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ACYCLIC SPACES

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§1. STATEMENT OF THE MAIN RESULTS

LET X be a connected space. If $\tilde{H}_j(X, \mathbb{Z}) \simeq 0$ for $j \leq n-1$ and $\pi_1 X \simeq 0$, then the Hurewicz map $\pi_j X \simeq \tilde{H}_j X$ is an isomorphism for $0 \leq j \leq n$. The assumption $\pi_1 X = 0$ is crucial and whenever it fails one encounters a space with vanishing integral homology up to dimension n, but which may have very rich homotopy in these dimensions. Important examples for this are homology spheres.

The purpose of this paper is to analyze the homotopy structure of an arbitrary *acyclic* space (i.e. a space X with $\tilde{H}_* X \simeq 0$ where $\tilde{H}_*$ denotes the reduced *integral* homology functor). This is done by means of a Postnikov-like decomposition into a tower of acyclic spaces which successively approximate X. The analysis leads to the construction of all acyclic spaces, and thus of many new examples of homology spheres. In addition, the constructions give a firmer hold on the algebraic K-groups.

Definition 1.1. Let X be an acyclic space. By an *acyclic decomposition* of X we mean a tower of fibrations

(*) $\varprojlim E_n = E_\infty \to \cdots E_n \to E_{n-1} \to \cdots E_0 = (\text{pt.})$

together with a (weak) homotopy equivalence $e\colon E_\infty \xrightarrow{\sim} X$ which satisfies the following conditions for all $n \geq 0$:

(i) The n-stage, E_n, is acyclic;
(ii) The n-stage, E_n, is j-simple for all $j > n$ (i.e. $\pi_1 E_n$ acts trivially on $\pi_j E_n$); and
(iii) The fibre of $E_n \to E_{n-1}$ is $(n-1)$-connected.

This definition is sensible in view of the following uniqueness and existence theorems.

1.2. Uniqueness of acyclic decompositions

Let X be acyclic. Then, given any two acyclic decompositions

$$\begin{tikzcd} E^i_\infty \arrow[r] \arrow[d, "e^i"', "\sim"] & \cdots E^i_n \arrow[r, "p^i_n"] & E^i_{n-1} \arrow[r] & \cdots E^i_1 \arrow[r] & E^i_0 (i = 1, 2) \\ X \end{tikzcd}$$

there exists, in a natural way, a third acyclic decomposition tower: $p_n : E_n \to E_{n-1}$ with $e : E_\infty \overset{\sim}{\to} X$, and homotopy equivalences $f_n^i : E_n \to E_n^i$ ($i \le n \le \infty$, $i = 1, 2$) which render all diagrams commutative, i.e. $f_{n-1}^i p_n = p_n^i f_n^i$ and $f_\infty^i e^i = e$ for $i = 1, 2$. Furthermore, the homotopy class of equivalences $[f_n^2] \circ [f_n^1]^{-1}$ which commutes with $[e^2]^{-1}_\circ [e^1]$ is unique.

EXISTENCE THEOREM 1.3. *Let X be an acyclic space. Then there exists a natural acyclic decomposition tower of X, denoted by*

$$\begin{array}{l} AX = \varprojlim AP_n X \longrightarrow \cdots AP_n X \xrightarrow{p_n} AP_{n-1} X \longrightarrow \cdots\ AP_1 X \longrightarrow \text{pt.} \\ \quad a \Big\downarrow \sim \\ \quad X \end{array}$$

The main advantage of the acyclic tower 1.3 over the usual Postnikov tower is that one can easily construct acyclic decomposition towers, whereas it is hard to see how to proceed in constructing Postnikov towers which will yield, in the limit, acyclic spaces.

1.4. Construction of acyclic decomposition towers

For a given $(n-1)$-stage ($n \ge 1$) in (1.1), i.e. an acyclic space E_{n-1} which is j-simple for all $j > n-1$, we analyze all the choices for an n-stage over it: $p_n : E_n \to E_{n-1}$.

We start with 1-stages:

1.4 (a). *The case $n = 1$.* The basic property of the fundamental group of an acyclic space is given by the following proposition; furthermore π_1 gives a full classification of 1-stages:

PROPOSITION. *Let E_1 be a 1-stage. Then $H_i(\pi_1 E_1) \simeq 0$ for $i = 1, 2$.*

UNIQUENESS. *Given two 1-stages E_1^1, E_1^2 and an isomorphism $\pi_1 E_1^1 \to \pi_1 E_1^2$, then one can construct, in a natural way, a diagram $E_1^1 \overset{\sim}{\leftarrow} E_1 \overset{\sim}{\to} E_1^2$ of homotopy equivalences, which induces the given isomorphism on π_1.*

EXISTENCE. *Let π be a group with $H_i \pi \simeq 0$ for $i = 1, 2$. Then there exists, in a natural way, an acyclic space $A(\pi)$ with $\pi_1 A(\pi) \simeq \pi$, which is j-simple for all $j > 1$.*

1.4(b). *The case $n > 1$.* For a given acyclic decomposition tower of "height" $(n-1)$: $E_{n-1} \to E_{n-2} \to \cdots E_1$, we analyze the choices of adding an n-stage $E_n \to E_{n-1}$.

These choices depend on the choice of a group and a k-"invariant," which are defined as follows:

Definition. For a given n-stage over $E_{n-1} p_n : E_n \to E_{n-1}$, we define:

(i) *The acyclic homotopy group.* These are groups which play a role analogous to that played by the homotopy groups in the Postnikov decomposition. They are defined by

$$\alpha_n E_n = \pi_n(\text{fibre of } p_n).$$

We will consider $\alpha_n E_n$ to be a $\pi_1 E_1$-group.

(ii) *The acyclic k-"invariant"*. $a^{n+1}(p_n)$, is defined to be the obstruction to a cross-section of p_n. It follows from 1.1(iii) that $a^{n+1}(p_n)$ lies in the cohomology group with twisted coefficients $H^{n+1}(E_{n-1}, \alpha_n E_n)$.

For the acyclic homotopy groups we have the following analogue of Proposition 1.4(a).

PROPOSITION. *For any n-stage E_n one has $H_i(\pi_1 E_1, \alpha_n E_n) \simeq 0$ for $i = 0, 1$* (homology is taken with twisted coefficients).

The acyclic homotopy groups and k-"invariants" classify $E_n \to E_{n-1}$ as follows. Given an $(n-1)$-stage, E_{n-1} one has:

UNIQUENESS. *Any two n-stages $p^i\colon E_n^i \to E_{n-1}$ $(i = 1, 2)$ are fibre-homotopy equivalent* [2] *if and only if there exists an isomorphism of $\pi_1 E_1$-groups $\alpha_n E_n^1 \to \alpha_n E_n^2$ which carries the cohomology class $a^{n+1}(p_n^1)$ to $a^{n+1}(p_n^2)$.*

EXISTENCE. *With notation as above, given any $\pi_1 E_1$-module α, with $H_i(\pi_1 E_1, \alpha) \simeq 0$ for $i = 0, 1$ and any (twisted-)cocycle $c^{n+1} \in Z^{n+1}(E_{n-1}, \alpha)$ there exists, in a natural way, a fibre map: $p_n\colon E_n \to E_{n-1}$ such that E_n is an n-stage, $\alpha_n E_n \simeq \alpha$ as $\pi_1 E_1$-groups, and $c^{n+1} \in a^{n+1}(p_n)$.*

1.5. The acyclic homotopy groups

Some remarks on the connection between $\pi_n X$ and $\alpha_n AP_n X$ are in order here. In fact, it follows from 1.2 that the groups $\alpha_n E_n$ are independent of the specific decomposition tower, and thus are determined up to unique isomorphism by X. Thus one can denote them by $\alpha_n X$. Further, it is not hard to see that $\alpha_n X$ depends only on $\pi_n X$ as a $\pi_1 X$-module.

Thus one can conclude that acyclic spaces are built out of acyclic homotopy groups and acyclic k-"invariants" in the same way general spaces are built out of their homotopy groups and k-"invariants". Further, any perfect group π or π-perfect module M (i.e. module M with $H_0(\pi, M) \simeq 0$) can be extended to get groups which can serve as acyclic homotopy groups of acyclic spaces. Thus there are many candidates for acyclic homotopy groups.

1.6. Organization of the Paper

We work in the category S_* of pointed simplicial Kan complexes [6]. This is more convenient for our constructions. The nature of the proofs, however, is not combinatorial.

In Section 2 we introduce the basic tool of our analysis, the *acyclic functor*, which is used in Section 3 to construct the acyclic decomposition and to prove 1.2 and 1.3. We deal with the proofs of 1.4(a) and (b) in Sections 4 and 5, respectively.

§2. THE ACYCLIC FUNCTOR

The main tool in our work is an acyclic functor which "kills" in a natural way the integral homology of any $K \in S_*$:

THEOREM 2.1. *There exists a functor*

$$A\colon S_* \to S_*$$

and for any $K \in S_*$ *a natural fibre map* $a\colon AK \to K$ *such that:*

(i) AK *is acyclic for all* K.

(ii) *The map* $a\colon AK \to K$ *is, up to homotopy, universal for maps of acyclic complexes into* K.

Further, any such functor enjoys the properties:

(iii) *Let* n *be* $1 \le n \le \infty$. *If* $\tilde{H}_j K \simeq 0$ *for all* $1 \le j \le n$, *then the fibre of* $a\colon AK \to K$ *is* $(n-1)$*-connected. In particular, if* K *is acyclic then* a *is an equivalence.*

(iv) *If* K *is* j*-simple for some* $j \ge 1$ *then so is* AK.

Our functor A *satisfies, in addition, the following technical properties:*

(v) *If* $p\colon E \to B$ *is a (Kan-)fibre map then so is* Ap.

(vi) *Let* K_∞ *be the inverse limit of the tower:*

$$K_\infty = \varprojlim K_n \to \cdots K_n \xrightarrow{p_n} K_{n-1} \cdots \longrightarrow K_1 .$$

Assume that for any s-skeleton $(K_n)_s$ *the restriction* $p_n \mid (K_n)_s$ *is an isomorphism for* n *big enough. Then* $AK_\infty = \varprojlim AK_n$.

2.2. The construction of A

Let $K \in S_*$. We define AK by

$$AK = \varprojlim A_n K$$

where

$$\cdots \to A_n K \cdots \to A_1 K \to A_0 K = K$$

is a tower of fibrations defined as follows:

(i) $p_1\colon A_1 K \to K$ is the cover of K which corresponds to the maximal perfect subgroup of $\pi_1 K$ [3, p. 144]—one can easily check that any group G has a (unique) maximal subgroup PG which is perfect, i.e. equal to its own commutator subgroup, and that PG is a normal subgroup.

(ii) $A_n K (n > 1)$ is defined as the pull back in the diagram

$$\begin{tikzcd} A_n K \arrow[r, dashed] \arrow[d, dashed] & \Lambda P_n Z A_{n-1} K \arrow[d] \\ A_{n-1} K \arrow[r] & P_n Z A_{n-1} K. \end{tikzcd}$$

Here Z denotes the reduced free abelian group functor [6, p. 98], P_n the n-stage of the Postnikov tower [8], and Λ the simplicial path functor [1, I.2.2].

2.3. Proof of Theorem 2.1

It is obvious from the construction that A is a functor and $AK \to K$ is a natural fibre map. Notice that since $\pi_* ZK \simeq \tilde{H}_* K$, the complex ZK is n-connected under conditions (iii) of Theorem 2.1. Hence $A_n K \to K$ is an equivalence and thus the fibre of $AK \to K$ is $(n-1)$-connected. This proves (iii).

To prove part (i) it is enough to show that for all $n \geq 0$ $\tilde{H}_j A_n K \simeq 0$ for $j \leq n$. This will imply that the fibre of $A_n K \to A_{n-1} K$ is $(n-2)$-connected and thus $\tilde{H}_* AK \simeq \varprojlim H_* A_n K \simeq 0$.

The case $n = 1$. By construction $H_1 A_1 K \simeq 0$.

The case $n > 1$. Assume $H_j A_{n-1} K \simeq 0$ for $1 \leq j \leq n-1$. Consider the diagram $A_n K \to A_{n-1} K \xrightarrow{p} P_n Z A_{n-1} K$ as in 2.2(ii), in which the base-space has the homotopy type of a $K(H_n A_{n-1} K, n)$. It is clear that $H_n p$ is an isomorphism. Since $H_1 A_n K \simeq 0$, Lemma 2.4 below implies that $\tilde{H}_j A_n K \simeq 0$ for $0 \leq j \leq n$.

LEMMA 2.4. *Let $F \to E \to B$ be a fibration where B has the homotopy type of a $K(\Pi; n)$ for some $n > 1$, then $H_j E \xrightarrow{\simeq} H_j B$ for $j \leq n-2$, and one has an exact sequence:*

$$H_n(B, H_1 F) \to H_n F \to H_n E \to H_n B \to H_{n-1} F \to H_{n-1} E \to 0.$$

Proof. This follows from the usual arguments in the Serre spectral sequence using the fact that $H_{n+1} B \simeq 0$.

2.5. Continuation of the proof of 2.1

Part (ii) follows easily from (iii) and (i) using 3.4 and 3.5 below, (iv) follows from 2.6 below and (v) is easily verified. As for (vi)

It follows from our construction that if $K \to L$ is an isomorphism on the s-skeletons, then $A_r L \to A_r K$ is an isomorphism on the $(s-r)$-skeletons. Thus $\varprojlim_n A_r K_n \simeq A_r \varprojlim_n K_n$ in 2.1 (vi). The result follows.

LEMMA 2.6. *Let $A \rightarrowtail B \twoheadrightarrow C$ be an extension of π-modules for a perfect group π. Then π acts trivially on B if and only if it acts trivially on both A and C.*

Proof. One has an exact sequence

$$H_1(\pi, C) \to H_0(\pi, A) \to H_0(\pi, B) \to H_0(\pi, C) \to 0$$

whose left-most term vanishes for trivial C. The "only if" part is obvious. The "if" par follows from the five-lemma.

§3. THE ACYCLIC DECOMPOSITION TOWER

In this section we construct a natural acyclic decomposition tower, thus proving Theorem 1.3. Using this tower we prove the uniqueness of acyclic decompositions 1.2.

3.1. Construction

Given an acyclic Kan complex X, consider the usual Postnikov tower of X:

$$X = \varprojlim P_n X \to \ldots \; P_n X \to P_{n-1} X \to \ldots \; P_1 X \to (\text{pt.}).$$

The acyclic decomposition tower is gotten by applying the acyclic functor A to this tower. One gets a natural tower

$$\begin{tikzcd} AX \arrow[r] \arrow[d, "a", "\sim"'] & \dots AP_n X \arrow[r] & AP_{n-1}X \arrow[r] & \dots AP_1 X \arrow[r] & (\text{pt.}). \\ X \end{tikzcd}$$

In which,

(i) all maps are (Kan-)fibre maps, by 2.1(v),

(ii) AX is the inverse limit of $AP_n X$, since the tower $P_n X$ satisfies the assumption of 2.1(vi).

It remains to prove that our tower satisfies 1.1(i)–(iii).

3.2. Proof of 1.3

Parts (i) and (ii) follow from 2.1(i) and (iv), respectively. As for part (iii), first observe that $p_n\colon AP_n X \to AP_{n-1}X$ induces an isomorphism on π_j for $j \leq n-1$, thus the fibre of p_n is $(n-2)$-connected. *It remains to show that* $\pi_n AP_n X \to \pi_n AP_{n-1}X$ *is onto.*

Consider the diagram of fibrations, where the maps are the obvious ones:

$$\begin{tikzcd} F_1 \arrow[r, "\tilde{f}"] \arrow[d] & F_2 \arrow[d] \\ AP_n X \arrow[r, "f"] \arrow[d] & AP_{n-1}X \arrow[d] \\ P_{n-1}X \arrow[r, phantom, "="] & P_{n-1}X. \end{tikzcd}$$

It is clear that for acyclic X, $H_j P_n X \simeq 0$ for all $j \leq n+1$. Hence by 2.1(iii) and (iv), F_1 and F_2 are $(n-1)$-connected and the fibration on the right is orientable. This implies that $\pi_n f$ is surjective iff $H_n \tilde{f}_n$ is. But by Lemma 3.5 below $H_0(\pi_1 P_{n-1}X, H_n \tilde{f})$ is bijective. This completes the proof.

3.3. Proof of 1.2

One constructs E_n from the diagrams $E_n^i \longleftarrow E_\infty^i \xrightarrow{\sim} X\ (i = 1, 2)$ by taking the "total" fibre-product of the diagram

$$\begin{tikzcd}[column sep=small] & AE_n^1 \arrow[dl, "a"'] \arrow[dr] & & AP_n E_\infty^1 \arrow[dl] \arrow[dr] & & AP_n E_\infty^2 \arrow[dl] \arrow[dr] & & AE_n^2 \arrow[dl] \arrow[dr, "a"] & \\ E_n^1 & & AP_n E_n^1 & & AP_n X & & AP_n E_n^2 & & E_n^2 . \end{tikzcd}$$

To prove 1.2 it is enough to show that all the maps in this diagram are homotopy equivalences.

It follows from 1.1(ii) (iii) that these maps induce isomorphisms on π_j for $j \leq n$ and that all these spaces are j-simple for all $j > n$. This implies that the maps are homotopy equivalences by the following Lemma 3.4. Note that the uniqueness of $[f_n^2] \circ [f_n^1]^{-1}$ follows from the obvious universal properties of an n-stage.

LEMMA 3.4. *Let f: $X \to Y$ be a map of acyclic complexes. Assume that whenever X or Y is not j-simple, $\pi_j f$ is an isomorphism. Then $\pi_n f$ is an isomorphism for all n.*

Proof. This follows easily by induction on n: Assume $\pi_i f$ isomorphic for all $j \leq n - 1$; if both X and Y are n-simple then

$$\pi_n X \simeq H_{n+1}P_{n-1}X \xrightarrow{\simeq} H_{n+1}P_{n-1}Y \simeq \pi_n Y,$$

where the first and last isomorphisms follow from Lemma 3.5 below.

LEMMA 3.5. *Let $F \to E \to B$ be a fibration of connected complexes with fibre F a $K(M, n)$ for some $n > 1$. Then M is a $\pi_1 B$-module and one has an exact sequence:*

$$H_{n+2}E \to H_{n+2}B \to H_1(\pi_1 B; M) \to H_{n+1}E \to H_{n+1}B \to H_0(\pi_1 B; M) \to H_n E \to H_n B \to 0.$$

If one only assumes that F is $(n-1)$-connected then one should omit the three groups on the left.

Proof. This is immediate from the Serre spectral sequence since $H_{n+1}K(M, n) \simeq 0$.

This completes the general functorial constructions. We now turn to a step by step construction of acyclic decomposition towers.

§4. SIMPLE ACYCLIC SPACES

In this section we prove the statements of 1.4(a). We call an acyclic space *simple* if it is j-simple for all $j > 1$ (i.e. if it is a 1-stage in the terminology of 1.4). Simple acyclic spaces are the analogue in the category of acyclic spaces of the $K(\Pi, 1)$'s in S_*.

4.1. Proof of Proposition 1.4(a)

Observe that for any $K \in S_*$ one has $H_1 K \simeq H_1 \pi_1 K$. Further, the natural map $H_2 K \to H_2 \pi_1 K$ is onto. The last fact is due to [4], and follows easily from the spectral sequence of the fibration $\tilde{X} \to X \to K(\pi_1 X_1, 1)$. Thus if $H_1 K \simeq H_2 K \simeq 0$, one must have $H_1 \pi_1 K \simeq H_2 \pi_1 K \simeq 0$.

4.2. A classification theorem

Uniqueness and existence in 1.4(a) are obvious from the following theorem: Let G denote the category of all groups σ which satisfy $H_i \sigma \simeq 0$ for $i = 1, 2$; and let $\mathbf{a}^s$ denote the homotopy category of all simple acyclic complexes in S_*.

THEOREM. *The fundamental group functor π_1: $\mathbf{a}^s \to G$ gives an equivalence of categories.*

Proof. We define an inverse to π_1 by $A\,\overline{W}$: $G \to \mathbf{a}^s$, where $\overline{W}$ is as in [8]. It is clear from 2.1(iv) that $\mathbf{a}^s$ is the range of $A\,\overline{W}$ and that $\pi_1 A\,\overline{W}\sigma = \sigma$. Thus $A\,\overline{W}$ is a right inverse. Since the canonical map $AX \to A\,\overline{W}\pi_1 X$ induces isomorphism on π_1, it follows from 3.5 and 2.1(ii) that for $X \in \mathbf{a}^s$ one has natural equivalences $X \xleftarrow{\simeq} AX \xrightarrow{\simeq} A\,\overline{W}\pi_1 X$. Thus, up to homotopy, $A\,\overline{W}$ is a left inverse to π_1. This completes the proof.

§5. PROOF OF 1.4(b)

First observe that Proposition 1.4(b) is obvious from the Serre spectral sequence of the fibration $F_n \to E_n \xrightarrow{p_n} E_{n-1}$, because F_n is $(n-1)$-connected and $\tilde{H}_* p_n \colon \tilde{H}_* E_n \to H_* E_{n-1}$, is an isomorphism.

We next give a brief review of "twisted-k-invariants" as developed in [7, 9].

The problem is to classify, for a given Kan complex B, all possible fibrations

$$K(M, n) \longrightarrow E \xrightarrow{p} B$$

where $n \geq 1$ and M is a given $\pi_1 B$-module.

Let $\mathrm{Ob}(p) \in H^{n+1}(B, M)$ be the (twisted) obstruction to a cross-section of p. Then one has:

5.1. A classification theorem for nonoriented fibration

(i) Any two fibrations

$$K(M^i, m) \longrightarrow E^i \xrightarrow{p^i} B \qquad (i = 1, 2)$$

are fibre homotopy equivalent if and only if there exists an isomorphism of $\pi_1 B$-modules $M^1 \to M^2$ which carries $\mathrm{Ob}(p^1)$ to $\mathrm{Ob}(p^2)$.

(ii) For any (twisted-)cocycle $c^{n+1} \in Z^{n+1}(B, M)$ and any $\pi_1 B$-module M there exists a fibration $K(M, n) \longrightarrow E \xrightarrow{p} B$ with $c^{n+1} \in \mathrm{Ob}(p)$ and $\pi_n K(M, n) \simeq M$ as $\pi_1 B$-modules.

Proof. See [6, 7, 8]. A few words on the construction of $p \colon E \to B$. Let $L_\phi(M, n+1)$ be the classifying complex for twisted coefficient cohomology, where $\phi \colon \pi = \pi_1 B \to \operatorname{aut} M$. One can construct $L_\phi(M, n+1)$ as follows: The constant simplicial group $\pi = K(\pi, 0)$ acts via ϕ on $\overline{W}^n M$, thus one can form the twisted product to get a group complex: $\overline{W}^n M \rightarrowtail \overline{W}^n M \tilde{\times} \pi \overset{\leftarrow}{\twoheadrightarrow} \pi$. $L_\phi(M, n+1)$ is defined to be $\overline{W}(\overline{W}^n M \tilde{\times} \pi)$ and retracts to $\overline{W}\pi$. According to [9] one has: The elements of $Z^{n+1}(B,M)$ (twisted) are in a natural 1–1 correspondence with diagrams

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f"'] & L_\phi(M, n+1) \arrow[dl, shift left] \\ \overline{W}\pi_1 B \arrow[ur, shift left, "s"'] & \end{tikzcd}$$

and $H^{n+1}(B, M)$ is naturally isomorphic to the group of homotopy classes of such diagrams.

Now given a cocycle as in 5.1(ii), one constructs a corresponding pull-back diagram which defines E:

$$\begin{tikzcd}
E \arrow[rr, dashed] \arrow[dd, "p"'] & & \Lambda_\pi L_\phi(M, n+1) \arrow[dl] \arrow[dd, "p'"] \\
& \overline{W}\pi_1 B & \\
B \arrow[ur] \arrow[rr] & & L_\phi(M, n+1) \arrow[ul]
\end{tikzcd}$$

where Λ_π is the "path over $\overline{W}\pi$" functor [7] (Up to homotopy, p' is the cross-section s.) Clearly $c^{n+1} \in \mathrm{Ob}(p)$.

5.2. Construction of the n-stage

We now prove the existence theorem 1.4(b). We are given a $\pi_1 E_1$-module α with $H_i(\pi_1 E_1, \alpha) \simeq 0$ for $i = 0, 1$ and a cocycle $c^{n+1} \in Z^{n+1}(E_{n-1}, \alpha)$. Let

$$K(\alpha, n) \longrightarrow E_n' \xrightarrow{p_n'} E_{n-1}$$

be the corresponding fibration from 5.1(ii). We define $p_n\colon E_n \to E_{n-1}$ for 1.4(b) by

$$E_n = AE_n' \xrightarrow{a} E_n' \xrightarrow{p_n'} E_{n-1}.$$

The most important property of E_n' is:

LEMMA. *If* $0 \leq j \leq n+1$ *then* $\tilde{H}_j E_n' \simeq 0$.

Proof. This is an immediate corollary of the exact sequence 3.5 since $H_i(\pi_1 E_{n-1}, \alpha) \simeq 0$ for $i = 0, 1$.

The last lemma implies that the fibre of $AE_n' \to E_n'$ is n-connected (see 2.1(iii)), thus the first obstruction to a cross section of p_n is the same as that of p_n'. Hence, by 5.1(ii) $c^{n+1} \in a^{n+1}(p_n)$.

CRUCIAL PROPOSITION. *The complex* $E_n = AE_n'$ *is an n-stage, and* $\alpha_n E_n \simeq \alpha$ *as a* $\pi_1 E_1$*-module.*

Proof. Clearly E_n' is j-simple for all $j \geq n+1$, because E_{n-1} is. By 2.1(iv) E_n is an n-stage. To prove the second part we consider the diagram of fibrations

$$\begin{tikzcd}
F_n \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d] & E_{n-1} \arrow[d] \\
K(\alpha, n) \arrow[r] & E_n' \arrow[r] & E_{n-1}
\end{tikzcd}$$

which gives the ladder of $\pi_1 E_1$-groups:

$$\begin{tikzcd}
\pi_{n+1} E_n \arrow[r] \arrow[d, "\pi_{n+1} a"] & \pi_{n+1} E_{n-1} \arrow[r] \arrow[d, equal] & \alpha_n E_n \arrow[r] \arrow[d] & \pi_n E_n \arrow[r] \arrow[d, "\pi_n a", "\simeq"'] & \pi_n E_{n-1} \arrow[d, equal] \\
\pi_{n+1} E_n' \arrow[r] & \pi_{n+1} E_{n-1} \arrow[r] & \alpha \arrow[r] & \pi_n E_n' \arrow[r] & \pi_n E_{n-1}.
\end{tikzcd}$$

Since $H_j E_n' \simeq 0$ for $j \leq n + 1$ (see lemma on previous page), one gets from 2.1(iii) that $\pi_n a$ is bijective and $\pi_{n+1} a$ is surjective. By the five-lemma $\alpha_n E_n \simeq \alpha$.

5.3. Proof of uniqueness

Let $f: L_\phi(\alpha^1, n + 1) \to L_\phi(\alpha^2, n + 1)$ be the map which corresponds to a given map map $\alpha^1 \to \alpha^2$ where $\alpha^i = \alpha_n E_n{}^i$. One has a diagram

$$
\begin{tikzcd}
E_n{}^1 \arrow[r, dashed, "\tilde{f}"] \arrow[d, "p"] & E_n{}^2 \arrow[d] \\
E_{n-1} \arrow[r, equal] \arrow[d, "\mathrm{Ob}(p^1)"] & E_{n-1} \arrow[d, "\mathrm{Ob}(p^2)"] \\
L_\phi(\alpha^1, n + 1) \arrow[r, "f"] & L_\phi(\alpha^2, n + 1)
\end{tikzcd}
$$

in which $p \circ \mathrm{Ob}(p^1) \sim 0$ and thus $p \circ \mathrm{Ob}(p^2) \sim 0$. So the first obstruction to a lifting $\tilde{f}$ vanishes; all the higher obstructions lie in $H^{n+i}(E_{n-1}, \pi_{n+i-1} E_n{}^2)$ $(i \geq 2)$ which is the zero group since E_{n-1} is acyclic and $E_n{}^2$ is j-simple for $j \geq n + 1$.

Since $\tilde{f}$ induces isomorphism on π_j for $j \leq n + 1$, it follows from Lemma 3.5 that f is a homotopy equivalence. Thus $E_n{}^1$ is fibre homotopy equivalent to $E_n{}^2$.

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