CROSSED MODULES AS HOMOTOPY NORMAL MAPS

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ABSTRACT. In this note we consider crossed modules of groups $(N \to G, G \to \mathrm{Aut}(N))$, as a homotopy version of the inclusion $N \subset G$ of a normal subgroup. Our main observation is a characterization of the underlying map $N \to G$ of a crossed module, in terms of a simplicial group structure on the associated bar construction. This approach allows for "natural" generalizations to other monoidal categories, in particular we consider briefly what we call 'normal maps' between simplicial groups.

1. Introduction and main results

In this note we consider a well-know relation between crossed modules and simplicial groups [Lo, B, Con, CC]. Our aim is to reformulate this association in terms of *homotopy co-limits*. More specifically, we associate to a given crossed module *an explicit simplicial group structure on the bar construction*. This allows for natural definitions and results for similar concepts in the category of loop spaces and in fact, in any monoidal category. It yields, for example, a better understanding of the localization of principal fibrations under monoidal functors [P]. Further, we find in §6 that this reformulation makes the higher versions of crossed modules such as Loday's diagram of groups easier to approach.

Our main Theorem 1.1 below states that a crossed module structure $G \to \mathrm{Aut}(N)$ on a group map $N \to G$, yields directly a simplicial group structure on the usual bar construction (taken here as a model of the homotopy quotient), namely on the simplicial set $\mathrm{Bar}(G, N) = (G \times N^k)_{k \geq 0}$. Thus $\mathrm{Bar}(G, N)$ is isomorphic, as a simplicial set, to a simplicial group which, moreover, is *compatible with the natural action* of G on the bar construction (see Remark 2.4.3). Moreover this process is reversible. In fact, we give an explicit constructive recipes to pass from the cross module structure to the group structure on the bar construction and back.

From this point of view, it is tempting to regard a map of groups $N \to G$ which underlies a crossed module, as a (homotopy) *normal map*, since in some sense it generalizes the inclusion of a normal subgroup $N \subset G$ (see Lemma 2.5.3 and the discussion in Remark 2.5.2). Indeed

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we show that the crossed module map $\ell : G \to \mathrm{Aut}(N)$, which we call a *normal structure* on the map $N \to G$ is inversely associated with a compatible group structure on the homotopy quotient $G//N := hocolim_N G$. As noted above, we take $G//N$ to be a specific simplicial set, namely the bar construction $\mathrm{Bar}(G, N)$.

In subsection 2.4 we define the notion of a *normal simplicial group structure* on $\mathrm{Bar}(G, N)$. In the following theorem we give two mutually inverse associations: let $n\colon N \to G$ be a group homomorphism. Given a normal structure on the map n we construct in §4 a normal simplicial group structure on $\mathrm{Bar}(G, N)$. *Conversely*, given a normal simplicial group structure on $\mathrm{Bar}(G, N)$ we show in §3 that it yields a crossed module structure on the map n. The short subsection 4.2 shows that the above associations are mutual inverses.

Theorem 1.1. *Given a group homomorphism $n : N \to G$, a crossed module structure on the map n gives a normal simplicial group structure on $\mathrm{Bar}(G, N)$, and conversely, any normal simplicial group structure on $\mathrm{Bar}(G, N)$ determines a crossed module structure on the map n. These two explicit associations are mutual inverses.*

Notice that the homotopy type of the underlying space of the simplicial group associated to a cross module (i.e., taken as a simplicial set, forgetting the simplicial group structure) depends only on the underlying map $N \to G$, and not on the normal structure. The normal structure determines the loop-type or the homotopy type of $G//N$ as a simplicial group. See also Remark 2.1.1 below.

Remark 1.2. There is a rich literature on crossed modules in the context of discrete groups, it goes back to the relation between the first and second homotopy groups of a pointed pair of spaces: Here the main example of a crossed module is the boundary map $\pi_2(X, A) \to \pi_1 A$ naturally associated to any inclusion of pointed spaces $A \subseteq X$. In this context, the discussion above is closely related to results and proofs in [Con, Lo].

1.3. **Extensions and natural questions.** The above characterization of a crossed module structure suggests a generalization of the notions of a normal map to maps between say n-loop spaces or E_∞-spaces, or even for a normal map between two normal maps considered in [Lo, (5.3)], and of a normal structure on these maps.

In §5 below we do this generalization in the category of simplicial groups. This can be extended to a notion of a normal map of $n-$loop spaces $\Omega^n : \Omega^n X \to \Omega^n Y$: it is a map such that the associated homotopy quotient $\Omega^n Y//\Omega^n X$ has a "compatible" $n-$ loop space structure. A topological result analogous to the above would characterize normal maps in terms of simplicial spaces. Thus, *any functor that preserves finite products* up to homotopy, will be shown to preserve normal maps between n-loop spaces. This would alow one to consider questions such as: Given a normal map of loop spaces $\Omega X \to \Omega Y$, is the induced map gotten by taking the n-Postnikov sections $P_n\Omega X \to P_n\Omega Y$ also normal – as defined in Definition 5.1 below? What functors would yield a normal map, when applied to a normal map $\Omega X \to \Omega Y$ of loop space? These results have direct implications regarding the behavior of principle fibrations under such functors, in particular localizations and completions. This last question was the origin of the present note, see [DF, P].

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2. Some background material

In this section we describe the basic notions that this paper is involved with. We note that throughout this paper maps are always applied from left to right and are often applied on the right of the variable. We start with:

2.1. **The homotopy quotient as a space.**

For *any* map of groups $h\colon H \to G$ the space $hocolim_H G$ which is the homotopy colimit of the discrete space G under the action of H, (via $H \ni a\colon g \mapsto g(ah)$) is denoted here by $G//H$. The Borel construction on G, namely $EH \times_H G$ gives one possible version of $G//H$. It has the homotopy type of a disjoint union of identical copies of $K(\pi, 1)$-spaces:

$$G//H \approx \coprod_{G/\mathrm{Im}(h)} K(\ker h, 1).$$

The quotient $G/\mathrm{Im}(h)$ above is the quotient set (i.e. the set of cosets). Of course, for a general map $H \to G$, this homotopy quotient space is not equivalent to a loop space, namely it is not homotopy equivalent to a topological (or a simplicial) group.

Remark 2.1.1. We note that the mere fact that the homotopy quotient $G//H$ is homotopically equivalent as a simplicial set to a simplicial group is only a necessary condition for having a crossed module structure (see subsection 2.5 ahead) on $h\colon H \to G$, *but it is not sufficient.* It is not sufficient since for *any* inclusion of groups $H \subseteq G$, the action of H on G is free, so the homotopy quotient is the usual quotient set G/H (i.e. the set of cosets), taken as a constant simplicial set, and hence it trivially posses some simplicial group structure, but if H is not a normal subgroup of G, there is no crossed module structure on h.

2.2. **Simplicial sets and groups.**

A simplicial set $\mathcal{X}$ consists of a collection of sets

$$X_k, \qquad k = 0, 1, 2, \ldots$$

together with *face maps* $d_i = d_i^{(k)}\colon X_k \to X_{k-1}$, $k \geq 1$ and $i \in [0, k]$, and *degeneracy maps* $s_i = s_i^{(k)}\colon B_k \to B_{k+1}$, $k \geq 0$ and $i \in [0, k]$, such that the following *simplicial identities* hold. (Recall that we are composing maps from left to right.)

(1) $d_j d_i = d_i d_{j-1}$, if $i < j$.
(2) $s_j d_i = d_i s_{j-1}$, if $i < j$.
(3) $s_j d_j = id = s_j d_{j+1}$.

(4) $s_j d_i = d_{i-1} s_j \quad$ if $i > j+1$.
(5) $s_j s_i = s_i s_{j+1} \quad$ if $i \leq j$.

We say that the simplicial set $\mathcal{X}$ is a *simplicial group* if X_k is a group, for all k and the face and degeneracy maps are homomorphisms.

2.3. **The simplicial set** $\mathrm{Bar}(X, N)$.

Let N be a group acting on a set X. Denote this action by $a\colon x \to xa$, where $a \in N$ and $x \in X$. Recall that the bar construction (see e.g. [Cur, Mac])

$$B := \mathrm{Bar}(X, N),$$

is the simplicial set consisting of the following data.

(1) For each integer $k \geq 0$, a set B_k which is defined by $B_0 = X$, and $B_k := X \times N^k$, for $k \geq 1$, together with
(2) the *face maps* $d_i^{(k)} \equiv d_i\colon B_k \to B_{k-1}$, for all $k \geq 1$ and $0 \leq i \leq k$, defined by:
   (i) $d_0\colon (x, a_1, \dots, a_k) \mapsto (xa_1, a_2, \dots, a_k)$;
   (ii) $d_i\colon (x, a_1, \dots, a_i, a_{i+1}, \dots, a_k) \mapsto (x, a_1, \dots, a_i \cdot a_{i+1}, \dots, a_k)$, for $1 \leq i < k$;
   (iii) $d_k\colon (x, a_1, \dots, a_k) \mapsto (x, a_1, \dots, a_{k-1})$,
(3) and together with *degeneracy maps* $s_i^{(k)} = s_i\colon B_k \to B_{k+1}$, defined by
$s_i\colon (x, a_1, \dots, a_k) \mapsto (x, a_1, \dots, a_i, 1, a_{i+1}, \dots, a_k)$, for all $k \geq 0$ and $0 \leq i \leq k$.

It is easy to check that $\mathrm{Bar}(X, N)$ is a simplicial set.

Remark 2.3.1. It is "well-known" that under the present conditions (i.e. X, N are discrete) the simplicial set $\mathrm{Bar}(X, N)$ has only two non-trivial homotopy sets: The set $\pi_0 \mathrm{Bar}(X, N)$ and the fundamental groups of the components, all of which are isomorphic to the stabilizer groups of the action of G on X, see also section 2.1 above. In the case where this simplicial set admits a simplicial group structure, as is claimed for normal maps, see below, then its classifying space is connected and has only two non-trivial homotopy groups in dimension one and two. Further, it is known that any such space comes, up to homotopy, from a simplicial group associated to some cross module. We are interested in underlying the direct relation to the bar construction that is associated to any map of groups $N \to G$; where G takes the role of X and N acts on G as in the next paragraph.

Suppose that $X = G$ is a group and N acts on G via a homomorphism $n\colon N \to G$, i.e. the action is $a\colon g \mapsto g(an)$, for all $a \in N$ and $g \in G$, we denote the resulting simplicial set by

$$\mathrm{Bar}(G, N), \tag{2.3.1}$$

suppressing the map n since it is understood from the context.

2.4. A normal simplicial group structure on $\mathrm{Bar}(G,N)$.

In this subsection N and G are groups,

$$n\colon N \to G,$$

is a group homomorphism, and $\mathrm{Bar}(G,N)$ is the simplicial set defined in subsection 2.3 (see eq. (2.3.1)).

Definition 2.4.1. Let $B := \mathrm{Bar}(G,N)$. By a *normal simplicial group structure on* B we mean the following.

(i) $B_0 = G$ is the group G.
(ii) B_k is endowed with a group structure for all $k \geq 1$, we denote the multiplication in B_k by

$$(g, a_1, \dots, a_k) * (h, b_1, \dots, b_k). \tag{2.4.1}$$

(iii) The face maps $d_i^{(k)}$ and the degeneracy maps $s_i^{(k)}$ are group homomorphisms.
(iv) $(g, 1, \dots, 1) * (h, a_1, \dots, a_k) = (gh, a_1, \dots, a_k)$, for all $g \in G$ and $(h, a_1, \dots, a_k) \in B_k$, where multiplication takes place in B_k.

Remark 2.4.2. By straight-forward calculations one shows that condition (iv) of Definition 2.4.1 follows from the same condition for $k = 1$. We omit these calculations for the sake of brevity.

Remark 2.4.3. By the *natural action* of G on $\mathrm{Bar}(G,N)$ we mean $g\colon (h, a_1, \dots, a_k) \mapsto (gh, a_1, \dots, a_k)$, for all $k \geq 0$, $(h, a_1, \dots, a_k) \in B_k$ and $g \in G$. When we say that the multiplication in $\mathrm{Bar}(G,N)$ is *compatible with the natural action* of G, we mean that condition (iv) of Definition 2.4.1 holds.

Notation 2.4.4. Let $k \geq 1$, we denote

(1) $G_k := \{(g, 1, \dots, 1) \mid g \in G\} \leq B_k$.
(2) $N_k := \{(1, a_1, \dots, a_k) \mid a_i \in N\} \leq B_k$.

Lemma 2.4.5. *Suppose that* $\mathrm{Bar}(G,N)$ *is endowed with a normal simplicial group structure. Let* $k \geq 1$*, then* G_k *is a subgroup of* B_k *which is isomorphic to* G*,* N_k *is a normal subgroup of* B_k *,* $B_k = G_k N_k$ *and* $G_k \cap N_k = 1$*.*

Proof. G_k is the image of G_{k-1} under s_{k-1}, so, by induction it is a subgroup of B_k, and since s_{k-1} is injective, it is isomorphic to G. Also, N_k is the kernel of $d_k \circ d_{k-1} \circ \cdots \circ d_1$, so it is a normal subgroup of B_k. Clearly $G_k \cap N_k = 1$ and $B_k = G_k N_k$ by condition (iv) of Definition 2.4.1. □

2.5. **Crossed modules, normal maps and normal structure.**

A crossed module consists of a group homomorphism

$$n\colon N \to G,$$

which we call here a *normal map* (see Remark 2.5.2 below), together with a homomorphism

$$\ell\colon G \to \mathrm{Aut}(N),$$

which we call here *a normal structure* (or *a crossed module structure*) on n, such that when denoting by a^g the image of $a \in N$ under $\ell(g)$ for $g \in G$, the following two requirement are satisfied.

(NM1) $(a^g)n = g^{-1}(an)g$, for all $g \in G$ and $a \in N$.
(NM2) $a^{bn} = b^{-1}ab$, for all $a, b \in N$.

Remark 2.5.1. The following (seemingly not "well-known") diagram may clarify these formulae. Given a map of groups $n : N \to G$ one can always associate with it the following solid arrows square:

$$\begin{tikzcd} N \arrow[r, "conj"] \arrow[d, "n"] & \mathrm{Aut}(N) \arrow[d] \\ G \arrow[r, "conj"] \arrow[ur, dashed, "\ell"] & \mathrm{Hom}_{\mathrm{gps}}(N, G) \end{tikzcd}$$

where the map $G \xrightarrow{conj} \mathrm{Hom}_{\mathrm{gps}}(N, G)$ is the map $G \ni g \mapsto (a \mapsto g^{-1}(an)g,\ \forall a \in N)$ and the map $\mathrm{Aut}(N) \longrightarrow \mathrm{Hom}_{\mathrm{gps}}(N, G)$ is the map $\mathrm{Aut}(N) \ni \sigma \mapsto (a \mapsto (a\sigma)n,\ \forall a \in N)$. A cross module structure $\ell : G \to \mathrm{Aut}(N)$ is simply a (dashed) diagonal lift ℓ rendering the diagram commutative where the lower right triangle is (NM1) and the upper left triangle is (NM2).

Remark 2.5.2. Let G be a group and let N be a subgroup of G. Let $n\colon N \to G$ be the inclusion map and let G/N be the set of left cosets of N in G. Then there is a *natural action* of G on the set G/N via left multiplication and it is easy to check that the following are equivalent.

(i) $N \lhd G$.
(ii) There exists $\ell\colon G \to \mathrm{Aut}(N)$ such that (NM1) and (NM2) hold.
(iii) There exists a group structure $(G/N, *)$ on G/N which is compatible with the natural group action of G on G/N, i.e.

$$g \cdot (hN) = (gN) * (hN), \quad \text{for all } g, h \in G,$$

where $g \cdot (hN)$ is of course $(gh)N$.

We mention this since we indicated in the introduction that Theorem 1.1 is a generalization of the above observation to the case where n is not injective. This also explains our terminology: *normal map* and *normal structure*.

Notice the following well known basic property:

Lemma 2.5.3. *Let $n\colon N \to G$ be a normal map and let $\ell\colon G \to \mathrm{Aut}(N)$, be a normal structure on n, then*

1. *$n(N)$ is normal in G;*
2. $\ker n \leq Z(N)$*;*
3. $\ker n$ *is invariant under $\ell(G)$.*

3. From a normal simplicial group structure on $\mathrm{Bar}(G, N)$ to a normal structure on the map $n\colon N \to G$

In this section N and G are groups and

$$n\colon N \to G,$$

is a group homomorphism. The purpose of this section is to prove that we can recover the normal structure on a map, from a normal simplicial group structure on the associated bar construction. The following lemma shows how to define the crossed module structure $\ell : G \to \mathrm{Aut}(N)$.

Proposition 3.1. *Assume that* $\mathrm{Bar}(G, N)$ *is endowed with a normal simplicial group structure. Then*

1. $(1, a) * (1, b) = (1, ab)$*, for all $a, b \in N$, where multiplication takes place in N_1 (see notation 2.4.1, 2.4.4).*
2. *The map $\ell\colon G \to \mathrm{Aut}(N)$ defined by: $\ell(g)\colon a \to a^g$, where*
$$(1, a^g) := (g^{-1}, 1) * (1, a) * (g, 1).$$
is a normal structure on n (see subsection 2.5).

Note that Proposition 3.1(1) together with Lemma 2.4.5 imply that the map ℓ above is a well defined action of G on N.

We assume that $\mathrm{Bar}(G, N)$ is endowed with a normal simplicial group structure as defined in subsection 2.4, and we adopt the notation of that subsection (see eq. (2.4.1) and Notation 2.4.4).

Lemma 3.2. *Let $k \geq 0$ and let $a, b \in N$. Then*

1. *The identity element of B_k is $(1, \dots, 1)$;*
2. $(1, a^{-1}, a) * (1, 1, b) = (1, a^{-1}, ab)$*;*
3. $(n(a^{-1}), a) * (1, b) = (n(a^{-1}), ab)$*;*
4. $(1, a) * (1, b) = (1, ab)$.

Proof. (1): By definition, the identity element of $B_0 = G$ is the identity element 1 of G. Then, by induction, since $s_0 \colon B_k \to B_{k+1}$ is a group homomorphism, for all $k \geq 0$, part (1) follows.

(2& 3): Applying $d_2^{(2)}$ and using (1) we get that $(1, a^{-1}, a) * (1, 1, b) = (1, a^{-1}, x)$. Applying $d_1^{(2)}$ and using (1) again we get that $(1, b) = (1, a^{-1}x)$, so $x = ab$ and (2) holds. Part (3) follows from (2) by applying $d_0^{(2)}$.

Part (4) follows by a similar calculation. □

Notice that $B_1 = G \ltimes_\ell N$ is a semidirect product of N by G, that is, the product in B_1 is given by

$$(h, a) * (g, b) = (hg, a^g b), \qquad g, h \in G,\ a, b \in N.$$

Recall hypotheses (NM1) and (NM2) from subsection 2.5.

Lemma 3.3. *The map $\varphi \colon G \ltimes_\ell N \to G$ defined by $\varphi \colon (g, a) \to g(an)$ is a homomorphism if and only if n satisfies* (NM1) *above.*

Proof. We have

$$[(h, a)(g, b)]\varphi = (hg, a^g b)\varphi = hg(a^g b)n,$$

while

$$(h, a)\varphi(g, a)\varphi = h(an)g(bn),$$

hence

$$\begin{aligned} [(h, a)(g, b)]\varphi = (h, a)\varphi(g, a)\varphi &\iff \\ (an)g = g(a^g)n \qquad &\iff \\ (a^g)n = (an)^g. \quad \square & \end{aligned}$$

Lemma 3.4. *Consider the action of N on itself via conjugation, $c \colon N \to \mathrm{Aut}(N)$, and form the semidirect product $N \ltimes_c N$ with respect to this action. Thus*

$$(a, b)(c, d) = (ac, b^c d), \qquad a, b, c, d \in N.$$

Then the map $\psi \colon N \ltimes_c N \to G \ltimes_\ell N$ defined by $(a, b) \mapsto (n(a), b)$, is a homomorphism, iff n satisfies (NM2).

Proof. This follows by a straightforward calculation. □

Lemma 3.5. *Let $a, b \in N$. Then*

(1) *for all $k \geq 1$ and all $a_1, \dots, a_k, b_1, \dots, b_k \in N$, we have:*

$$\begin{gathered}(1, a_1, \dots, a_k) * (1, b_1, \dots, b_k) = \\ (1, \quad a_1 b_1, \quad a_2^{b_1} b_2, \quad a_3^{b_1 b_2} b_3, \dots, \quad a_k^{b_1 \cdots b_{k-1}} b_k).\end{gathered}$$

(2) *Let $g \in G$ and let $(1, a_1, \dots, a_k) \in N_k$. Then*

$$(1, a_1, \dots, a_k)^{(g, 1, \dots, 1)} = (1, a_1^g, \dots, a_k^g).$$

Proof. We prove (1) by induction on k. For $k=1$, this is Lemma 3.2(4). Then applying d_k using induction we see that

$$(1, a_1, \dots, a_k) * (1, b_1, \dots, b_k) = (1,\ a_1b_1,\ \dots,\ a_{k-1}^{b_1\cdots b_{k-2}}b_{k-1},\ x),$$

Applying d_{k-1} using induction once more we get that

$$\begin{aligned}(1,\ a_1b_1,\ \dots,\ a_{k-1}^{b_1\cdots b_{k-2}}b_{k-1}x) &= (1, a_1, \dots, a_{k-1}a_k) * (1, b_1, \dots, b_{k-1}b_k)\\ &= (1,\ a_1b_1,\ \dots,\ a_{k-2}^{b_1\cdots b_{k-3}}b_{k-2},\quad (a_{k-1}a_k)^{b_1\cdots b_{k-2}}b_{k-1}b_k).\end{aligned}$$

It follows that $b_{k-1}x = a_k^{b_1\cdots b_{k-2}}b_{k-1}b_k$, so $x = a_k^{b_1\cdots b_{k-1}}b_k$.

Part (2) follows by a similar calculation. □

Lemma 3.6. *The homomorphisms n and ℓ satisfy* (NM1) *and* (NM2)*.*

Proof. Since $B_1 = G \ltimes_\ell N$, and since the homomorphism $d_0\colon B_1 \to B_0$ is defined by $d_0\colon (g,a) \mapsto g(an)$, Lemma 3.3 implies that (NM1) holds.

Next notice that by Lemma 3.5(1), the group N_2 (see Notation 2.4.4) is isomorphic to $N \ltimes_c N$, where $N \ltimes_c N$ is as in Lemma 3.4. Further the map d_0 restricted to N_2 is given by $d_0\colon (1,a,b) \mapsto (n(a), b)$, and it is a homomorphism from $N \ltimes_c N$ to $G \ltimes_\ell N$. Hence, by Lemma 3.4, (NM2) holds. □

Proposition 3.7. *Let $(g, a_1, \dots, a_k), (h, b_1, \dots, b_k) \in B_k$, then*

$$\begin{gathered}(g, a_1, \dots, a_k) * (h, b_1, \dots, b_k) =\\ (gh,\quad a_1^h b_1,\quad a_2^{h(b_1n)}b_2,\quad a_3^{h(b_1b_2)n}b_3, \dots,\quad a_k^{h(b_1\cdots b_{k-1})n}b_k).\end{gathered}$$

Proof. First, by Lemma 3.6, (NM1) and (NM2) hold for n and ℓ. Using Lemma 3.5(1) we get

$$(1, a_1, \dots, a_k) * (1, b_1, \dots, b_k) = (1,\quad a_1b_1,\quad a_2^{b_1n}b_2,\quad a_3^{(b_1b_2)n}b_3, \dots,\quad a_k^{(b_1\cdots b_{k-1})n}b_k).$$

Then the proposition follows using condition (iv) of Definition 2.4.1 and Lemma 3.5(2). □

4. From a normal structure on $n\colon N \to G$ to a normal simplicial group structure on $\mathrm{Bar}(G, N)$

In this section G and N are groups and

$$n\colon N \to G, \qquad \ell\colon G \to \mathrm{Aut}(N),$$

are group homomorphisms. Recall that we denote

$$\ell(g)\colon a \mapsto a^g, \qquad a \in N \text{ and } g \in G.$$

We assume that ℓ is a normal structure on n (see subsection 2.5). We let $\mathrm{Bar}(G, N)$ denote the bar construction using the permutation action of N on the set G via $g \mapsto g(an)$, for all $g \in G$ and $a \in N$ (see eq. (2.3.1)). Our aim in this section is to show that the normal structure ℓ leads to a normal simplicial group structure on $\mathrm{Bar}(G, N)$.

We start by defining multiplication on B_k, for all $k \geq 0$. For $k = 0$, $B_0 = G$ and the multiplication is as in G. For $k \geq 1$, in view of Proposition 3.7 we define multiplication by:

$$(g, a_1, \ldots, a_k) \circ (h, b_1, \ldots, b_k) = \tag{4.1}$$

$$= (gh, \quad a_1^h b_1, \quad a_2^{h(b_1 n)} b_2, \quad a_3^{h(b_1 b_2)n} b_3, \ldots, \quad a_k^{h(b_1 \cdots b_{k-1})n} b_k);$$

Theorem 4.1. *Let $k \geq 0$, then*

(1) *B_k is a group;*
(2) *the map $d_0\colon (g, a_1, \ldots a_k) \mapsto (g(a_1 n), a_2, \ldots, a_k)$ is a homomorphism, $B_k \to B_{k-1}$;*
(3) *the maps*
$$d_i\colon (g, a_1, \ldots, a_k) \mapsto (g, a_1, \ldots, a_{i-1}a_i, \ldots, a_k)$$
are homomorphism, $B_k \to B_{k-1}$, for all $i \in [1, k-1]$;
(4) *the map $d_k\colon (g, a_1, \ldots, a_k) \mapsto (g, a_1, \ldots, a_{k-1})$ is a homomorphism $B_k \to B_{k-1}$;*
(5) *the maps $s_i\colon (g, a_1, \ldots, a_i, 1, a_{i+1}, \ldots, a_k)$ are homomorphisms, for all $i \in [0, k]$.*

Proof. (1): For each $k \geq 1$ define

$$n_k\colon (g, a_1, \ldots, a_k) \to g(a_1 \cdots a_k)n, \qquad B_k \to G.$$

We prove simultaneously that B_k is a group and that n_k is a group homomorphism.

For $k = 1$ this is Lemma 3.3. Suppose this holds for $k - 1$. Then B_{k-1} acts on N via

$$(g, a_1, \ldots, a_{k-1})\colon a \to a^{g(a_1 \cdots a_{k-1})n}.$$

Notice that B_k is just the semidirect product of $B_{k-1} \ltimes N$ with respect to this action, so B_k is a group.

To show that n_k is a group homomorphism we compute

$$\begin{aligned}
&[(g, a_1, \ldots, a_k) \circ (h, b_1, \ldots, b_k)]n_k \\
&= gh[a_1^h b_1 \cdot a_2^{h(b_1 n)} b_2 \cdot a_3^{h(b_1 b_2)n} b_3 \cdots a_k^{h(b_1 \cdots b_{k-1})n} b_k]n \\
&= gh(a_1 n)^h (b_1 n) \cdot (a_2 n)^{h(b_1 n)} (b_2 n) \cdot (a_3 n)^{h(b_1 b_2)n} (b_3 n) \cdots (a_k n)^{h(b_1 \cdots b_{k-1})n} (b_k n) \\
&= (g(a_1 \cdots a_k)n)(h(b_1 \cdots b_k)n) \\
&= (g, a_1, \ldots, a_k)n_k (h, b_1, \ldots, b_k)n_k.
\end{aligned}$$

(2): Let $u, v \in B_k$, $u = (g, a_1, \ldots, a_k)$, $v = (h, b_1, \ldots, b_k)$, then

$$\begin{aligned}
(u \circ v)d_0 &= (gh, \quad a_1^h b_1, \quad a_2^{h(b_1 n)} b_2, \quad a_3^{h(b_1 b_2)n} b_3, \ldots, \quad a_k^{h(b_1 \cdots b_{k-1})n} b_k)d_0 \\
&= (gh(a_1^h b_1)n, \quad a_2^{h(b_1 n)} b_2, \quad a_3^{h(b_1 b_2)n} b_3, \ldots, \quad a_k^{h(b_1 \cdots b_{k-1})n} b_k) \\
&= (g(a_1 n)h(b_1 n), \quad a_2^{h(b_1 n)} b_2, \quad a_3^{h(b_1 b_2)n} b_3, \ldots, \quad a_k^{h(b_1 \cdots b_{k-1})n} b_k) \\
&= (g(a_1 n)h(b_1 n), a_2^{h(b_1 n)} b_2, a_3^{h(b_1 n)(b_2 n)} b_3, \ldots, a_k^{h(b_1 n)(b_2 \cdots b_{k-1})n} b_k) \\
&= (ud_0) \circ (vd_0).
\end{aligned}$$

so (2) holds.

(3): Since n satisfies (NM1) we have $a^{bn} = a^b$, for all $a, b \in N$ so

$$\begin{aligned}
&(u \circ v)d_i \\
&= (gh, \quad a_1^h b_1, \quad a_2^{h(b_1 n)} b_2, \quad a_3^{h(b_1 b_2)n} b_3, \dots, \quad a_k^{h(b_1 \cdots b_{k-1})n} b_k)d_i \\
&= (gh, a_1^h b_1, \dots, \quad a_i^{h(b_1 \cdots b_{i-1})n} b_i a_{i+1}^{h(b_1 \cdots b_i)n} b_{i+1}, \dots, \quad a_k^{h(b_1 \cdots b_{k-1})n} b_k) \\
&= (gh, a_1^h b_1, \dots, \quad (a_i a_{i+1})^{h(b_1 \cdots b_{i-1})n} b_i b_{i+1}, \dots, \quad a_k^{h(b_1 \cdots b_{k-1})n} b_k) \\
&= (g, a_1, \dots, a_i a_{i+1}, a_{i+2}, \dots, a_k) \circ (h, b_1, \dots, b_i b_{i+1}, b_{i+2}, \dots, b_k) \\
&= (ud_i) \circ (vd_i).
\end{aligned}$$

(4): In any semidirect product, projection onto the first coordinate is a homomorphism.
(5): This holds by a similar calculation. □

We conclude this section by showing that, as claimed in Theorem 1.1, the two associations given in §3 and in this section are mutual inverses.

4.2. **The mutual inverse relation between our two associations.**

Let $n\colon N \to G$ be a group homomorphism. In §3 we showed how to start with a normal simplicial group structure on $\mathrm{Bar}(G, N)$ and obtain a normal structure on n, and in this §4 we showed how to start with a normal structure on n and obtain a normal simplicial group structure on $\mathrm{Bar}(G, N)$. The purpose of this brief subsection is to make the observation that these two associations are mutual inverses.

Assume first that $\mathrm{Bar}(G, N)$ is endowed with a normal simplicial group structure, and denote the multiplication in B_k as in equation (2.4.1) (i.e. by $*$). By Proposition 3.1(2) the map $\ell\colon G \to \mathrm{Aut}(N)$ defined by: $\ell(g)\colon a \to a^g$, where $(1, a^g) := (g^{-1}, 1) * (1, a) * (g, 1)$, for all $a \in N$, is a normal structure on n. Further, given this normal structure on n, equation (4.1) tells us how to define a normal simplicial group structure on B_k (with multiplication denoted $\circ$). But now Proposition 3.7 shows that the multiplication $\circ$ is the same as the original multiplication $*$.

Conversely, let $\ell\colon G \to \mathrm{Aut}(N)$ be a normal structure on n (denoted $g \xrightarrow{\ell} (a \mapsto a^g,\ \forall a \in N)$. Let $\circ$ be the multiplication in B_k given in equation (4.1). Let $\ell'\colon G \to \mathrm{Aut}(N)$ be the normal structure on n as obtained in Proposition 3.1(2), where in that proposition $*$ should be replaced by $\circ$. That is for all $g \in G$, $\ell'(g)\colon a \to a'$, where $(1, a') = (g^{-1}, 1) \circ (1, a) \circ (g, 1)$. Now by the definition of $\circ$, we have

$$(g^{-1}, 1) \circ (1, a) \circ (g, 1) = (g^{-1}, a) \circ (g, 1) = (1, a^g).$$

We thus see that $a' = a^g$, for all $a \in N$, that is $\ell'(g) = \ell(g)$, for all $g \in G$.

This completes the observation that the two associations are mutual inverses.

5. NORMAL MAPS BETWEEN SIMPLICIAL GROUPS

The purpose of this section is to extend the definition of a normal map and of a normal structure (see subsection 2.5) from groups to simplicial groups. We refer to this general definition only in remarks about an application of the present approach to Loday's result [Lo, (5.4)] in §6 below.

Definition 5.1. A map $n : N_\bullet \to G_\bullet$ of simplicial groups is called (homotopy) *normal* if the induced map $BN_\bullet \to BG_\bullet$ (i.e. $\overline{W}N_\bullet \to \overline{W}G_\bullet$) can be extended to a homotopy fibration sequence of *connected and pointed* simplicial sets:

$$BN_\bullet \xrightarrow{Bn} BG_\bullet \longrightarrow W_\bullet.$$

A homotopy class of maps $BG_\bullet \to W_\bullet$ whose homotopy fibre inclusion map is equivalent to $BN_\bullet \to BG_\bullet$ is called a *normal structure* on the map n. A map of normal structures is a map of pairs of simplicial groups which is a part of a map of the associated fibration sequences. Here $BG_\bullet \cong \overline{W}G_\bullet$ for any simplicial group $G_\bullet$ denotes the standard classifying space construction [Cur], a space whose loop space is equivalent as a loop space to $G_\bullet$.

It has been observed that a map of discrete groups is normal if and only if there is a fibration sequence on the connected spaces $BN \to BG \to W$, or equivalently, if the given map has the form $\pi_1(E, e) \to \pi_1(X, p(e))$ for some principal fibration $p : E \to X$ of connected spaces [B].

5.2. **Associated Puppe fibration sequence.** Recall that a Puppe sequence is a sequence of pointed spaces and maps between them such that two adjacent arrows give a fibration sequence. Note that *any* map of simplicial groups $n : N_\bullet \to G_\bullet$ yields a five-term Puppe sequence (the first four arrows of eq. (5.1)). A *normal structure* on the map n, as in Definition 5.1, extends this sequence by one term

$$N_\bullet \xrightarrow{n} G_\bullet \to G_\bullet//N_\bullet \to BN_\bullet \to BG_\bullet \xrightarrow{\nu} W_\bullet = B(G_\bullet//N_\bullet) \tag{5.1}$$

The sequence is determined up to equivalence by the last map ν on the right, via successively taking homotopy fibers.

5.3. **A Canonical example.** This section uses some standard knowledge of the space of maps between two spaces. For discrete groups the conjugation map $N \to \mathrm{Aut}(N)$, is a typical example of a normal map, with the identity map $\mathrm{Aut}(N) \to \mathrm{Aut}(N)$ as the normal structure.

Similarly, to pass to topological or simplicial groups, an initially good candidate for the "canonical example" of a normal map of simplicial groups is the conjugation map:

$$G_\bullet \to aut\, G_\bullet.$$

Although this map is well defined for any simplicial group $G_\bullet$, to make good homotopy sense (since $G_\bullet$ is not free) we may proceed as follows:

One associate to each space X the space $aut\, X$ of all self homotopy equivalences $X \to X$. This space is known to be of the homotopy type of a topological group or alternatively it is equivalent to a loop space ΩY for some well defined connected and pointed space Y, the so called classifying space of $aut\, X$, which is denoted below by $Y = Baut\, X$, all spaces are well defined up to well defined homotopy equivalences. Given a simplicial group $G_\bullet$, the Puppe sequence required in the definition of normal maps is the well- known sequence of loop spaces and spaces:

$$G_\bullet \to aut^*\, BG_\bullet \to aut\, BG_\bullet \to BG_\bullet \to Baut^*\, BG_\bullet \to Baut\, BG_\bullet = W_\bullet,$$

where $aut\, BG_\bullet$ is an instance of $aut\, X$ as above, where $Baut^*\, BG_\bullet$ is the classifying space of *pointed* self-equivalence of $BG_\bullet$, and where the required space $W_\bullet$ is the classifying space of unpointed self-equivalences of $BG_\bullet$. This fibration (Puppe) sequence hides the equivalence of spaces of homotopy self equivalences: $aut\, G_\bullet \cong aut^*\, BG_\bullet$, that holds for a free simplicial group $G_\bullet$, where the LHS refers to space of loop equivalences and the RHS to space of pointed self-equivalence.

In other words the canonical topological example of a normal map of loop spaces is the map $\Omega X \to aut^* X$ for any pointed connected space X. Geometrically speaking, the last maps associates to any based loop λ in a well-pointed X, the pointed self-equivalence map $X \to X$ that drags a small neighborhood of the base point around λ and leaves the rest of X unmoved.

6. A remark on Loday's cubes of groups

In this section we make some remarks concerning [Lo, (5.4)]. The above homotopy quotient approach to crossed modules shade some light on Loday's presentation of spaces with a finite number of non-zero homotopy groups, i.e. Postnikov sections. Let us look at his presentation of a connected 3-stage. Namely, a connected space whose homotopy groups above dimension 3 vanish. Consider with Loday the following square of groups in which all maps are normal maps.

$$\begin{tikzcd} G \arrow[r] \arrow[d] & X \arrow[d] \\ H \arrow[r] & Y \end{tikzcd}$$

The normal structures on these maps are not written or named explicitly. We assume *in addition* that these maps define in both directions maps of normal structures, i.e., the maps preserve these structures. This implies that the induced map on the quotients $X//G \to Y//H$ is a simplicial group map (see subsection 2.1 and section 3.) We now can form the homotopy quotient

$$(Y//H)//(X//G),$$

to which we will refer as the homotopy double quotient. This is a not necessarily connected space with vanishing homotopy groups above dimension 2. In fact, it is equivalent to the homotopy fibre of the induced maps on classifying spaces. The first relevant question is what *non-connected 2-stages* can be gotten from a square of normal maps as above as a double quotient? We then ask further, under what condition the above homotopy double quotient can be endowed with a "compatible" simplicial group structures. Or, equivalently, under what conditions the induced map $X//G \to Y//H$ of simplicial groups is normal in the sense of Definition 5.1. Loday [Lo] gives a sufficient condition via a set of equations, some of which are quiet involved and the role of each is not very clear at the first sight. But in the present light it turns out that several of his equations allows the construction of the double quotient and the rest guarantee the existence of a "compatible" simplicial group structure on the double quotient. We have checked this for some but not all of Loday's equations.

We note that it is not hard to see that one has equivalence of simplicial sets (In fact they have isomorphic diagonals):

$$(Y//H)//(X//G) \cong (Y//X)//(H//G)$$

Our main point is that Loday's conditions are equivalent to the existence of a group structure, compatible with the action in the sense explained above, on the double quotient space as above. The double quotient is well defined as a simplicial set given the equations demanding the square to be a square of normal maps. It has a compatible simplicial group structure under Loday's extra conditions-equations. In that case its classifying space is proven by Loday to be a general *connected* 3-stage Postnikov piece.

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