Two(!) Good To Be True

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Joint work with

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Sergiu Hart and Phil Reny
“Revenue Maximization in Two Dimensions”
(2010, in preparation)
• Sergiu Hart and Phil Reny
  “Revenue Maximization in Two Dimensions”
  (2010, in preparation)

• Sergiu Hart and Phil Reny
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  Nonmonotonicity and Other Observations”
  (2011)
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(2014)
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A Simple Problem

1 SELLER
A Simple Problem

- 1 SELLER
- 1 BUYER
A Simple Problem

- 1 SELLER
- 1 BUYER
- \( k \) GOODS (ITEMS)
A Simple Problem

- 1 SELLER
- 1 BUYER
- \( k \) GOODS (ITEMS)

OBJECTIVE:
MAXIMIZE the REVENUE of the SELLER
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
A Simple Problem

1 SELLER

1 BUYER

\( k \) GOODS (ITEMS)

values of GOODS to BUYER:

\[ X = (X_1, X_2, ..., X_k) \]
A Simple Problem

- **1 SELLER**
- **1 BUYER**
- **$k$ GOODS (ITEMS)**
  - values of GOODS to BUYER: $X = (X_1, X_2, ..., X_k)$
  - additive valuation (good 1 and good 2 = $X_1 + X_2$)
A Simple Problem

1 SELLER

1 BUYER

$k$ GOODS (ITEMS)

values of GOODS to BUYER:

$$X = (X_1, X_2, \ldots, X_k)$$

additive valuation

(good 1 and good 2 = $X_1 + X_2$)

BUYER knows the value $X$
A Simple Problem

- **1 SELLER**
- **1 BUYER**
- **$k$ GOODS (ITEMS)**
  - **values** of GOODS to BUYER: \( X = (X_1, X_2, \ldots, X_k) \)
  - **additive** valuation
    (good 1 and good 2 = \( X_1 + X_2 \))
  - **BUYER knows** the value \( X \)
  - **SELLER does not know** the value \( X \)
A Simple Problem

1 SELLER

1 BUYER

$k$ GOODS (ITEMS)

values of GOODS to BUYER:

$$X = (X_1, X_2, \ldots, X_k)$$

additive valuation

(good 1 and good 2 = $X_1 + X_2$)

BUYER knows the value $X$

SELLER does not know the value $X$

$X$ distributed according to c.d.f. $\mathcal{F}$ on $\mathbb{R}_+^k$
A Simple Problem

1 SELLER

1 BUYER

$k$ GOODS (ITEMS)

values of GOODS to BUYER: $X = (X_1, X_2, ..., X_k)$

additive valuation

(good 1 and good 2 = $X_1 + X_2$)

BUYER knows the value $X$

SELLER does not know the value $X$

$X$ distributed according to c.d.f. $F$ on $\mathbb{R}_+^k$

SELLER knows the distribution $F$ of $X$
A Simple Problem

1 SELLER

1 BUYER

k GOODS (ITEMS)

values of GOODS to BUYER:

\[ X = (X_1, X_2, \ldots, X_k) \] (random variable)

additive valuation

(good 1 and good 2 = \( X_1 + X_2 \))

BUYER knows the value \( X \)

SELLER does not know the value \( X \)

\( X \) distributed according to c.d.f. \( F \) on \( \mathbb{R}_+^k \)

SELLER knows the distribution \( F \) of \( X \)
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)
A Simple Problem

1 SELLER

1 BUYER

$k$ GOODS (ITEMS)

SELLER and BUYER:

quasi-linear utilities (i.e., additive in monetary payments)
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)

SELLER and BUYER:

- **quasi-linear** utilities (i.e., additive in monetary payments)
- **risk-neutral** (i.e., linear in probabilities)
A Simple Problem

- 1 SELLER
- 1 BUYER
- $k$ GOODS (ITEMS)

SELLER and BUYER:
- quasi-linear utilities (i.e., additive in monetary payments)
- risk-neutral (i.e., linear in probabilities) (or: linear in quantities)
A Simple Problem

1 SELLER

1 BUYER

\( k \) GOODS (ITEMS)

SELLER and BUYER:

- **quasi-linear** utilities (i.e., additive in monetary payments)
- **risk-neutral** (i.e., linear in probabilities) (or: linear in quantities)

SELLER:

- **no value** and **no cost** for the GOODS
A Simple Problem

- 1 SELLER
- 1 BUYER
- \( k \) GOODS (ITEMS)
A Simple Problem

1 SELLER

1 BUYER

\( k \) GOODS (ITEMS)

OBJECTIVE:

MAXIMIZE the REVENUE of the SELLER
ONE GOOD ($k = 1$):
ONE GOOD ($k = 1$):

Myerson 1981
ONE GOOD ($k = 1$):

- SELLER posts a PRICE $p$
ONE GOOD ($k = 1$):

- **SELLER** posts a **PRICE** $p$
- **BUYER** chooses between:
  - get the good and pay $p$, or
  - get nothing and pay nothing

Myerson 1981
ONE GOOD ($k = 1$):

- **SELLER posts** a PRICE $p$
- **BUYER chooses** between:
  - get the good and pay $p$, or
  - get nothing and pay nothing
- $p$ such that \( R = p \cdot \text{Pr}[X > p] \) \( = p \cdot (1 - F(p)) \) is MAXIMAL

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Myerson 1981
ONE GOOD ($k = 1$):

- **SELLER** posts a **PRICE** $p$
- **BUYER** chooses between:
  - get the good and pay $p$, or
  - get nothing and pay nothing

$p$ such that **REVENUE** $R = p \cdot \Pr[X > p]$

$= p \cdot (1 - F(p))$ is **MAXIMAL**

$$\operatorname{Rev}(X) = \max_p p \cdot (1 - F(p))$$

Myerson 1981
One Good: Example

\[ X \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]
One Good: Example

\[ X \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases} \]

\[ p = 10 \rightarrow R = 10 \cdot 1 = 10 \]
One Good: Example

\[ X \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
22 & \text{with probability } \frac{1}{2} 
\end{cases} \]

- \( p = 10 \rightarrow R = 10 \cdot 1 = 10 \)
- \( p = 22 \rightarrow R = 22 \cdot \frac{1}{2} = 11 \)
One Good: Example

\[ X \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- \( p = 10 \rightarrow R = 10 \cdot 1 = 10 \)
- \( p = 22 \rightarrow R = 22 \cdot 1/2 = 11 \)
One Good: Example

\[ X \sim \left\{ \begin{array}{l}
10 \quad \text{with probability } \frac{1}{2} \\
22 \quad \text{with probability } \frac{1}{2}
\end{array} \right. \]

- \( p = 10 \rightarrow R = 10 \cdot 1 = 10 \)
- \( p = 22 \rightarrow R = 22 \cdot \frac{1}{2} = 11 \)

\[ \text{REV}(X) = 11 \quad p = 22 \]
Two Goods ($k = 2$)
Two Goods

Two Goods \((k = 2)\), Independent
Two Goods ($k = 2$), Independent

sell separately:
Two Goods ($k = 2$), Independent
sell separately:

\[
\text{PRICE} = p_1 \quad \text{for good 1}
\]
\[
\text{PRICE} = p_2 \quad \text{for good 2}
\]
Two Goods ($k = 2$), Independent
Two Goods: Example

Two Goods \((k = 2)\), Independent
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[ X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[ X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

\( \text{REV}(X_1) = \text{REV}(X_2) = 11 \)
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

\[\text{REV}(X_1) = \text{REV}(X_2) = 11\]

\[\max(10 \cdot 1, 22 \cdot 1/2) = 11\]
Two Goods: Example

Two Goods \( (k = 2) \), Independent

\[ X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- \( \text{REV}(X_1) = \text{REV}(X_2) = 11 \)
- \( \max(10 \cdot 1, 22 \cdot 1/2) = 11 \)
- \( \text{REV}(X_1) + \text{REV}(X_2) = 11 + 11 = 22 \)
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[ X_1, X_2 \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases} \]

- \(\text{Rev}(X_1) = \text{Rev}(X_2) = 11\)
- \(\max(10 \cdot 1, 22 \cdot 1/2) = 11\)
- \(\text{Rev}(X_1) + \text{Rev}(X_2) = 11 + 11 = 22\)
- sell the two goods together ("bundle") for the price \(p_{12} = 32\) :
Two Goods: Example

Two Goods ($k = 2$), Independent

$$X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases}$$

- $\text{REV}(X_1) = \text{REV}(X_2) = 11$
  $$\max(10 \cdot 1, 22 \cdot 1/2) = 11$$

- $\text{REV}(X_1) + \text{REV}(X_2) = 11 + 11 = 22$

- sell the two goods together ("bundle") for the price $p_{12} = 32$ :
  $$R = 32 \cdot 3/4 = 24 > 22$$
Two Goods: Example

Two Goods \((k = 2)\), Independent

\[ X_1, X_2 \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

\[ \text{Rev}(X_1) = \text{Rev}(X_2) = 11 \]

\[ \max(10 \cdot 1, 22 \cdot 1/2) = 11 \]

\[ \text{Rev}(X_1) + \text{Rev}(X_2) = 11 + 11 = 22 \]

sell the two goods together ("bundle") for the price \(p_{12} = 32\) :

\[ R = 32 \cdot 3/4 = 24 > 22 \]
OUTCOME:
OUTCOME:

$q_i = \text{probability that } \text{BUYER gets good } i$
OUTCOME:

- \( q_i \) = probability that **BUYER** gets good \( i \)
- \( q = (q_1, \ldots, q_k) \in [0, 1]^k \)
OUTCOME:

- $q_i = \text{probability that BUYER gets good } i$
- $q = (q_1, \ldots, q_k) \in [0, 1]^k$
- $s = \text{payment from BUYER to SELLER ("REVENUE")}$
OUTCOME:

- $q_i =$ probability that BUYER gets good $i$
  
  $q = (q_1, \ldots, q_k) \in [0, 1]^k$

- $s =$ payment from BUYER to SELLER ("REVENUE")

PAYOFF (utility) of BUYER when his valuation is $x = (x_1, \ldots, x_k)$:
**General Mechanism**

**OUTCOME:**
- $q_i =$ probability that **BUYER** gets good $i$
  \[ q = (q_1, \ldots, q_k) \in [0, 1]^k \]
- $s =$ payment from **BUYER** to **SELLER**
  ("REVENUE")

**PAYOFF** (utility) of **BUYER** when his valuation is
  \[ x = (x_1, \ldots, x_k) : \]
- $b = q_1 \cdot x_1 + \ldots + q_k \cdot x_k - s$
General Mechanism

**OUTCOME:**
- \( q_i \) = probability that **BUYER** gets good \( i \)
- \( q = (q_1, ..., q_k) \in [0, 1]^k \)
- \( s \) = payment from **BUYER** to **SELLER**
  ("REVENUE")

**PAYOFF** (utility) of **BUYER** when his valuation is \( x = (x_1, ..., x_k) \):
- \( b = q_1 \cdot x_1 + ... + q_k \cdot x_k - s = q \cdot x - s \)
Simple Mechanism

MENU $M$: a SET of possible OUTCOMES
**Simple Mechanism**

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$
**MENU** \( M \): a **SET** of possible **OUTCOMES**

\[ M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R} \]

**SELLER posts a MENU** \( M \)
**Simple Mechanism**

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$

- **SELLER** posts a **MENU** $M$
- **BUYER** chooses one **OUTCOME** in **MENU** $M$: 
**Simple Mechanism**

**MENU** $M$: a set of possible outcomes

$M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$

- **Seller posts** a **MENU** $M$
- **Buyer chooses** one outcome in **MENU** $M$:
  - **Outcome** chosen by **Buyer** when his valuation is $x$: $(q(x), s(x)) \in M$
Simple Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$

- **SELLER** posts a **MENU** $M$
- **BUYER** chooses one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$
  - payoff of **SELLER**: $s(x)$
**Simple Mechanism**

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0, 1]^{k} \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$
- **BUYER chooses** one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$
  - payoff of **SELLER**: $s(x)$
  - payoff of **BUYER**: $b(x) = q(x) \cdot x - s(x)$
"Menu" Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

\[ M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R} \]

- **SELLER** posts a **MENU** $M$

- **BUYER** chooses one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: \((q(x), s(x)) \in M\)
  - payoff of **SELLER**: $s(x)$
  - payoff of **BUYER**: $b(x) = q(x) \cdot x - s(x)$
"Menu" Mechanism

**MENU** $M$: a set of possible outcomes

$$M = \{ (q, s) \} \subset [0, 1]^k \times \mathbb{R}$$

- **Seller** posts a **MENU** $M$
- **Buyer** chooses one **outcome** in **MENU** $M$:
  - **Outcome** chosen by **Buyer** when his valuation is $x$: $$(q(x), s(x)) \in M$$
  - Payoff of **Seller**: $s(x)$
  - Payoff of **Buyer**: $$b(x) = q(x) \cdot x - s(x)$$

**The Revelation Principle:**
"Menu" Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$

- **SELLER posts** a **MENU** $M$
- **BUYER chooses** one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$
  - payoff of **SELLER**: $s(x)$
  - payoff of **BUYER**: $b(x) = q(x) \cdot x - s(x)$

*The Revelation Principle*: Every mechanism is equivalent to a **MENU MECHANISM**
"Menu" Mechanism

**MENU** $M$: a **SET** of possible **OUTCOMES**

$$M = \{(q, s)\} \subset [0, 1]^k \times \mathbb{R}$$

- **SELLER** posts a **MENU** $M$
- **BUYER** chooses one **OUTCOME** in **MENU** $M$:
  - **OUTCOME** chosen by **BUYER** when his valuation is $x$: $(q(x), s(x)) \in M$
  - payoff of **SELLER**: $s(x)$
  - payoff of **BUYER**: $b(x) = q(x) \cdot x - s(x)$

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**The Revelation Principle**: Every mechanism is equivalent to a **MENU MECHANISM** ("direct mechanism")
Incentive Compatibility (IC)
Incentive Compatibility (IC)

\[ q_1(x) \cdot x_1 + \cdots + q_k(x) \cdot x_k - s(x) \geq 0 \]
\[ q_1(y) \cdot x_1 + \cdots + q_k(y) \cdot x_k - s(y) \geq 0 \]  
(for all \( x \) and \( y \))
Incentive Compatibility (IC)

\[ q_1(x) \cdot x_1 + \ldots + q_k(x) \cdot x_k - s(x) \geq 0 \]

\[ q_1(y) \cdot x_1 + \ldots + q_k(y) \cdot x_k - s(y) \]

(for all \( x \) and \( y \))

Individual Rationality (IR) / Participation
Incentive Compatibility (IC)

\[ q_1(x) \cdot x_1 + \ldots + q_k(x) \cdot x_k - s(x) \geq 0 \]
\[ q_1(y) \cdot x_1 + \ldots + q_k(y) \cdot x_k - s(y) \geq 0 \]

(for all \( x \) and \( y \))

Individual Rationality (IR) / Participation

\[ q_1(x) \cdot x_1 + \ldots + q_k(x) \cdot x_k - s(x) \geq 0 \]

(for all \( x \))
Maximize Revenue:
Maximize Revenue:

\[
\text{maximize} \quad R = \mathbb{E}[s(X)] = \int s(x) d\mathcal{F}(x)
\]
Maximize Revenue:

maximize

$$R = \mathbb{E}[s(X)] = \int s(x) d\mathcal{F}(x)$$

subject to

$$(q, s) \text{ satisfies IC & IR}$$
Incentive Compatibility (IC)
Incentive Compatibility (IC)

\[ b(x) \] is a convex function of \( x \)
Incentive Compatibility (IC)

\[ b(x) \] is a convex function of \( x \)

Proof
Incentive Compatibility (IC)

- $b(x)$ is a convex function of $x$

Proof

- for every $y$: $q(y) \cdot x - s(y)$ affine in $x$
Incentive Compatibility

Incentive Compatibility (IC)

- $b(x)$ is a convex function of $x$

Proof

- for every $y$: $q(y) \cdot x - s(y)$ affine in $x$
- $b(x) = \max_y \{q(y) \cdot x - s(y)\}$ convex in $x$
Incentive Compatibility (IC)

- \( b(x) \) is a convex function of \( x \)

Proof

- for every \( y \): \( q(y) \cdot x - s(y) \) affine in \( x \)
- \( b(x) = \max_y \{ q(y) \cdot x - s(y) \} \) convex in \( x \)
- \( b(x) = q(x) \cdot x - s(x) \)
Incentive Compatibility (IC)

- $b(x)$ is a convex function of $x$

Proof

- for every $y$: \[ q(y) \cdot x - s(y) \] affine in $x$
- $b(x) = \max_y \{q(y) \cdot x - s(y)\}$ convex in $x$
- $b(x) = q(x) \cdot x - s(x) = \nabla b(x) \cdot x - s(x)$
Incentive Compatibility (IC) \iff

- $b(x)$ is a convex function of $x$ and
- $q_i(x) = \frac{\partial b(x)}{\partial x_i}$ for a.e. $x$ and all $i$
- $q(x) = \nabla b(x)$ for a.e. $x$

**Proof**

- for every $y$: $q(y) \cdot x - s(y)$ affine in $x$
- $b(x) = \max_y \{q(y) \cdot x - s(y)\}$ convex in $x$
- $b(x) = q(x) \cdot x - s(x) = \nabla b(x) \cdot x - s(x)$
Incentive Compatibility

Incentive Compatibility (IC) \iff

\begin{itemize}
  \item \( b(x) \) is a convex function of \( x \) and \[ b(x) = \max_y \{ q(y) \cdot x - s(y) \} \] convex in \( x \)
  \item \( q_i(x) = \frac{\partial b(x)}{\partial x_i} \) for a.e. \( x \) and all \( i \)
  \item \( q(x) = \nabla b(x) \) for a.e. \( x \)
\end{itemize}

Proof

\begin{itemize}
  \item for every \( y \): \( q(y) \cdot x - s(y) \) affine in \( x \)
  \item \( b(x) = \max_y \{ q(y) \cdot x - s(y) \} \) convex in \( x \)
  \item \( b(x) = q(x) \cdot x - s(x) = \nabla b(x) \cdot x - s(x) \)
  \item \( s(x) = \nabla b(x) \cdot x - b(x) \)
\end{itemize}
Incentive Compatibility (IC) \iff \begin{align*} b(x) & \text{ is a convex function of } x \quad \text{ and} \\ q_i(x) &= \frac{\partial b(x)}{\partial x_i} \quad \text{for a.e. } x \text{ and all } i \\ q(x) &= \nabla b(x) \quad \text{for a.e. } x \\ s(x) &= \nabla b(x) \cdot x - b(x) \end{align*}
Incentive Compatibility (IC) \[\iff\]

- \( b(x) \) is a convex function of \( x \) and
- \( q_i(x) = \frac{\partial b(x)}{\partial x_i} \) for a.e. \( x \) and all \( i \)
- \( q(x) = \nabla b(x) \) for a.e. \( x \)
- \( s(x) = \nabla b(x) \cdot x - b(x) \)
\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_{+}^k \rightarrow \mathbb{R}_{+} \text{ s.t.} \]
Maximal Revenue

$\mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \rightarrow \mathbb{R}_+ \text{ s.t.}$

$b$ is a convex function
\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)
Maximal Revenue

\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)
Maximal Revenue

\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)

\[ R(b, X) = \mathbb{E}[\nabla b(X) \cdot X - b(X)] \]
Maximal Revenue

\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)

\[ R(b, X) = \mathbb{E}[\nabla b(X) \cdot X - b(X)] = \mathbb{E}[b'(X; X) - b(X)] \]
Maximal Revenue

- \( \mathcal{B}^k \) = set of all functions \( b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \) s.t.
  - \( b \) is a convex function, \( b(0) = 0 \)
  - \( 0 \leq \partial b(x)/\partial x_i \leq 1 \) for a.e. \( x \)

- \( R(b, X) = \mathbb{E}[\nabla b(X) \cdot X - b(X)] \)
  = \( \mathbb{E}[b'(X; X) - b(X)] \)

- \( \text{REV}(X) = \max_{b \in \mathcal{B}^k} R(b, X) \)
Maximal Revenue

\( \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \text{ s.t.} \)

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)

\[ R(b, X) = \mathbb{E}[\nabla b(X) \cdot X - b(X)] \]
\[ = \mathbb{E}[b'(X; X) - b(X)] \]

\[ \text{REV}(X) = \max_{b \in \mathcal{B}^k} R(b, X) \]

- \( \mathcal{B}^k \) is a closed convex set
Maximal Revenue

\( \mathcal{B}^k = \) set of all functions \( b : \mathbb{R}^+_k \rightarrow \mathbb{R}_+ \) s.t.

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)

\[
R(b, X) = \mathbb{E}[\nabla b(X) \cdot X - b(X)] = \mathbb{E}[b'(X; X) - b(X)]
\]

\[
\text{REV}(X) = \max_{b \in \mathcal{B}^k} R(b, X)
\]

\( \mathcal{B}^k \) is a closed \textbf{convex} set

\( R(b, X) \) is \textbf{linear} in \( b \)
Maximal Revenue

- $\mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \to \mathbb{R}_+ \text{ s.t.}$
- $b$ is a convex function, $b(0) = 0$
- $0 \leq \partial b(x)/\partial x_i \leq 1$ for a.e. $x$

- $R(b, X) = \mathbb{E}[\nabla b(X) \cdot X - b(X)]$
  $= \mathbb{E}[b'(X; X) - b(X)]$

- $\text{REV}(X) = \max_{b \in \mathcal{B}^k} R(b, X)$
  - $\mathcal{B}^k$ is a closed convex set
  - $R(b, X)$ is linear in $b$

- $\text{REV}(X) = \max_{b \in \text{EXT}(\mathcal{B}^k)} R(b, X)$
  ($\text{EXT} = \text{set of extreme points}$)
Maximal Revenue

\[ \mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \) for a.e. \( x \)

\[ R(b, X) = \mathbb{E}[\nabla b(X) \cdot X - b(X)] = \mathbb{E}[b'(X; X) - b(X)] \]

\[ \text{REV}(X) = \max_{b \in \mathcal{B}^k} R(b, X) \]

- \( \mathcal{B}^k \) is a closed convex set
- \( R(b, X) \) is linear in \( b \)

\[ \text{REV}(X) = \max_{b \in \text{EXT}(\mathcal{B}^k)} R(b, X) \]

Manelli & Vincent 2007
Maximal Revenue: One Good

- $\mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \rightarrow \mathbb{R}_+ \text{ s.t. }$
  - $b$ is a convex function, $b(0) = 0$
  - $0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1$ for a.e. $x$

- $R(b, X) = \mathbb{E}[\nabla b(X) \cdot X - b(X)]$

- $\text{REV}(X) = \max_{b \in \mathcal{B}^k} R(b, X)$
Maximal Revenue: One Good

- $\mathcal{B}^1 = \text{set of all functions } b : \mathbb{R}_+^1 \rightarrow \mathbb{R}_+ \text{ s.t.}$

- $b$ is a convex function, $b(0) = 0$

- $0 \leq \frac{\partial b(x)}{\partial x_i} \leq 1 \text{ for a.e. } x$

- $R(b, X) = \mathbb{E}[\nabla b(X) \cdot X - b(X)]$

- $\text{REV}(X) = \max_{b \in \mathcal{B}^1} R(b, X)$
Maximal Revenue: One Good

\[ B^1 = \text{set of all functions } b : \mathbb{R}_+^1 \rightarrow \mathbb{R}_+ \text{ s.t.} \]

\[ b \text{ is a convex function, } b(0) = 0 \]

\[ 0 \leq b'(x) \leq 1 \text{ for a.e. } x \]

\[ R(b, X) = \mathbb{E}[\nabla b(X) \cdot X - b(X)] \]

\[ \text{REV}(X) = \max_{b \in B^1} R(b, X) \]
Maximal Revenue: One Good

\[ B^1 = \text{set of all functions } b : \mathbb{R}^1_+ \to \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq b'(x) \leq 1 \) for a.e. \( x \)

\[ R(b, X) = \mathbb{E}[b'(X) \cdot X - b(X)] \]

\[ \text{REV}(X) = \max_{b \in B^1} R(b, X) \]
Maximal Revenue: One Good

\[ \mathcal{B}^1 = \text{set of all functions } b : \mathbb{R}_+ \to \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq b'(x) \leq 1 \) for a.e. \( x \)

\[ \text{REV}(X) = \max_{b \in \mathcal{B}^1} \mathbb{E}[b'(X) \cdot X - b(X)] \]
Maximal Revenue: One Good

\[ \mathcal{B}^1 = \text{set of all functions } b : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq b'(x) \leq 1 \) for a.e. \( x \)

\[ \text{REV}(X) = \max_{b \in \mathcal{B}^1} \mathbb{E}[b'(X) \cdot X - b(X)] \]
Maximal Revenue: One Good

\[ B^1 = \text{set of all functions } b : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ s.t.} \]

- \( b \) is a convex function, \( b(0) = 0 \)
- \( 0 \leq b'(x) \leq 1 \) for a.e. \( x \)

\[ \text{REV}(X) = \max_{b \in B^1} E[b'(X) \cdot X - b(X)] \]

\[ B^1 = \text{closed convex hull of } \{b_p\}_{p \geq 0} \text{ where } \]
\[ b_p(x) = \max\{0, x - p\} \]
Maximal Revenue: One Good

- \( \mathcal{B}^1 = \) set of all functions \( b : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) s.t.
  - \( b \) is a convex function, \( b(0) = 0 \)
  - \( 0 \leq b'(x) \leq 1 \) for a.e. \( x \)

\[
\text{REV}(X) = \max_{b \in \mathcal{B}^1} E[b'(X) \cdot X - b(X)]
\]

- \( \mathcal{B}^1 = \) closed convex hull of \( \{b_p\}_{p \geq 0} \) where
  \( b_p(x) = \max\{0, x - p\} \)

\[
\text{REV}(X) = \max_{p \geq 0} E[b'_p(X) \cdot X - b_p(X)]
\]
Maximal Revenue: One Good Good

- \( \mathcal{B}^1 \) = set of all functions \( b : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) s.t.
  - \( b \) is a convex function, \( b(0) = 0 \)
  - \( 0 \leq b'(x) \leq 1 \) for a.e. \( x \)

\[ \text{REV}(X) = \max_{b \in \mathcal{B}^1} \mathbb{E}[b'(X) \cdot X - b(X)] \]

\[ \mathcal{B}^1 = \text{closed convex hull of } \{b_p\}_{p \geq 0} \text{ where } \]
\[ b_p(x) = \max\{0, x - p\} \]

\[ \text{REV}(X) = \max_{p \geq 0} \mathbb{E}[b'_p(X) \cdot X - b_p(X)] \]
\[ = \max_{p \geq 0} \mathbb{E}[(X - (X - p))1_{X \geq p}] \]
Maximal Revenue: One Good Good

- $\mathcal{B}^1$ = set of all functions $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ s.t.
- $b$ is a convex function, $b(0) = 0$
- $0 \leq b'(x) \leq 1$ for a.e. $x$

$$\text{REV}(X) = \max_{b \in \mathcal{B}^1} \mathbb{E}[b'(X) \cdot X - b(X)]$$

- $\mathcal{B}^1$ = closed convex hull of $\{b_p\}_{p \geq 0}$ where $b_p(x) = \max\{0, x - p\}$

$$\text{REV}(X) = \max_{p \geq 0} \mathbb{E}[b'_p(X) \cdot X - b_p(X)]$$

$$= \max_{p \geq 0} \mathbb{E}[(X - (X - p))1_{X \geq p}]$$

$$= \max_{p \geq 0} p \cdot (1 - F(p))$$
Maximal Revenue: $k \geq 2$ Goods
Maximal Revenue: $k \geq 2$ Goods

$\mathcal{B}^k = \text{set of all functions } b : \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \text{ s.t.}

- b \text{ is a convex function, } b(0) = 0
- 0 \leq \partial b(x)/\partial x_i \leq 1 \text{ for a.e. } x$
Maximal Revenue: $k \geq 2$ Goods

- $\mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \rightarrow \mathbb{R}_+$ s.t.
  - $b$ is a convex function, $b(0) = 0$
  - $0 \leq \partial b(x)/\partial x_i \leq 1$ for a.e. $x$

- $\mathbf{REV}(X) = \max_{b \in \text{EXT}(\mathcal{B}^k)} R(b, X)$
Maximal Revenue: $k \geq 2$ Goods

- $\mathcal{B}^k =$ set of all functions $b : \mathbb{R}^k_+ \to \mathbb{R}_+$ s.t.
- $b$ is a convex function, $b(0) = 0$
- $0 \leq \partial b(x) / \partial x_i \leq 1$ for a.e. $x$

- $\text{REV}(X) = \max_{b \in \text{EXT}(\mathcal{B}^k)} R(b, X)$

- EXTREME points of $\mathcal{B}^k =$ ?
Maximal Revenue: $k \geq 2$ Goods

- $\mathcal{B}^k = \text{set of all functions } b : \mathbb{R}^k_+ \to \mathbb{R}_+ \text{ s.t.}$
- $b$ is a convex function, $b(0) = 0$
- $0 \leq \partial b(x)/\partial x_i \leq 1$ for a.e. $x$

$$\text{REV}(X) = \max_{b \in \text{EXT}(\mathcal{B}^k)} R(b, X)$$

- EXTREME points of $\mathcal{B}^k = ?$
- EXTREMELY COMPLEX!
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

Separate:
\[ \text{REV}(Y) + \text{REV}(Z) \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

Separate:
\[ \text{Rev}(Y) + \text{Rev}(Z) \]
\[ \max(10 \cdot 1, 22 \cdot 1/2) = 11 \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
22 & \text{with probability } \frac{1}{2}
\end{cases} \]

Separate:

\[ \text{REV}(Y) + \text{REV}(Z) = 11 + 11 = 22 \]
\[ \max(10 \cdot 1, 22 \cdot \frac{1}{2}) = 11 \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
22 & \text{with probability } \frac{1}{2} 
\end{cases} \]

Separate:

\[ \text{REV}(Y) + \text{REV}(Z) = 11 + 11 = 22 \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
22 & \text{with probability } \frac{1}{2}
\end{cases} \]

- **Separate:**
  \[ \text{REV}(Y) + \text{REV}(Z) = 11 + 11 = 22 \]

- **Bundled:**
  \[ \text{REV}(Y + Z) \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases} \]

- **Separate:**
  \[ \text{Rev}(Y) + \text{Rev}(Z) = 11 + 11 = 22 \]

- **Bundled:**
  \[ \text{Rev}(Y + Z) \]
  \[ \max(20 \cdot 1, 32 \cdot 3/4, 44 \cdot 1/4) = 24 \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- **Separate:**
  \[ \text{REV}(Y) + \text{REV}(Z) = 11 + 11 = 22 \]

- **Bundled:**
  \[ \text{REV}(Y + Z) = 32 \cdot 3/4 = 24 \]
  \[ \max(20 \cdot 1, 32 \cdot 3/4, 44 \cdot 1/4) = 24 \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
22 & \text{with probability } 1/2 
\end{cases} \]

- **Separate:**
  \[ \text{REV}(Y) + \text{REV}(Z) = 11 + 11 = 22 \]

- **Bundled:**
  \[ \text{REV}(Y + Z) = 32 \cdot 3/4 = 24 \]
Two Goods: Example 1

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 22 & \text{with probability } 1/2 \end{cases} \]

- **Separate:**
  \[ \text{Rev}(Y) + \text{Rev}(Z) = 11 + 11 = 22 \]

- **Bundled:**
  \[ \text{Rev}(Y + Z) = 32 \cdot \frac{3}{4} = 24 \]

**PRICE FOR THE BUNDLE**
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

- Separate:
  \[ \text{Rev}(Y) + \text{Rev}(Z) \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

Separate:

\[
\text{Rev}(Y) + \text{Rev}(Z) = \max(10 \cdot 1, 50 \cdot 1/2) = 25
\]
Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
50 & \text{with probability } \frac{1}{2} 
\end{cases} \]

Separate:

\[
\text{Rev}(Y) + \text{Rev}(Z) = 25 + 25 = 50
\]

\[
\max(10 \cdot 1, 50 \cdot 1/2) = 25
\]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

Separate:

\[ \text{Rev}(Y) + \text{Rev}(Z) = 25 + 25 = 50 \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } \frac{1}{2} \\
50 & \text{with probability } \frac{1}{2}
\end{cases} \]

- **Separate:**
  \[ \text{REV}(Y) + \text{REV}(Z) = 25 + 25 = 50 \]

- **Bundled:**
  \[ \text{REV}(Y + Z) \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

- Separate:
  \[ \text{Rev}(Y) + \text{Rev}(Z) = 25 + 25 = 50 \]

- Bundled:
  \[ \text{Rev}(Y + Z) \]
  \[ \max(20 \cdot 1, 60 \cdot 3/4, 100 \cdot 1/4) = 45 \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

- **Separate:**
  \[ \text{Rev}(Y) + \text{Rev}(Z) = 25 + 25 = 50 \]

- **Bundled:**
  \[ \text{Rev}(Y + Z) = 60 \cdot 3/4 = 45 \]
  \[ \max(20 \cdot 1, 60 \cdot 3/4, 100 \cdot 1/4) = 45 \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 10 & \text{with probability } 1/2 \\ 50 & \text{with probability } 1/2 \end{cases} \]

- **Separate:**
  \[ \text{REV}(Y) + \text{REV}(Z) = 25 + 25 = 50 \]

- **Bundled:**
  \[ \text{REV}(Y + Z) = 60 \cdot \frac{3}{4} = 45 \]
Two Goods: Example 2

Independent and Identically Distributed (IID)

\[ Y, Z \sim \begin{cases} 
10 & \text{with probability } 1/2 \\
50 & \text{with probability } 1/2 
\end{cases} \]

- **Separate:**
  \[ \text{REV}(Y) + \text{REV}(Z) = 25 + 25 = 50 \]

- **Bundled:**
  \[ \text{REV}(Y + Z) = 60 \cdot \frac{3}{4} = 45 \]

PRICE FOR EACH GOOD
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \quad \text{(IID)} \\
2 & \text{w/probability } 1/3 
\end{cases} \]
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)} \]

Separate:

\[
\max(0 \cdot 1, 1 \cdot \frac{2}{3}, 2 \cdot \frac{1}{3}) = \frac{2}{3}
\]
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)} \]

Separate: 
\[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]
\[ \max(0 \cdot 1, 1 \cdot \frac{2}{3}, 2 \cdot \frac{1}{3}) = \frac{2}{3} \]
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 & \text{w/probability } \frac{1}{3} \\
1 & \text{w/probability } \frac{1}{3} \\
2 & \text{w/probability } \frac{1}{3} 
\end{cases} \quad \text{(IID)} \]

Separate: \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 0 & \text{w/probability } 1/3 \\ 1 & \text{w/probability } 1/3 \quad \text{(IID)} \\ 2 & \text{w/probability } 1/3 \end{cases} \]

**Separate:** \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)

**Bundled:**
\[
\max(0 \cdot 1, 1 \cdot \frac{8}{9}, 2 \cdot \frac{6}{9}, 3 \cdot \frac{3}{9}, 4 \cdot \frac{1}{9})
\]
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 \text{ w/probability } 1/3 \\
1 \text{ w/probability } 1/3 \quad \text{(IID)} \\
2 \text{ w/probability } 1/3 
\end{cases} \]

- **Separate:** \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)
- **Bundled:**
  \[
  \max(0 \cdot 1, 1 \cdot \frac{8}{9}, 2 \cdot \frac{6}{9}, 3 \cdot \frac{3}{9}, 4 \cdot \frac{1}{9}) = \frac{4}{3}
  \]
Two Goods: Example 3

\[\begin{align*}
Y, Z \sim & \begin{cases}
0 \text{ w/probability } 1/3 \\
1 \text{ w/probability } 1/3 \\
2 \text{ w/probability } 1/3
\end{cases} \quad \text{(IID)}
\end{align*}\]

- **Separate:** \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)
- **Bundled:** \( R = \frac{4}{3} \)

\[\max(0 \cdot 1, 1 \cdot \frac{8}{9}, 2 \cdot \frac{6}{9}, 3 \cdot \frac{3}{9}, 4 \cdot \frac{1}{9}) = \frac{4}{3}\]
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 & \text{w/probability } \frac{1}{3} \\
1 & \text{w/probability } \frac{1}{3} \\
2 & \text{w/probability } \frac{1}{3} 
\end{cases} \quad \text{(IID)} \]

- **Separate:** \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)
- **Bundled:** \( R = \frac{4}{3} \)
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 & \text{w/ probability } \frac{1}{3} \\
1 & \text{w/ probability } \frac{1}{3} \quad \text{(IID)} \\
2 & \text{w/ probability } \frac{1}{3} 
\end{cases} \]

- **Separate:** \[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]
- **Bundled:** \[ R = \frac{4}{3} \]

\[ b(y, z) = \max(0, y - 2, z - 2, y + z - 3) \]
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)} \]

- **Separate:** \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)
- **Bundled:** \( R = \frac{4}{3} \)

\[ b(y, z) = \max(0, y - 2, z - 2, y + z - 3) \]

\[ s(2, 0) = s(0, 2) = 2 \]
\[ s(2, 1) = s(1, 2) = s(2, 2) = 3 \]
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)} \]

- **Separate:** \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)
- **Bundled:** \( R = \frac{4}{3} \)

\( b(y, z) = \max(0, y - 2, z - 2, y + z - 3) \)

\[
\begin{align*}
s(2, 0) &= s(0, 2) = 2 \\
s(2, 1) &= s(1, 2) = s(2, 2) = 3 \\
R &= 2 \cdot \frac{2}{9} + 3 \cdot \frac{3}{9} = \frac{13}{9}
\end{align*}
\]
Two Goods: Example 3

\[
Y, Z \sim \begin{cases} 
0 & \text{w/probability } \frac{1}{3} \\
1 & \text{w/probability } \frac{1}{3} \\
2 & \text{w/probability } \frac{1}{3}
\end{cases} \quad \text{(IID)}
\]

Separate: \[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]

Bundled: \[ R = \frac{4}{3} \]

\[ b(y, z) = \max(0, y - 2, z - 2, y + z - 3) \]

\[ R(b) = \frac{13}{9} \]
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \\
2 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)} \]

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- **Separate:** \( R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \)
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\[ b(y, z) = \max(0, y - 2, z - 2, y + z - 3) \]

\[ R(b) = \frac{13}{9} = \text{REV}(Y, Z) \]

**THE UNIQUE OPTIMAL MECHANISM**
Two Goods: Example 3

\[ Y, Z \sim \begin{cases} 
0 & \text{w/probability } 1/3 \\
1 & \text{w/probability } 1/3 \quad \text{(IID)} \\
2 & \text{w/probability } 1/3
\end{cases} \]

- **Separate:** \[ R = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \]
- **Bundled:** \[ R = \frac{4}{3} \]

\[ b(y, z) = \max(0, y - 2, z - 2, y + z - 3) \]

\[ R(b) = \frac{13}{9} = \text{REV}(Y, Z) \]

**THE UNIQUE OPTIMAL MECHANISM**

**PRICE FOR EACH GOOD AND FOR BUNDLE**
Two Goods: Example 4

\[(Y, Z) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases}\]
Two Goods: Example 4

\[(Y, Z) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases}\]

\[b(y, z) = \max(0, \frac{1}{2}y - \frac{1}{2}, z - 2, y + z - 5)\]
Two Goods: Example 4

\[(Y, Z) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3
\end{cases}\]

\[b(y, z) = \max(0, \frac{1}{2} y - \frac{1}{2}, z - 2, y + z - 5)\]

\[R(b) = 2.5\]
Two Goods: Example 4

\[ (Y, Z) \sim \begin{cases} 
(1, 0) \text{ w/probability } 1/3 \\
(0, 2) \text{ w/probability } 1/3 \\
(3, 3) \text{ w/probability } 1/3 
\end{cases} \]

\[ b(y, z) = \max(0, \frac{1}{2}y - \frac{1}{2}, z - 2, y + z - 5) \]

\[ R(b) = 2.5 = \text{REV}(Y, Z) \]
Two Goods: Example 4

\[(Y, Z) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases} \]

\[b(y, z) = \max(0, \frac{1}{2}y - \frac{1}{2}, z - 2, y + z - 5)\]

\[R(b) = 2.5 = \text{REV}(Y, Z)\]

THE UNIQUE OPTIMAL MECHANISM
Two Goods: Example 4

\[(Y, Z) \sim \begin{cases} 
(1, 0) \quad \text{w/probability } 1/3 \\
(0, 2) \quad \text{w/probability } 1/3 \\
(3, 3) \quad \text{w/probability } 1/3 
\end{cases}\]

\[b(y, z) = \max(0, \frac{1}{2}y - \frac{1}{2}, z - 2, y + z - 5)\]

\[R(b) = 2.5 = \text{REV}(Y, Z)\]

THE UNIQUE OPTIMAL MECHANISM

\[b_1(y, z) = \max(0, y - 1, z - 2, y + z - \_\_)\]
Two Goods: Example 4

\[(Y, Z) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases} \]

\[b(y, z) = \max(0, \frac{1}{2}y - \frac{1}{2}, z - 2, y + z - 5)\]

\[R(b) = 2.5 = \REV(Y, Z)\]

THE UNIQUE OPTIMAL MECHANISM

\[b_1(y, z) = \max(0, y - 1, z - 2, y + z - )\]

\[b_0(y, z) = \max(0, z - 2, y + z - )\]
Two Goods: Example 4

\((Y, Z) \sim \begin{cases} 
(1, 0) \text{ w/probability } 1/3 \\
(0, 2) \text{ w/probability } 1/3 \\
(3, 3) \text{ w/probability } 1/3 
\end{cases} \)

\[
b(y, z) = \max(0, \frac{1}{2} y - \frac{1}{2}, z - 2, y + z - 5)
\]

\[
R(b) = 2.5 = \text{REV}(Y, Z)
\]

THE UNIQUE OPTIMAL MECHANISM

\[
b_1(y, z) = \max(0, y - 1, z - 2, y + z - 4)
\]

\[
b_0(y, z) = \max(0, z - 2, y + z - \quad)
\]
Two Goods: Example 4

\[(Y, Z) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases}\]

\[b(y, z) = \max(0, \frac{1}{2}y - \frac{1}{2}, z - 2, y + z - 5)\]

\[R(b) = 2.5 = \text{REV}(Y, Z)\]

THE UNIQUE OPTIMAL MECHANISM

\[b_1(y, z) = \max(0, y - 1, z - 2, y + z - 4)\]

\[b_0(y, z) = \max(0, z - 2, y + z - 5)\]
Two Goods: Example 4

\[(Y, Z) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases}\]

\[b(y, z) = \max(0, \frac{1}{2} y - \frac{1}{2}, z - 2, y + z - 5)\]

\[R(b) = 2.5 = \text{REV}(Y, Z)\]

THE UNIQUE OPTIMAL MECHANISM

\[b_1(y, z) = \max(0, y - 1, z - 2, y + z - 4)\]

\[R(b_1) = 2.33\ldots\]

\[b_0(y, z) = \max(0, z - 2, y + z - 5)\]
Two Goods: Example 4

\[(Y, Z) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3 
\end{cases} \]

\[b(y, z) = \max(0, \frac{1}{2}y - \frac{1}{2}, z - 2, y + z - 5)\]

\[R(b) = 2.5 = \text{Rev}(Y, Z)\]

**THE UNIQUE OPTIMAL MECHANISM**

\[b_1(y, z) = \max(0, y - 1, z - 2, y + z - 4)\]

\[R(b_1) = 2.33...\]

\[b_0(y, z) = \max(0, z - 2, y + z - 5)\]

\[R(b_0) = 2.33...\]
Two Goods: Example 4

\[(Y, Z) \sim \begin{cases} 
(1, 0) & \text{w/probability } 1/3 \\
(0, 2) & \text{w/probability } 1/3 \\
(3, 3) & \text{w/probability } 1/3
\end{cases}\]

\[b(y, z) = \max(0, \frac{1}{2}y - \frac{1}{2}, z - 2, y + z - 5)\]

\[R(b) = 2.5 = \text{REV}(Y, Z)\]

THE UNIQUE OPTIMAL MECHANISM

PRICE FOR LOTTERIES ON GOODS
Two Goods: Example 4’
Two Goods: Example 4’

\[ Y, Z \sim \begin{cases} 
1 & \text{w/probability } 1/6 \\
2 & \text{w/probability } 1/2 \\
4 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)} \]
Two Goods: Example 4’

\[ Y, Z \sim \begin{cases} 
1 & \text{w/probability } 1/6 \\
2 & \text{w/probability } 1/2 \\
4 & \text{w/probability } 1/3 
\end{cases} \quad \text{(IID)}
\]

THE UNIQUE OPTIMAL MECHANISM:

\[ b(y, z) = \max(0, \frac{1}{2}y - 1, \frac{1}{2}z - 1, y + z - 4) \]
Two Goods: Example 4’

\[ Y, Z \sim \begin{cases} 
1 & \text{w/probability } \frac{1}{6} \\
2 & \text{w/probability } \frac{1}{2} \quad \text{(IID)} \\
4 & \text{w/probability } \frac{1}{3} 
\end{cases} \]

THE UNIQUE OPTIMAL MECHANISM:

\[ b(y, z) = \max(0, \frac{1}{2}y - 1, \frac{1}{2}z - 1, y + z - 4) \]

PRICE FOR LOTTERIES ON GOODS
Revenue maximizing mechanisms:
Revenue maximizing mechanisms:
1. post a price for each good separately
Multiple Goods

Revenue maximizing mechanisms:
1. post a price for each good separately
2. post a price for the bundle
Revenue maximizing mechanisms:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
Multiple Goods

Revenue maximizing mechanisms:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries
Revenue maximizing mechanisms:

1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries

1 – 3: deterministic mechanisms
4: stochastic mechanisms
Revenue maximizing mechanisms:
1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries
Multiple Goods

Revenue maximizing mechanisms:
1. post a price for each good separately
2. post a price for the bundle
3. post prices for each good separately and for the bundle
4. post prices for various lotteries

$X_1, X_2, ..., X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.}$
Multiple Goods, I.I.D. Uniform

\(X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.}\)

\(k = 1: \quad b(x) = \max(0, \ x_1 - \frac{1}{2})\)
Multiple Goods, I.I.D. Uniform

\[ X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \]

- \( k = 1 \): \[ b(x) = \max(0, x_1 - \frac{1}{2}) \]
- \( k = 2 \): \[ b(x) = \max(0, x_i - \frac{2}{3}, x_1 + x_2 - \frac{4-\sqrt{2}}{3}) \]
Multiple Goods, I.I.D. Uniform

\( X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \)

- \( k = 1 \): \( b(x) = \max(0, x_1 - \frac{1}{2}) \)
- \( k = 2 \):
  \[
  b(x) = \max(0, x_i - \frac{2}{3}, x_1 + x_2 - \frac{4-\sqrt{2}}{3})
  \]
- \( k = 3 \):
  \[
  b(x) = \max(0, x_i - \frac{3}{4}, x_i + x_j - \frac{6-\sqrt{2}}{4}, x_1 + x_2 + x_3 - s)
  \]
Multiple Goods, I.I.D. Uniform

\(X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.}\)

- \(k = 1: b(x) = \max(0, x_1 - \frac{1}{2})\)
- \(k = 2:\)
  \[b(x) = \max(0, x_i - \frac{2}{3}, x_1 + x_2 - \frac{4-\sqrt{2}}{3})\]
- \(k = 3: b(x) = \max(0, x_i - \frac{3}{4}, x_i + x_j - \frac{6-\sqrt{2}}{4}, x_1 + x_2 + x_3 - s)\)

where \(s = \frac{9}{4} - \frac{\sqrt{6}}{4} \cos\left(\frac{1}{3} \arctan\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right) - \frac{3\sqrt{2}}{4} \sin\left(\frac{1}{3} \arctan\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right)\)
Multiple Goods, I.I.D. Uniform

\[ X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \]

- \( k = 1: \quad b(x) = \max(0, \ x_1 - \frac{1}{2}) \)
- \( k = 2: \quad b(x) = \max(0, \ x_i - \frac{2}{3}, \ x_1 + x_2 - \frac{4-\sqrt{2}}{3}) \)
- \( k = 3: \quad b(x) = \max(0, \ x_i - \frac{3}{4}, \ x_i + x_j - \frac{6-\sqrt{2}}{4}, \ x_1 + x_2 + x_3 - s) \)

where \( s \approx 1.2257... = \text{solution of 3rd degree equation with coefficients in } \mathbb{Q}[^{\sqrt{2}}] \)
\( X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \)

- \( k = 1: b(x) = \max(0, x_1 - \frac{1}{2}) \)
- \( k = 2: \)
  \[ b(x) = \max(0, x_i - \frac{2}{3}, x_1 + x_2 - \frac{4-\sqrt{2}}{3}) \]
- \( k = 3: b(x) = \max(0, x_i - \frac{3}{4}, x_i + x_j - \frac{6-\sqrt{2}}{4}, x_1 + x_2 + x_3 - s) \)
  
  \[ \ldots \]
Multiple Goods, I.I.D. Uniform

\[ X_1, X_2, \ldots, X_k \sim \text{Uniform } [0, 1], \text{ i.i.d.} \]

- \( k = 1 \): \( b(x) = \max(0, x_1 - \frac{1}{2}) \)
- \( k = 2 \):
  \[ b(x) = \max(0, x_i - \frac{2}{3}, x_1 + x_2 - \frac{4-\sqrt{2}}{3}) \]
- \( k = 3 \): \( b(x) = \max(0, x_i - \frac{3}{4}, x_i + x_j - \frac{6-\sqrt{2}}{4}, x_1 + x_2 + x_3 - s) \)
  
  \[ \ldots \]

---

Monotonicity

If valuations of **BUYER** increase
Monotonicity

If valuations of **BUYER** *increase* then maximal revenue of **SELLER** *increases* (weakly)
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** \( k = 1 \):
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $x' > x$. 
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $x' > x$.

$$q(x)x - s(x) \geq q(x')x - s(x')$$  (IC: $x \rightarrow x'$)
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $x' > x$.

\[ q(x) x - s(x) \geq q(x') x - s(x') \]  
\[ (IC: x \rightarrow x') \]

\[ q(x') x' - s(x') \geq q(x) x' - s(x) \]  
\[ (IC: x' \rightarrow x) \]
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $x' > x$.

$q(x) \cdot x - s(x) \geq q(x') \cdot x - s(x')$ (IC: $x \rightarrow x'$)

$q(x') \cdot x' - s(x') \geq q(x) \cdot x' - s(x)$ (IC: $x' \rightarrow x$)

$\Rightarrow (q(x') - q(x))(x' - x) \geq 0$ (add)
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $x' > x$.

$$q(x) \cdot x - s(x) \geq q(x') \cdot x - s(x') \quad \text{(IC: } x \rightarrow x')$$

$$q(x') \cdot x' - s(x') \geq q(x) \cdot x' - s(x) \quad \text{(IC: } x' \rightarrow x)$$

$$\Rightarrow (q(x') - q(x))(x' - x) \geq 0 \quad \text{(add)}$$

$$\Rightarrow q(x') \geq q(x) \quad \text{ (} x' > x)$$
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** \( k = 1 \): Let \( x' > x \).

\[
q(x) x - s(x) \geq q(x') x - s(x') \quad \text{(IC: } x \rightarrow x')
\]
\[
q(x') x' - s(x') \geq q(x) x' - s(x) \quad \text{(IC: } x' \rightarrow x)
\]
\[
\Rightarrow (q(x') - q(x))(x' - x) \geq 0 \quad \text{(add)}
\]
\[
\Rightarrow q(x') \geq q(x) \quad \text{ (} x' > x \text{)}
\]
\[
s(x') - s(x) \geq (q(x') - q(x)) x \quad \text{(IC: } x \rightarrow x')
\]
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $x' > x$.

\[ q(x)x - s(x) \geq q(x')x - s(x') \quad (\text{IC: } x \rightarrow x') \]
\[ q(x')x' - s(x') \geq q(x)x' - s(x) \quad (\text{IC: } x' \rightarrow x) \]
\[ \Rightarrow (q(x') - q(x))(x' - x) \geq 0 \quad \text{(add)} \]
\[ \Rightarrow q(x') \geq q(x) \quad (x' > x) \]
\[ s(x') - s(x) \geq (q(x') - q(x))x \quad (\text{IC: } x \rightarrow x') \]
\[ \Rightarrow s(x') - s(x) \geq 0 \]
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$: Let $x' > x$.

$$\Rightarrow s(x') - s(x) \geq 0$$
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** $k = 1$:

- $x' > x \implies s(x') \geq s(x)$
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** \( k = 1 \):

- \( x' > x \Rightarrow s(x') \geq s(x) \)
- Every IC mechanism has **monotonic** \( s \)
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** \( k = 1 \):

- \( x' > x \Rightarrow s(x') \geq s(x) \)
- Every **IC** mechanism has **monotonic** \( s \)
- \( \Rightarrow \) Revenue of every **IC** mechanism is **monotonic** w.r.t. to **BUYER** valuations
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** \( k = 1 \):

- \( x' > x \Rightarrow s(x') \geq s(x) \)
- Every **IC** mechanism has monotonic \( s \)
- \( \Rightarrow \) Revenue of every **IC** mechanism is **monotonic** w.r.t. to **BUYER** valuations
- \( \Rightarrow \) Maximal revenue is **monotonic** w.r.t. **BUYER** valuations
Monotonicity

If valuations of **BUYER** increase then maximal revenue of **SELLER** increases (weakly)

**Proof for** \( k = 1 \):

- \( x' > x \Rightarrow s(x') \geq s(x) \)

- Every **IC** mechanism has **monotonic** \( s \)

- \( \Rightarrow \) Revenue of every **IC** mechanism is **monotonic** w.r.t. to **BUYER** valuations

- \( \Rightarrow \) Maximal revenue is **monotonic** w.r.t. **BUYER** valuations

**Proof for** \( k > 1 \) ?
Non-Monotonicity: Example

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]
Non-Monotonicity: Example

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]
Non-Monotonicity: Example

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]
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\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]
Non-Monotonicity: Example

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]
Non-Monotonicity: Example

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]

\[
\begin{align*}
& (10, 23) : z - 20 \\
& (20, 27) : y - 10
\end{align*}
\]
Non-Monotonicity: Example

\[
b(y, z) = \max(0, y - 10, z - 20, y + z - 40)
\]

\[(10, 23) : z - 20 \]

\[(20, 27) : y - 10 \]

\[y \text{ increases} \]
\[z \text{ increases} \]
\[s \text{ DECREASES!} \]
Non-Monotonicity: Example

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]
Non-Monotonicity: Example

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]

There exist 2-good valuations \( X = (Y, Z) \) for which this \( b \) MAXIMIZES REVENUE
Non-Monotonicity: Example

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]

There exist 2-good valuations \( X = (Y, Z) \) for which this \( b \) MAXIMIZES REVENUE (moreover: unique maximizer; robust)
Non-Monotonicity

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]

- There exist 2-good valuations \( X = (Y, Z) \) for which this \( b \) MAXIMIZES REVENUE (moreover: unique maximizer; robust)

- There exist 2-good valuations \( X, X' \) s.t.
  \[ X' \geq X \text{ but } Rev(X') < Rev(X) \]
Non-Monotonicity

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]

- There exist 2-good valuations \( X = (Y, Z) \) for which this \( b \) **MAXIMIZES REVENUE** (moreover: unique maximizer; robust)

- There exist 2-good valuations \( X, X' \) s.t.
  \[ X' \geq X \text{ but } \text{Rev}(X') < \text{Rev}(X) \]

- There exists 2-good **I.I.D.** valuations \( X, X' \)
Non-Monotonicity

\[ b(y, z) = \max(0, y - 10, z - 20, y + z - 40) \]

- There exist 2-good valuations \( X = (Y, Z) \) for which this \( b \) **MAXIMIZES REVENUE** (moreover: unique maximizer; robust)

- There exist 2-good valuations \( X, X' \) s.t.
  \[ X' \geq X \] but \( \text{Rev}(X') < \text{Rev}(X) \)

- There exists 2-good **I.I.D.** valuations \( X, X' \)
  \[ X_1, X_2 \sim \text{i.i.d.-}F, \quad X'_1, X'_2 \sim \text{i.i.d.-}G \]
  \[ X'_1 \geq X_1, X'_2 \geq X_2, \quad \text{Rev}(X') < \text{Rev}(X) \]
Problem:

- Maximize revenue
- 2 goods, 1 buyer
- Each good has 10 possible values
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**Solution:**
- $3 \times 10 \times 10 = 300$ numbers
  (Linear Programming)
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What is the **STRUCTURE** of the solution?
Does \textsc{Computational Complexity} capture all the difficulty of a problem?
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Even after computing the precise solution, one may not understand what it is, what it means, what it represents ...
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⇒ "**CONCEPTUAL COMPLEXITY**"
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⇒

**“CONCEPTUAL COMPLEXITY”**

= complexity of the **STRUCTURE** of the solution
Summary: Multiple Goods
Maximizing revenue with multiple goods:
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many of the results for ONE GOOD
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Summary: Multiple Goods

Maximizing revenue with multiple goods:

- many of the results for **ONE GOOD** are **FALSE** for **MULTIPLE GOODS**
- is an extremely complex problem (even for simple distributions)
- “what we have learned from one good is too good to be true for two goods”
- ?
- HOW GOOD are **SIMPLE** mechanisms for **MULTIPLE GOODS**?
Two Independent Goods
Two Independent Goods

Theorem 1. If $X_1$ and $X_2$ are INDEPENDENT AND IDENTICALLY DISTRIBUTED:
Two Independent Goods

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\[ \text{REV}(X_1) + \text{REV}(X_2) \geq \frac{e}{e+1} \text{REV}(X_1, X_2) \]
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$$= 2p^* \cdot (1 - F(p^*))$$
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Posting the optimal one-good price per unit guarantees at least 73% of the optimal revenue.
Theorem 2. If $Y$ and $Z$ are INDEPENDENT:

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Proof.
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E[s(Y, Z)] \leq E[s(Y, Z) 1_{Y \geq Z}] + E[s(Y, Z) 1_{Z \geq Y}]
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- $E[s(Y, Z)] \leq E[s(Y, Z) 1_{Y \geq Z}] + E[s(Y, Z) 1_{Z \geq Y}]$

- Claim. $E[s(Y, Z) 1_{Y \geq Z}] \leq 2 \text{REV}(Y)$
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\( \implies \mathbb{E}[s(Y, Z)] \leq 2 \text{REv}(Y) + 2 \text{REv}(Z) \)

\( \implies \text{REv}(Y, Z) \leq 2 \text{REv}(Y) + 2 \text{REv}(Z) \)
Claim. \( \mathbb{E} [s(Y, Z) 1_{Y \geq Z}] \leq 2 \text{Rev}(Y) \)
Theorem 2: Proof

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Claim. \( E[s(Y, Z) 1_{Y \geq Z}] \leq 2 \text{REV}(Y) \)

Proof.
Theorem 2: Proof

Claim. \[ \mathbb{E}[s(Y, Z) 1_{Y \geq Z}] \leq 2 \text{REV}(Y) \]

Proof. For every fixed \( z \):
Theorem 2: Proof

Claim. $\mathbb{E}[s(Y, Z) 1_{Y \geq Z}] \leq 2 \text{Rev}(Y)$

Proof. For every fixed $z$:

- Instead of giving $z$ with probability $q_2$, give a "monetary refund" of $q_2 \cdot z$
Theorem 2: Proof

Claim. \( \mathbb{E}[s(Y, Z) 1_{Y \geq Z}] \leq 2 \text{Rev}(Y) \)

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- Instead of giving \( z \) with probability \( q_2 \), give a "monetary refund" of \( q_2 \cdot z \), i.e.
  - \( \tilde{q}(y) := q_1(y, z) \)
  - \( \tilde{s}(y) := s(y, z) - q_2(y, z) \cdot z \)
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Then: $(\tilde{q}, \tilde{s})$ is IC&IR for good $y$. 
Theorem 2: Proof

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  Then: \( (\tilde{q}, \tilde{s}) \) is IC&IR for good \( y \).

- \[
  s(y, z) = \tilde{s}(y) + q_2(y, z) \cdot z \leq \tilde{s}(y) + z
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Claim. \( \mathbb{E}[s(Y, Z) 1_{Y \geq Z}] \leq 2 \text{Rev}(Y) \)

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- Then: \((\tilde{q}, \tilde{s})\) is IC&IR for good \( y \).
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Proof. For every fixed $z$:

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  \[ \leq \text{Rev}(Y) + zP[Y \geq z] \]
  \[ \leq \text{Rev}(Y) + \text{Rev}(Y) \]
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- $(\tilde{q}, \tilde{s})$ is IC&IR for good $y$.
- $s(y, z) \leq \tilde{s}(y) + z$
- $\mathbb{E}[s(Y, z) 1_{Y \geq z}] \leq \mathbb{E}[\tilde{s}(Y) 1_{Y \geq z}] + \mathbb{E}[z 1_{Y \geq z}]$
  \[\leq \text{REV}(Y) + z \mathbb{P}[Y \geq z]\]
  \[\leq \text{REV}(Y) + \text{REV}(Y)\]

- Take expectation over the values $z$ of $Z$
A class of IC&IR mechanisms $\mathbf{\checkmark}$
A class of IC&IR mechanisms $\mathcal{N}$

A family of valuations $X$ (distributions)
A class of IC&IR mechanisms $\mathcal{N}$

A family of valuations $\mathcal{X}$ (distributions)

**Guaranteed Fraction of Optimal Revenue**
A class of IC&IR mechanisms $\mathcal{N}$
A family of valuations $X$ (distributions)

GUARANTEED FRACTION OF OPTIMAL REVENUE
A class of IC&IR mechanisms $\mathcal{N}$

A family of valuations $\mathbb{X}$ (distributions)

**Guaranteed Fraction of Optimal Revenue**

= maximal fraction $\alpha$ in $[0, 1]$ such that for every valuation $X$ in $\mathbb{X}$ there is a mechanism $\nu$ in $\mathcal{N}$ satisfying

$$R(\nu, X) \geq \alpha \cdot \text{Rev}(X)$$
A class of IC&IR mechanisms $\mathcal{N}$

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$$\text{GFOR} = \inf_{X \in \mathbf{X}} \frac{\mathcal{N} - \text{REV}(X)}{\text{REV}(X)}$$
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$$R(\nu, X) \geq \alpha \cdot \text{REV}(X)$$

$$\text{GFOR} = \inf_{X \in X} \frac{\mathcal{N} \cdot \text{REV}(X)}{\text{REV}(X)} = \inf_{X \in X} \frac{\sup_{\nu \in \mathcal{N}} R(\nu, X)}{\sup_{\mu \in M} R(\mu, X)}$$

($M = \text{class of all IC&IR mechanisms}$)
GFOR: Two Goods

- SEPARATE selling of I.I.D. goods:
SEPARATE selling of I.I.D. goods:

\[ 73\% \leq \text{GFOR} \]
GFOR: Two Goods

- **SEPARATE** selling of **I.I.D.** goods:
  
  \[ 73\% \leq GFOR \]

- **SEPARATE** selling of **INDEPENDENT** goods:
GFOR: Two Goods

- **SEPARATE** selling of I.I.D. goods:
  \[ 73\% \leq \text{GFOR} \]

- **SEPARATE** selling of INDEPENDENT goods:
  \[ 50\% \leq \text{GFOR} \]
GFOR: Two Goods

1 BUYER, 2 GOODS

● SEPARATE selling of I.I.D. goods:

\[ 73\% \leq \text{GFOR} \]

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GFOR: Two Goods

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\( n \) independent BUYERS, 2 GOODS
GFOR: Two Goods

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GFOR: Two Goods

1 BUYER, 2 GOODS

- SEPARATE selling of I.I.D. goods:
  \[ 73\% \leq \text{GFOR} \leq 78\% \]

- SEPARATE selling of INDEPENDENT goods:
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GFOR: $n$ Buyers, Two Goods

$n$ independent BUYERS, 2 GOODS

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$50\% \leq \text{GFOR}$
GFOR: $n$ Buyers, Two Goods

$n$ independent **BUYERS**, 2 **GOODS**

- **SEPARATE** selling of **INDEPENDENT** goods:

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Holds for:
GFOR: \( n \) Buyers, Two Goods

\( n \) independent \textbf{BUYERS}, 2 \textbf{GOODS}

- \textbf{SEPARATE} selling of \textbf{INDEPENDENT} goods:

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Holds for:

- \textbf{BAYESIAN-NASH} implementation
GFOR: $n$ Buyers, Two Goods

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Holds for:

- **BAYESIAN-NASH** implementation
- **DOMINANT-STRATEGY** implementation
\[ \text{n independent BUYERS, 2 GOODS} \]

- SEPARATE selling of INDEPENDENT goods:
  \[ 50\% \leq \text{GFOR} \]

Holds for:
- BAYESIAN-NASH implementation
- DOMINANT-STRATEGY implementation

(in each case: use the same implementation for one good and for two goods)
2 GOODS, ARBITRARY DEPENDENCE
2 GOODS, ARBITRARY DEPENDENCE

- SEPARATE selling:
2 GOODS, ARBITRARY DEPENDENCE

- SEPARATE selling: \[ \text{GFOR} = 0 \]
2 GOODS, ARBITRARY DEPENDENCE

- SEPARATE selling: \( \text{GFOR} = 0 \)

for every \( \varepsilon > 0 \) there is a valuation \( X \) in \([0, 1]^2\) such that: \( S\text{REV}(X) < \varepsilon \cdot R\text{EV}(X) \)
2 GOODS, ARBITRARY DEPENDENCE

- **SEPARATE** selling: \( \text{GFOR} = 0 \)
- **BUNDLED** selling:
GFOR: Correlated Goods

2 GOODS, ARBITRARY DEPENDENCE

- SEPARATE selling: $\text{GFOR} = 0$
- BUNDLED selling: $\text{GFOR} = 0$
2 GOODS, ARBITRARY DEPENDENCE

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2 GOODS, ARBITRARY DEPENDENCE

- **SEPARATE** selling: \( \text{GFOR} = 0 \)
- **BUNDLED** selling: \( \text{GFOR} = 0 \)
- **DETERMINISTIC** mechanisms:
2 GOODS, ARBITRARY DEPENDENCE

- **SEPARATE** selling: \( \text{GFOR} = 0 \)
- **BUNLED** selling: \( \text{GFOR} = 0 \)
- **DETERMINISTIC** mechanisms: \( \text{GFOR} = 0 \)
GFOR: Correlated Goods

2 Goods, Arbitrary Dependence

- **SEPARATE** selling: \( \text{GFOR} = 0 \)
- **BUNDLED** selling: \( \text{GFOR} = 0 \)
- **DETERMINISTIC** mechanisms: \( \text{GFOR} = 0 \)

For every \( \varepsilon > 0 \) there is a valuation \( X \) in \([0, 1]^2\) such that \( \text{DRev}(X) < \varepsilon \cdot \text{Rev}(X) \)
**GFOR: Correlated Goods**

$k$ GOODS, ARBITRARY DEPENDENCE

- **SEPARATE** selling: $\text{GFOR} = 0$
- **BUNDLED** selling: $\text{GFOR} = 0$
- **DETERMINISTIC** mechanisms: $\text{GFOR} = 0$

The same holds for any $k \geq 2$ goods

for every $\varepsilon > 0$ there is a valuation $X$ in $[0, 1]^k$ such that $\text{DR}_{\text{Rev}}(X) < \varepsilon \cdot \text{Rev}(X)$
$$\text{MRev}^m = \text{maximal REVENUE from mechanisms with AT MOST } m \text{ OUTCOMES (i.e., with MENU SIZE } \leq m)$$
Menu Size

\[ \text{MRev}^{[m]} = \text{maximal REVENUE from mechanisms with AT MOST } m \text{ OUTCOMES} \]

\( \text{(i.e., with MENU SIZE } \leq m \) \)

- \( \text{MRev}^{[m]} \) for fixed \( m \): \( \text{GFOR} = 0 \)
Menu Size

\[ \text{MRev}^m = \text{maximal REVENUE from mechanisms with AT MOST } m \text{ OUTCOMES (i.e., with MENU SIZE } \leq m) \]

- \( \text{MRev}^m \) for fixed \( m \): \( \text{GFOR} = 0 \)
- \( \text{MRev}^m \) increases with \( m \) (polynomially)
Menu Size

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\( \text{DETERMINISTIC-Rev} \sim \text{MRev}^{2^k} \)
**Menu Size Complexity**

\[ \text{MRev}^m = \text{maximal REVENUE from mechanisms with AT MOST } m \text{ OUTCOMES (i.e., with MENU SIZE } \leq m) \]

- \( \text{MRev}^m \) for fixed \( m \): GFOR = 0
- \( \text{MRev}^m \) increases with \( m \) (polynomially)
- DETERMINISTIC-REV \( \sim \) MRev\(^{2^k}\)

**MENU SIZE** = measure of the COMPLEXITY of mechanisms
GUARANTEED FRACTION OF OPTIMAL REVENUE of SIMPLE mechanisms for two goods:
GUARANTEED FRACTION OF OPTIMAL REVENUE

of SIMPLE mechanisms for two goods:

- INDEPENDENT AND IDENTICALLY DISTRIBUTED (I.I.D.) goods:
Summary: GFOR

**Guaranteed Fraction of Optimal Revenue**
of SIMPLE mechanisms for two goods:

- **Independent and Identically Distributed (I.I.D.)** goods:

  \[ \text{GFOR} \geq 73\% \]
GUARANTEED FRACTION OF OPTIMAL REVENUE
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**Guaranteed Fraction of Optimal Revenue**

of **SIMPLE** mechanisms for two goods:

- **Independent and Identically Distributed (I.I.D.)** goods:
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- **Independent** goods:
  \[ \text{GFOR} \geq 50\% \]
Summary: GFOR

**GUARANTEED FRACTION OF OPTIMAL REVENUE**
of SIMPLE mechanisms for two goods:

- **INDEPENDENT AND IDENTICALLY DISTRIBUTED (I.I.D.)** goods:
  \[ \text{GFOR} \geq 73\% \]

- **INDEPENDENT** goods:
  \[ \text{GFOR} \geq 50\% \]

- **CORRELATED** goods:
Summary: GFOR

Guaranteed Fraction of Optimal Revenue of Simple mechanisms for two goods:

- Independent and identically distributed (I.I.D.) goods:
  \[ \text{GFOR} \geq 73\% \]

- Independent goods:
  \[ \text{GFOR} \geq 50\% \]

- Correlated goods:
  \[ \text{GFOR} = 0\% \]
Summary: Multiple Goods
Maximizing revenue with multiple goods:
Maximizing revenue with multiple goods:

- many of the results for ONE GOOD are FALSE for MULTIPLE GOODS
Maximizing revenue with multiple goods:

- many of the results for **ONE GOOD** are **FALSE** for **MULTIPLE GOODS**
- is an extremely complex problem (even for simple distributions)
Maximizing revenue with multiple goods:

- many of the results for ONE GOOD are FALSE for MULTIPLE GOODS
- is an extremely complex problem (even for simple distributions)
- “what we have learned from one good is too good to be true for two goods”
Maximizing revenue with multiple goods:

- many of the results for **ONE GOOD** are **FALSE** for **MULTIPLE GOODS**
- is an extremely complex problem (even for simple distributions)
- “what we have learned from one good is too good to be true for two goods”
- **SIMPLE** mechanisms **MAY** yield **UNIFORM APPROXIMATION**
"Are you trying to auction your Brussels sprouts again?"