X_1) + \ldots + \mathbb{E}(X_k) = k \mathbb{E}(F) \]
Proof of Theorem 3

\[ \lim_{k \to \infty} \frac{\text{BREV}(F \times k)}{\text{REV}(F \times k)} = 1 \]

Proof. Let \( X_i \) be i.i.d.-\( F \).

- \( \text{BREV}(F \times k) \leq \text{REV}(F \times k) \leq k \ E(F) \)
Proof of Theorem 3

\[ \lim_{k \to \infty} \frac{\text{BREV}(F \times k)}{\text{REV}(F \times k)} = 1 \]

Proof. Let \( X_i \) be i.i.d.-\( F \). Let \( \epsilon > 0 \).

\[ \text{BREV}(F \times k) \leq \text{REV}(F \times k) \leq k \ E(F) \]
Proof of Theorem 3

\[ \lim_{k \to \infty} \frac{\text{BREV}(F \times^k)}{\text{REV}(F \times^k)} = 1 \]

Proof. Let \( X_i \) be i.i.d.-\( F \). Let \( \epsilon > 0 \).

- \( \text{BREV}(F \times^k) \leq \text{REV}(F \times^k) \leq k \cdot \mathbb{E}(F) \)
- \( \text{BREV}(F \times^k) \geq p_k \cdot \alpha_k \)
Proof of Theorem 3

\[
\lim_{k \to \infty} \frac{\text{BREV}(F \times k)}{\text{REV}(F \times k)} = 1
\]

Proof. Let \( X_i \) be i.i.d.-\( F \). Let \( \epsilon > 0 \).

\[
\text{BREV}(F \times k) \leq \text{REV}(F \times k) \leq k \ E(F)
\]

\[
\text{BREV}(F \times k) \geq p_k \cdot \alpha_k
\]

where

\[
p_k = (1 - \epsilon) \ k \ E(F) : \text{ price for bundle}
\]
Proof of Theorem 3

\[
\lim_{k \to \infty} \frac{\text{BREV}(F^{\times k})}{\text{REV}(F^{\times k})} = 1
\]

Proof. Let \( X_i \) be i.i.d. \(-F\). Let \( \epsilon > 0 \).

\begin{itemize}
  \item \( \text{BREV}(F^{\times k}) \leq \text{REV}(F^{\times k}) \leq k \, \text{E}(F) \)
  \item \( \text{BREV}(F^{\times k}) \geq p_k \cdot \alpha_k \)
\end{itemize}

where

\( p_k = (1 - \epsilon) \, k \, \text{E}(F) \) : price for bundle
\( \alpha_k = \text{Pr} [X_1 + \ldots + X_k \geq p_k] \)
Proof of Theorem 3

\[ \lim_{k \to \infty} \frac{\text{BR}_{\text{EV}}(F \times k)}{\text{REV}(F \times k)} = 1 \]

**Proof.** Let \( X_i \) be i.i.d.-\( F \). Let \( \epsilon > 0 \).

- \( \text{BR}_{\text{EV}}(F \times k) \leq \text{REV}(F \times k) \leq k \ E(F) \)
- \( \text{BR}_{\text{EV}}(F \times k) \geq p_k \cdot \alpha_k \)

where

\[
p_k = (1 - \epsilon) k \ E(F) : \text{ price for bundle}
\]

\[
\alpha_k = \Pr [X_1 + \ldots + X_k \geq p_k]
\]

\[
= \Pr \left[ \frac{1}{k} (X_1 + \ldots + X_k) \geq (1 - \epsilon)E(F) \right]
\]
Proof of Theorem 3

\[
\lim_{k \to \infty} \frac{\text{BREV}(F \times k)}{\text{REV}(F \times k)} = 1
\]

Proof. Let \( X_i \) be i.i.d.-\( F \). Let \( \epsilon > 0 \).

\[\bullet \; \text{BREV}(F \times k) \leq \text{REV}(F \times k) \leq k \; E(F)\]
\[\bullet \; \text{BREV}(F \times k) \geq p_k \cdot \alpha_k\]

where

\[p_k = (1 - \epsilon) \; k \; E(F) : \text{price for bundle}\]
\[\alpha_k = \Pr [X_1 + \ldots + X_k \geq p_k]\]
\[= \Pr \left[ \frac{1}{k}(X_1 + \ldots + X_k) \geq (1 - \epsilon)E(F) \right] \to 1\]
Proof of Theorem 3

\[
\lim_{k \to \infty} \frac{\text{BREV}(F \times k)}{\text{REV}(F \times k)} = 1
\]

Proof. Let \(X_i\) be i.i.d.-\(F\). Let \(\epsilon > 0\).

\[ \text{BREV}(F \times k) \leq \text{REV}(F \times k) \leq k \ E(F) \]
\[ \text{BREV}(F \times k) \geq p_k \cdot \alpha_k \sim p_k \]

where
\[ p_k = (1 - \epsilon) k \ E(F) : \text{price for bundle} \]
\[ \alpha_k = \text{Pr} [X_1 + \ldots + X_k \geq p_k] \]
\[ = \text{Pr} \left[ \frac{1}{k}(X_1 + \ldots + X_k) \geq (1 - \epsilon)E(F) \right] \to 1 \]
Proof of Theorem 3

\[
\lim_{k \to \infty} \frac{\text{BREV}(F \times k)}{\text{REV}(F \times k)} = 1
\]

Proof. Let \(X_i\) be i.i.d.-\(F\). Let \(\epsilon > 0\).

- \(\text{BREV}(F \times k) \leq \text{REV}(F \times k) \leq k \ E(F)\)
- \(\text{BREV}(F \times k) \geq p_k \cdot \alpha_k \sim p_k = (1 - \epsilon) \ k \ E(F)\)

where

\[
p_k = (1 - \epsilon) \ k \ E(F) : \text{ price for bundle}
\]
\[
\alpha_k = \Pr [X_1 + \ldots + X_k \geq p_k]
\]
\[
= \Pr \left[ \frac{1}{k} (X_1 + \ldots + X_k) \geq (1 - \epsilon) E(F) \right] \to 1
\]
Proof of Theorem 3

\[ \lim_{k \to \infty} \frac{\text{BR}_{\text{EV}}(F \times k)}{\text{REV}(F \times k)} = 1 \]

Proof. Let \( X_i \) be i.i.d. \(-F\). Let \( \epsilon > 0 \).

- \( \text{BR}_{\text{EV}}(F \times k) \leq \text{REV}(F \times k) \leq k \ E(F) \)
- \( \text{BR}_{\text{EV}}(F \times k) \geq p_k \cdot \alpha_k \sim p_k = (1 - \epsilon) k \ E(F) \)
1 BUYER, $k$ GOODS
1 BUYER, $k$ GOODS

- SEPARATE selling of INDEPENDENT goods:
1 **BUYER**, \( k \) **GOODS**

Separate selling of **INDEPENDENT** goods:

\[
\frac{c_1}{\log^2 k} \leq \text{GFOR} \leq \frac{c_2}{\log k}
\]
GFOR: $k$ Goods

1 BUYER, $k$ GOODS

• SEPARATE selling of INDEPENDENT goods:

$$\frac{c_1}{\log^2 k} \leq \text{GFOR} \leq \frac{c_2}{\log k}$$

(same for IID goods)
1 BUYER, $k$ GOODS

- **SEPARATE** selling of **INDEPENDENT** goods:

$$\frac{c_1}{\log^2 k} \leq \text{GFOR} \leq \frac{c_2}{\log k}$$

(same for **IID** goods)

- **BUNDLED** selling of **INDEPENDENT** goods:
GFOR: $k$ Goods

1 BUYER, $k$ GOODS

- **SEPARATE** selling of INDEPENDENT goods:
  \[
  \frac{c_1}{\log^2 k} \leq \text{GFOR} \leq \frac{c_2}{\log k}
  \]

  (same for IID goods)

- **BUNDLED** selling of INDEPENDENT goods:
  \[
  \frac{c_3}{k} \leq \text{GFOR} \leq \frac{1}{k} + \varepsilon
  \]
GFOR: $k$ Goods

1. **BUYER, $k$ GOODS**

- **SEPARATE** selling of **INDEPENDENT** goods:

\[
\frac{c_1}{\log^2 k} \leq \text{GFOR} \leq \frac{c_2}{\log k}
\]

(same for **IID** goods)

- **BUNDLED** selling of **IID** goods:
GFOR: $k$ Goods

1 BUYER, $k$ GOODS

- **SEPARATE** selling of INDEPENDENT goods:

  \[
  \frac{c_1}{\log^2 k} \leq \text{GFOR} \leq \frac{c_2}{\log k}
  \]

  (same for IID goods)

- **BUNDLED** selling of IID goods:

  \[
  \frac{c_4}{\log k} \leq \text{GFOR} \leq 0.57 + \varepsilon
  \]
2 GOODS, ARBITRARY DEPENDENCE
2 GOODS, ARBITRARY DEPENDENCE

- SEPARATE selling:
2 GOODS, ARBITRARY DEPENDENCE

SEPARATE selling: $\text{GFOR} = 0$. 
for every $\varepsilon > 0$ there is a distribution $\mathcal{F}$ on $[0, 1]^2$ such that: $S\text{Rev}(\mathcal{F}) < \varepsilon \text{Rev}(\mathcal{F})$
GFOR: General

2 GOODS, ARBITRARY DEPENDENCE

- **SEPARATE** selling: \[ \text{GFOR} = 0. \]
- **BUNDLED** selling:
2 GOODS, ARBITRARY DEPENDENCE

- **SEPARATE** selling: \( \text{GFOR} = 0 \).
- **BUNDLED** selling: \( \text{GFOR} = 0 \).
2 GOODS, ARBITRARY DEPENDENCE

- **SEPARATE** selling: \( \text{GFOR} = 0. \)
- **BUNDLED** selling: \( \text{GFOR} = 0. \)

For every \( \varepsilon > 0 \) there is a distribution \( \mathcal{F} \) on \( [0, 1]^2 \) such that \( \text{BRev}(\mathcal{F}) < \varepsilon \text{ Rev}(\mathcal{F}) \)
2 GOODS, ARBITRARY DEPENDENCE

- **SEPARATE** selling: \( \text{GFOR} = 0 \).
- **BUNDLED** selling: \( \text{GFOR} = 0 \).
- **DETERMINISTIC** mechanisms:
2 GOODS, ARBITRARY DEPENDENCE

- SEPARATE selling: $\text{GFOR} = 0$.
- BUNDLED selling: $\text{GFOR} = 0$.
- DETERMINISTIC mechanisms: $\text{GFOR} = 0$. 
2 GOODS, ARBITRARY DEPENDENCE

- **SEPARATE** selling: \( \text{GFOR} = 0. \)
- **BUNDLED** selling: \( \text{GFOR} = 0. \)
- **DETERMINISTIC** mechanisms: \( \text{GFOR} = 0. \)

for every \( \varepsilon > 0 \) there is a distribution \( F \) on \([0, 1]\) such that \( \text{DRev}(F) < \varepsilon \text{ Rev}(F) \)
GFOR: General

2 GOODS, ARBITRARY DEPENDENCE

- **SEPARATE** selling: \( \text{GFOR} = 0 \).
- **BUNDLED** selling: \( \text{GFOR} = 0 \).
- **DETERMINISTIC** mechanisms: \( \text{GFOR} = 0 \).

(same for \( k \) goods)

For every \( \varepsilon > 0 \) there is a distribution \( \mathcal{F} \) on \([0, 1]^2\) such that \( \text{DRev}(\mathcal{F}) < \varepsilon \text{ Rev}(\mathcal{F}) \).
$[m]$-Rev$(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES}$
\([m]-\text{Rev}(\mathcal{F})\) = maximal revenue over all mechanisms with \text{AT MOST} \(m\) \text{ OUTCOMES}

\[1]-\text{Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F})\] all \(\mathcal{F}\)
$[m]\text{-Rev}(\mathcal{F})$ = maximal revenue over all mechanisms with at most $m$ outcomes

- $[1]\text{-Rev}(\mathcal{F}) = B\text{Rev}(\mathcal{F})$  all $\mathcal{F}$
- $[m]\text{-Rev}(\mathcal{F}) \leq m \cdot B\text{Rev}(\mathcal{F})$  all $\mathcal{F}$
[\textit{m}]-\text{Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES}

\textbullet\; [1]-\text{Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F}

\textbullet\; [\textit{m}]-\text{Rev}(\mathcal{F}) \leq m \cdot \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F}

\textbullet\; [\textit{m}]-\text{Rev}(\mathcal{F}) \geq c m^{1/7} \cdot \text{BRev}(\mathcal{F}) \quad \text{some } \mathcal{F}
Menu Size and GFOR

$[m]_{\text{Rev}}(\mathcal{F}) =$ maximal revenue over all mechanisms with at most $m$ outcomes

- $[1]_{\text{Rev}}(\mathcal{F}) = BR_{\text{Rev}}(\mathcal{F})$ for all $\mathcal{F}$
- $[m]_{\text{Rev}}(\mathcal{F}) \leq m \cdot BR_{\text{Rev}}(\mathcal{F})$ for all $\mathcal{F}$
- $[m]_{\text{Rev}}(\mathcal{F}) \geq cm^{1/7} \cdot BR_{\text{Rev}}(\mathcal{F})$ for some $\mathcal{F}$

Corollary. For $BR_{\text{Rev}}$: $GFOR = 0$. 

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$[m]\text{-Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES}$

- $[1]\text{-Rev}(\mathcal{F}) = \text{BRRev}(\mathcal{F}) \quad \text{all } \mathcal{F}$
- $[m]\text{-Rev}(\mathcal{F}) \leq m \cdot \text{BRRev}(\mathcal{F}) \quad \text{all } \mathcal{F}$
- $[m]\text{-Rev}(\mathcal{F}) \geq cm^{1/7} \cdot \text{BRRev}(\mathcal{F}) \quad \text{some } \mathcal{F}$

**Corollary.** For $\text{BRRev}$: $\text{GFOR} = 0$.

**Proof.**

$$\inf_{\mathcal{F}} \frac{\text{BRRev}}{\text{Rev}}$$
Menu Size and GFOR

\[ [m]-\text{Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES} \]

\[ [1]-\text{Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]

\[ [m]-\text{Rev}(\mathcal{F}) \leq m \cdot \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]

\[ [m]-\text{Rev}(\mathcal{F}) \geq cm^{1/7} \cdot \text{BRev}(\mathcal{F}) \quad \text{some } \mathcal{F} \]

Corollary. For \( \text{BRev} \): \( \text{GFOR} = 0 \).

Proof. For every \( m \)

\[ \inf_{\mathcal{F}} \frac{\text{BRev}}{\text{Rev}} \leq \inf_{\mathcal{F}} \frac{\text{BRev}}{[m]-\text{Rev}} \]
Menu Size and GFOR

\([m]^{-Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES}\)

- \([1]^{-Rev}(\mathcal{F}) = BRev(\mathcal{F}) \quad \text{all } \mathcal{F}\)
- \([m]^{-Rev}(\mathcal{F}) \leq m \cdot BRev(\mathcal{F}) \quad \text{all } \mathcal{F}\)
- \([m]^{-Rev}(\mathcal{F}) \geq cm^{1/7} \cdot BRev(\mathcal{F}) \quad \text{some } \mathcal{F}\)

Corollary. For \(BRev\): \(GFOR = 0\).

Proof. For every \(m\)

\[\inf_{\mathcal{F}} \frac{BRev}{Rev} \leq \inf_{\mathcal{F}} \frac{BRev}{[m]^{-Rev}} \leq \frac{1}{cm^{1/7}} \rightarrow m \quad 0\]
Menu Size and GFOR

\[ [m]-\text{Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES} \]

- \[ [1]-\text{Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]
- \[ [m]-\text{Rev}(\mathcal{F}) \leq m \cdot \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]
- \[ [m]-\text{Rev}(\mathcal{F}) \geq c m^{1/7} \cdot \text{BRev}(\mathcal{F}) \quad \text{some } \mathcal{F} \]

Corollary. For BRev: GFOR = 0.
Menu Size and GFOR

\([m]-\text{Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES}\]

- \([1]-\text{Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F})\)  
  all \(\mathcal{F}\)
- \([m]-\text{Rev}(\mathcal{F}) \leq m \cdot \text{BRev}(\mathcal{F})\)  
  all \(\mathcal{F}\)
- \([m]-\text{Rev}(\mathcal{F}) \geq cm^{1/7} \cdot \text{BRev}(\mathcal{F})\)  
  some \(\mathcal{F}\)

**Corollary.** For \(\text{BRev}:\) \(\text{GFOR} = 0\).

**Corollary.** For \([m]-\text{Rev}:\) \(\text{GFOR} = 0\).
[$m$]-$\text{Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES}$

- $[1]-\text{Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F})$ \quad all $\mathcal{F}$
- $[m]-\text{Rev}(\mathcal{F}) \leq m \cdot \text{BRev}(\mathcal{F})$ \quad all $\mathcal{F}$
- $[m]-\text{Rev}(\mathcal{F}) \geq c m^{1/7} \cdot \text{BRev}(\mathcal{F})$ \quad some $\mathcal{F}$

Corollary. For $\text{BRev}$: \quad $\text{GFOR} = 0$.

Corollary. For $[m]-\text{Rev}$: \quad $\text{GFOR} = 0$.

Proof.

$$\inf_{\mathcal{F}} \frac{[m]-\text{Rev}}{\text{Rev}}$$
Menu Size and GFOR

\[ [m] - \text{Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES} \]

\[ [1] - \text{Rev}(\mathcal{F}) = B\text{rev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]

\[ [m] - \text{Rev}(\mathcal{F}) \leq m \cdot B\text{rev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]

\[ [m] - \text{Rev}(\mathcal{F}) \geq c \cdot m^{1/7} \cdot B\text{rev}(\mathcal{F}) \quad \text{some } \mathcal{F} \]

Corollary. For \( B\text{rev} \): \( G\text{FOR} = 0 \).

Corollary. For \([m] - \text{Rev} \): \( G\text{FOR} = 0 \).

Proof.

\[ \inf_{\mathcal{F}} \frac{[m] - \text{Rev}}{\text{Rev}} \leq m \cdot \inf_{\mathcal{F}} \frac{B\text{rev}}{\text{Rev}} \]
Menu Size and GFOR

Let \([m]-\text{Rev}(\mathcal{F})\) denote the maximal revenue over all mechanisms with at most \(m\) outcomes.

- \([1]-\text{Rev}(\mathcal{F}) = B\text{Rev}(\mathcal{F})\) \quad \text{all } \mathcal{F}
- \([m]-\text{Rev}(\mathcal{F}) \leq m \cdot B\text{Rev}(\mathcal{F})\) \quad \text{all } \mathcal{F}
- \([m]-\text{Rev}(\mathcal{F}) \geq c m^{1/7} \cdot B\text{Rev}(\mathcal{F})\) \quad \text{some } \mathcal{F}

Corollary. For \(B\text{Rev}:\) \quad GFOR = 0.

Corollary. For \([m]-\text{Rev}:\) \quad GFOR = 0.

Proof.

\[
\inf_{\mathcal{F}} \frac{[m]-\text{Rev}}{\text{Rev}} \leq m \cdot \inf_{\mathcal{F}} \frac{B\text{Rev}}{\text{Rev}} = 0
\]
Menu Size and GFOR

\[ [m] \text{-Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with at most } m \text{ outcomes} \]

- \[ [1] \text{-Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]
- \[ [m] \text{-Rev}(\mathcal{F}) \leq m \cdot \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]
- \[ [m] \text{-Rev}(\mathcal{F}) \geq c \cdot m^{1/7} \cdot \text{BRev}(\mathcal{F}) \quad \text{some } \mathcal{F} \]

Corollary. For \( \text{BRev} \): \( \text{GFOR} = 0 \).

Corollary. For \( [m] \text{-Rev} \): \( \text{GFOR} = 0 \).
Menu Size and GFOR

\([m]\)-\text{Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES}

- \([1]\)-\text{Rev}(\mathcal{F}) = \text{BR}(\mathcal{F}) \quad \text{all } \mathcal{F}
- \([m]\)-\text{Rev}(\mathcal{F}) \leq m \cdot \text{BR}(\mathcal{F}) \quad \text{all } \mathcal{F}
- \([m]\)-\text{Rev}(\mathcal{F}) \geq cm^{1/7} \cdot \text{BR}(\mathcal{F}) \quad \text{some } \mathcal{F}

Corollary. For \text{BR}: \quad \text{GFOR} = 0.
Corollary. For \([m]\)-\text{Rev}: \quad \text{GFOR} = 0.
Corollary. For \text{DRev}: \quad \text{GFOR} = 0.
Menu Size and GFOR

\([m] \cdot \operatorname{REV}(\mathcal{F}) = \) maximal revenue over all mechanisms with AT MOST \(m\) OUTCOMES

- \(\operatorname{REV}^1(\mathcal{F}) = \operatorname{BR}(\mathcal{F})\) all \(\mathcal{F}\)
- \(\operatorname{REV}(\mathcal{F}) \leq m \cdot \operatorname{BR}(\mathcal{F})\) all \(\mathcal{F}\)
- \(\operatorname{REV}(\mathcal{F}) \geq c m^{1/7} \cdot \operatorname{BR}(\mathcal{F})\) some \(\mathcal{F}\)

Corollary. For \(\operatorname{BR}\): \(\) GFOR = 0.
Corollary. For \([m] \cdot \operatorname{REV}\): \(\) GFOR = 0.
Corollary. For \(\operatorname{DREV}\): \(\) GFOR = 0.

Proof.

\(\operatorname{DREV} \leq [2^k] \cdot \operatorname{REV}\)
[\text{\textit{m}}]-\text{REV}(F) = \text{maximal revenue over all mechanisms with AT MOST } \text{\textit{m}} \text{ OUTCOMES}

- \text{\textit{1}}]-\text{REV}(F) = \text{BRev}(F) \quad \text{all } F
- \text{\textit{m}}]-\text{REV}(F) \leq \text{\textit{m}} \cdot \text{BRev}(F) \quad \text{all } F
- \text{\textit{m}}]-\text{REV}(F) \geq \text{c} \cdot \text{\textit{m}}^{1/7} \cdot \text{BRev}(F) \quad \text{some } F

Corollary. For \text{BRev}: \quad \text{GFOR} = 0.
Corollary. For \text{\textit{m}}]-\text{REV}: \quad \text{GFOR} = 0.
Corollary. For \text{DRev}: \quad \text{GFOR} = 0.
$[m]-\text{Rev}(F) =$ maximal revenue over all mechanisms with AT MOST $m$ OUTCOMES

- $[1]-\text{Rev}(F) = B\text{Rev}(F)$
- $[m]-\text{Rev}(F) \leq m \cdot B\text{Rev}(F)$
- $[m]-\text{Rev}(F) \geq cm^{1/7} \cdot B\text{Rev}(F)$
Menu Size

\([m]-\text{Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES}\)

- \([1]-\text{Rev}(\mathcal{F}) = B\text{Rev}(\mathcal{F}) \quad \text{all } \mathcal{F}\)
- \([m]-\text{Rev}(\mathcal{F}) \leq m \cdot B\text{Rev}(\mathcal{F}) \quad \text{all } \mathcal{F}\)
- \([m]-\text{Rev}(\mathcal{F}) \geq c \cdot m^{1/7} \cdot B\text{Rev}(\mathcal{F}) \quad \text{some } \mathcal{F}\)
- \(D\text{Rev}(\mathcal{F}) \leq 2^k \cdot B\text{Rev}(\mathcal{F}) \quad \text{all } \mathcal{F}\)
[\text{Menu Size}]

\[ [m]-\text{Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES} \]

- \[ [1]-\text{Rev}(\mathcal{F}) = \text{BR}(\mathcal{F}) \quad \text{all } \mathcal{F} \]
- \[ [m]-\text{Rev}(\mathcal{F}) \leq m \cdot \text{BR}(\mathcal{F}) \quad \text{all } \mathcal{F} \]
- \[ [m]-\text{Rev}(\mathcal{F}) \geq c m^{1/7} \cdot \text{BR}(\mathcal{F}) \quad \text{some } \mathcal{F} \]
- \[ \text{DRev}(\mathcal{F}) \leq 2^k \cdot \text{BR}(\mathcal{F}) \quad \text{all } \mathcal{F} \]
- \[ \text{DRev}(\mathcal{F}) \geq \frac{c}{k} 2^k \cdot \text{BR}(\mathcal{F}) \quad \text{some } \mathcal{F} \]
Menu Size and Complexity

\([m]-\text{Rev}(\mathcal{F}) = \text{maximal revenue over all mechanisms with AT MOST } m \text{ OUTCOMES}\)

\[ [1]-\text{Rev}(\mathcal{F}) = \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]

\[ [m]-\text{Rev}(\mathcal{F}) \leq m \cdot \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]

\[ [m]-\text{Rev}(\mathcal{F}) \geq c m^{1/7} \cdot \text{BRev}(\mathcal{F}) \quad \text{some } \mathcal{F} \]

\[ \text{DRev}(\mathcal{F}) \leq 2^k \cdot \text{BRev}(\mathcal{F}) \quad \text{all } \mathcal{F} \]

\[ \text{DRev}(\mathcal{F}) \geq \frac{c}{k} 2^k \cdot \text{BRev}(\mathcal{F}) \quad \text{some } \mathcal{F} \]

**MENU SIZE** = measure of the **COMPLEXITY** of mechanisms
$[m] \cdot \text{Rev}(\mathcal{F}) = \text{maximal revenue from mechanisms with at most } m \text{ outcomes}$
$[m]\text{-Rev}(F) = \text{maximal revenue from mechanisms with at most } m \text{ outcomes}$

- $[m]\text{-Rev}$ for fixed $m$: $G_{\text{FOR}} = 0$
$[m]-\text{Rev}(F) = \text{maximal REVENUE from mechanisms with AT MOST } m \text{ OUTCOMES}$

- $[m]-\text{Rev}$ for fixed $m$: $\text{GFOR} = 0$
- $[m]-\text{Rev}$ increases with $m$ (polynomially)
$[m]\text{-Rev}(\mathcal{F})$ = maximal REVENUE from mechanisms with AT MOST $m$ OUTCOMES

- $[m]\text{-Rev}$ for fixed $m$: GFOR $= 0$
- $[m]\text{-Rev}$ increases with $m$ (polynomially)
- DETERMINISTIC-REV $\sim [2^k]\text{-Rev}$
Menu Size and Complexity

\[ [m]\text{-Rev}(\mathcal{F}) = \text{maximal revenue from mechanisms with AT MOST } m \text{ OUTCOMES} \]

- \([m]\text{-Rev}\) for fixed \(m\): \(G\text{FOR} = 0\)
- \([m]\text{-Rev}\) increases with \(m\) (polynomially)
- Deterministic-Rev \(\sim [2^k]\text{-Rev}\)

**MENU SIZE** = measure of the **COMPLEXITY** of mechanisms
Summary: Multiple Goods
Maximizing revenue with multiple goods:
Maximizing revenue with multiple goods:

- many of the results for ONE GOOD are FALSE for MULTIPLE GOODS
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- is an extremely complex problem (even for simple distributions)
Maximizing revenue with multiple goods:

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Maximizing revenue with multiple goods:

- many of the results for ONE GOOD are FALSE for MULTIPLE GOODS
- is an extremely complex problem (even for simple distributions)
- “what we have learned from one good is too good to be true for two goods”
- SIMPLE mechanisms MAY yield UNIFORM APPROXIMATION
“Are you trying to auction your Brussels sprouts again?”