Ergodic theory of the space of measured laminations

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1 Introduction

Let S be a surface of genus $g = g(S)$ with $n = n(S)$ boundary components. The mapping class group $\mathrm{Mod}(S)$ is defined as the group of diffeomorphisms $S \to S$ quotioned by the subgroup of diffeomorphisms homotopic to the identity map. It acts naturally on the space $\mathcal{ML}(S)$ of *measured laminations on* S: a piecewise linear space associated to S, whose quotient by the scalars $P\mathcal{ML}(S)$ can be viewed as a boundary of the Teichmüller space $\mathcal{T}(S)$ (see Section 2 for more details on measured laminations and Teichmüller spaces). The space $\mathcal{ML}(S)$ has a piecewise linear integral structure of dimension $6g(S) - 6 + 2n(S)$.

There is a natural $\mathrm{Mod}(S)$-invariant locally finite measure on $\mathcal{ML}(S)$, the *Thurston measure* μ_{Th}, given by this piecewise linear integral structure. This measure is by no means the only $\mathrm{Mod}(S)$-invariant measure on $\mathcal{ML}(S)$, and we give an explicit construction of many more such measures below. In this paper we classify all locally finite $\mathrm{Mod}(S)$-invariant measures on $\mathcal{ML}(S)$. We also classify all possible orbit closures for this action.

There is a special type of measured laminations that plays an important role in this classification, namely (non-intersecting) *multicurves*. Recall that $\gamma = \sum_{i=1}^{k} c_i \gamma_i$ is a multicurve on S if γ_i's are disjoint, essential, non-peripheral simple closed curves, no two of which are in the same homotopy class, and $c_i \geq 0$ for $1 \leq i \leq k$. The integral points of $\mathcal{ML}(S)$ are precisely those multicurves with all $c_i \in \mathbb{Z}^+$.

We extend the definition of $\mathcal{ML}$ to disconnected surfaces $R = \bigsqcup_i R_i$ in the obvious way, and set $\mathrm{Mod}(R) = \prod_i \mathrm{Mod}(R_i)$ (note that since 0 is not considered to be a measured lamination, $\mathcal{ML}(R)$ is slightly larger than $\prod_i \mathcal{ML}(R_i)$. Unless otherwise mentioned, surfaces are assumed to be connected, but subsurfaces are allowed to have several connected components.

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1.1 Statements of the main results.

Given a subsurface $R \subset S$, there is a natural $\mathrm{Mod}(R)$ equivariant embedding

$$\mathcal{I}_R : \mathcal{ML}(R) \to \mathcal{ML}(S),$$

such that $\mathcal{I}_R(t \cdot \lambda) = t \cdot \mathcal{I}_R(\lambda)$. This map gives rise to a family of locally finite ergodic $\mathrm{Mod}(S)-$invariant measures on $\mathcal{ML}(S)$ as follows.

We say a pair $\mathcal{R} = (R, \gamma)$ is *complete* iff γ is a multicurve and $R \subset S$ is a union of connected components of $S(\gamma)$. Here $S(\gamma)$ denote the surface obtained by cutting S along connected components of γ. In this case, $\partial(R) \subset \mathrm{Support}(\gamma)$, and for any simple closed curve α on R, $i(\alpha, \gamma) = 0$. Define

$$\mathcal{G}^{(R,\gamma)} = \{\gamma + \lambda \mid \lambda \in \mathcal{I}_R(\mathcal{ML}(R))\}.$$

In §3, we show that any complete pair $\mathcal{R} = (R, \gamma)$ gives rise to a locally finite $\mathrm{Mod}(S)-$invariant measure $\mu_{Th}^{[\mathcal{R}]}$ supported on the closed set

$$\mathcal{G}^{[\mathcal{R}]} = \bigcup_{(h \cdot R, h \cdot \gamma)} \mathcal{G}^{(h \cdot R, h \cdot \gamma)}.$$

The measures $\mu^{[(S_1,\gamma_1)]}$ and $\mu^{[(S_2,\gamma_2)]}$ are in the same class if and only if $[(S_1, \gamma_1)] = [(S_2, \gamma_2)]$; in other words

$$(S_1, \gamma_1) = (h \cdot S_2, h \cdot \gamma_2) \qquad \text{for some } h \in \mathrm{Mod}(S).$$

For simplicity for $\gamma = 0, R = S$ let $\mu_{Th}^{[(R,0)]} = \mu_{Th}$, and for $R = \emptyset$, let $\mu_{Th}^{[(R,\gamma)]}$ denote the Dirac measure supported on the discrete set $\mathrm{Mod}(S) \cdot \gamma$.

A measured lamination λ is *filling* if for every non peripheral simple closed curve γ, on S, $i(\gamma, \lambda) > 0$. Note that any measured lamination λ can be written as $\gamma + \eta$ where γ is a multicurve and η is a filling measured lamination in a (not necessarily connected) subsurface $R \subset S$.

In this paper, We establish the following results:

Theorem 1.1. *Let μ be a locally finite $\mathrm{Mod}(S)-$invariant ergodic measure on $\mathcal{ML}(S)$. Then μ is a constant multiple of $\mu_{Th}^{[\mathcal{R}]}$ for a complete pair $\mathcal{R} = (R, \gamma)$.*

Theorem 1.2. *Given $\lambda \in \mathcal{ML}(S)$, there exists a complete pair $\mathcal{R} = (R, \gamma)$ such that*

$$\overline{\mathrm{Mod}(S) \cdot \lambda} = \mathcal{G}^{[\mathcal{R}]}.$$

In particular, we have that any orbit closure of the mapping class group is the support of a unique $\mathrm{Mod}(S)-$invariant ergodic measure on $\mathcal{ML}(S)$. Another consequence of Theorem 1.2, λ has a dense orbit in $\mathcal{ML}(S)$ if and only if its support does not contain any simple closed curves. Also, λ has a discrete orbit if and only if $R = \emptyset$, i.e. λ is a multicurve.

1.2 Translation to dynamics on the moduli space of unit area quadratic differentials.

The space $\mathcal{ML}(S)$ is closely related to another important space attached to S: moduli space $\mathcal{Q}^1\mathcal{M}(S)$ of unit area quadratic differentials with simple poles at marked points of S. In particular, there is a natural construction assigning to any $\mathrm{Mod}(S)-$invariant measure μ on $\mathcal{ML}(S)$ a locally finite measure $\widetilde{\mu}$ on $\mathcal{Q}^1\mathcal{M}(S)$. The group $\mathrm{SL}(2,\mathbb{R})$ acts naturally on $\mathcal{Q}^1\mathcal{M}(S)$, and in particular we single out the two one-parameter subgroups $g_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$ and $u_t = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ of $\mathrm{SL}(2,\mathbb{R})$.

The action of the group g_t gives rise to a flow called the *Teichmüller geodesics flow*. The space $\mathcal{Q}^1\mathcal{M}(S)$ possesses two natural mutually transverse g_t-invariant foliations $\mathcal{F}^-$ and $\mathcal{F}^+$ which can be identified as the strong unstable and strong stable manifolds for g_t. We will refer to the leaves of foliation as *contracting* resp. *expanding horospheres*. The group u_t gives rise to another flow on $\mathcal{Q}^1\mathcal{M}(S)$ - *Teichmüller unipotent flow*[1], and each orbits of this flow remains on a single contracting horosphere.

The measures $\widetilde{\mu}$ discussed above are (in a sense that can be made precise) invariant under the horospheric foliations $\mathcal{F}^-$, and in particular invariant under the Teichmüller unipotent flow. The Teichmüller geodesic flow g_t maps such a measure $\widetilde{\mu}$ to another measure of this form, and the corresponding action on $\mathcal{ML}(S)$ corresponds to multiplication by scalars.

Theorems 1.1 and 1.2 can therefore be rephrased as theorems regarding "horospheric invariant" measures (or orbit closures) on the moduli space $\mathcal{Q}^1\mathcal{M}(S)$. As such they are in close analogy with S. G. Dani's [Dani, Thm 9.1 and 8.2] respectively.

In Dani's work, as in this paper, a key difficulty is that the space is not compact, and we deal with this difficulty by applying a modification of the quantitative nondivergence results for the Teichmüller unipotent flow of Y. Minsky and B. Weiss [MW]). The exact form we need is somewhat different than what is proved by Minsky and Weiss and we give a self-contained treatment in the appendix to this paper.

Another difficulty, which is not present in the homogeneous spaces analogues, is the identification of the possible invariant measures as given in Theorem 1.1, which is one of the main novelties in this paper.

Note that unlike in [Dani], the horospheric foliation does not come from a group action. This in and of itself is not a major difficulty: a fairly general (but not applicable in our situation, both because the g_t-action is not uniformly hyperbolic and because the space is not compact) result about measures invariant under horospheric foliation was proved by R. Bowen and B. Marcus in [BM].

[1]Some authors refer to this flow as the Teichmüller horocyclic flow.

1.3 Idea of proof.

The proof of Theorem 1.1 has two main steps:

I): We first show that μ_{Th} is the unique locally finite Mod(S) invariant ergodic measure supported on the locus $\mathcal{G} \subset \mathcal{ML}(S)$ of *filling* measured laminations (Theorem 7.1).

Our strategy of proof by using the mixing properties of the Teichmüller geodesic flow is quite classical (though is slightly more involved technically because the dynamics is not uniformly hyperbolic). This type of reasoning was probably first used by Margulis in his thesis (see [Mar] and the references there). We deal with the noncompactness of $\mathcal{Q}^1\mathcal{M}(S)$ by the quantitative nondivergence estimates for the Teichmüller unipotent flow mentioned above.

II): We show that if μ assigns zero measure to the set of all filling measured laminations then

$$\mu(\{\gamma + \eta \mid i(\eta + \gamma) = 0, \gamma \neq 0 \text{ is a multicurve }\}) > 0$$

as follows:

1): In §8.3 (Lemma 8.6) by applying Theorem 1.1 for subsurfaces of S, we show that if the support of a μ typical point of $\mathcal{ML}(S)$ does not contain any simple closed curves then there exits $k < 6g(S) - 6 + 2n(S)$ such that $\mu(tU) = t^k\mu(U)$.

2): We study study Mod(S)-invariant measures which are quasi invariant under the action of $\mathbb{R}_+$ on $\mathcal{ML}(S)$. In Proposition 8.2, we show that if for any $t \in \mathbb{R}_+$, $\mu(t \cdot U) = t^k\mu(U)$, then we have $k \geq 6g(S) - 6 + 2n(S)$.

Combining (1) and (2) implies that if $\mu(\mathcal{G}) = 0$, then μ is induced by a Mod(R) invariant ergodic measure on $\mathcal{ML}(R)$ for some subsurface $R \subset S$. Moreover, μ is supported on $\mathcal{G}^{[\mathcal{R}]}$ for a complete pair $\mathcal{R} = (R, \gamma)$.

1.4 Notes and references.

(1) There are many other relevant works related to horospheric invariant measures. In particular, we mention the very general results of Roblin [Ro] and the work of F. Leddrappier and O. Sarig [LS] who classify horocycle-invariant measures on quotients $\Gamma \backslash \mathrm{SL}(2, \mathbb{R})$ which have *infinite* volume, specifically when Γ is a normal subgroup of a lattice. The proof of Leddrappier and Sarig is quite different from ours and is based on directly studying the action of Γ on $\mathrm{SL}(2, \mathbb{R}) / \{u_t : t \in \mathbb{R}\}$ — which is the analog in this context to the action of Mod(S) on $\mathcal{ML}(S)$ - similar to Furstenberg's original proof of the unique ergodicity of the horocycle flow on compact quotients of $\mathrm{SL}(2, \mathbb{R})$.
(2) U. Hamenstädt has independently obtained a classification of Mod(S)-invariant measures on $\mathcal{ML}(S)$ supported on the set of recurrent measured laminations (a

subset of the set of filling measured laminations) [Ha], with an application to showing that the Teichmüller geodesics flow is Bernoulli in the measure theoretic sense. Her proof follows the outline of [LS]. It is quite interesting to see whether her argument gives any information regarding the action of subgroups of Mod(S) on this space.
(3) A much harder question is the classification of $\{u_t\}$ invariant measures on $\mathcal{Q}^1\mathcal{M}(S)$ in analogy with Ratner's measure classification theorem [Ra1]. [Ra1]. Even understanding of what are the "nice" invariant measures in this case is a deep and complicated question [Ca],[Mc]. We note that Ratner has her own version of the horocyclic argument [Ra2].

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The proof of Theorem 1.1 was obtained during the spring of 2006. Theorem 1.2 was added during the writeup of this paper but is based on similar ideas.

2 Background

In this section we briefly recall basic properties of the space of measured laminations. For more details see [Th] and [HP] for more.

2.1 Teichmüller space.

Let S be a surface of genus $g(S)$ with $n(S)$ marked points. A point in the *Teichmüller space* $\mathcal{T}(S)$ is a complete hyperbolic surface X of genus g(S) with n(S) cusps equipped with a diffeomorphism $f : S \to X$. The map f provides a *marking* on X by S. Two marked surfaces $f : S \to X$ and $g : S \to Y$ define the same point in $\mathcal{T}(S)$ if and only if $f \circ g^{-1} : Y \to X$ is isotopic to a conformal map. When ∂S is nonempty, consider hyperbolic Riemann surfaces homeomorphic to S with cusps at the marked points. Let Mod(S) denote the mapping class group of S, or in other words the group of isotopy classes of orientation preserving self homeomorphisms of S leaving each marked point fixed. The mapping class group Mod(S) acts on $\mathcal{T}(S)$ by changing the marking. The quotient space

$$\mathcal{M}(S) = \mathcal{T}(S)/\operatorname{Mod}(S)$$

is the moduli space of Riemann surfaces homeomorphic to S with $n(S)$ cusps.

2.2 Space of measured lamination.

A geodesic measured lamination λ consists of a closed subset of $X \in \mathcal{T}(S)$ foliated by complete simple geodesics and a measure on every arc k transverse to λ. For understanding measured laminations, it is helpful to consider the lift to the universal cover of X. A directed geodesic is determined by a pair of points on the boundary $(x_1, x_2) \in S^\infty \times S^\infty - \Delta$, where Δ is the diagonal $\{(x,x) |\ x \in S^\infty\}$. Given a measured geodesic lamination λ, the preimage of of its underlying geodesic lamination $A \subset D$ is decomposed as a union of geodesics of D. Then geodesic laminations on two homeomorphic hyperbolic surfaces can be compared by passing to the circle at ∞. So the notion of a measured lamination only depends on the topology of the surface X. The weak topology on measures induce the measure topology on the space $\mathcal{ML}(S)$ of measured laminations on S; in other words, this topology is induced by the weak topology on the space of measured on a given arc which is transverse to each lamination from an open subset of $\mathcal{ML}(S)$.

Given two closed curves γ_1, γ_2 on S, the intersection number $i(\gamma_1, \gamma_2)$ is the minimum number of points in which representatives of γ_1 and γ_2 must intersect. The intersection pairing extends to a continuous map

$$i : \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}_+.$$

Given $\lambda \in \mathcal{ML}(S)$, let

$$\mathcal{ML}(\lambda) = \{\eta \ |\forall \alpha \in \mathcal{ML}(S), i(\alpha, \eta) + i(\alpha, \lambda) > 0\}. \tag{2.1}$$

Then we have ([Pap]):

Theorem 2.1. *Let $\lambda \in \mathcal{ML}(S)$. Then the intersection pairing with λ*

$$\mathcal{ML}(\lambda) \to \mathbb{R}_+$$

$$\eta \to i(\lambda, \eta)$$

is a piecewise linear map.

Recall that a measured lamination λ is *filling* if for every non peripheral simple closed curve γ, on S, $i(\gamma, \lambda) > 0$. In this case, the complementary regions of λ are ideal polygons. It is easy to show that almost every $\lambda \in \mathcal{ML}(S)$ is filling.

Lemma 2.2. *Given $\lambda \in \mathcal{ML}(S)$, either*

1. *λ is a multicurve, or*

2. *there is a subsurface $S_1 \subset S$ such that λ is filling in S_1. In this case, we can write $\lambda = \gamma + \lambda_1$ such that $\mathcal{I}_{S_1}^{-1}(\lambda) \in \mathcal{ML}(S_1)$ is filling, and $i(\gamma, \eta) = 0$ for any $\eta \in \mathcal{I}_{S_1}(\mathcal{ML}(S_1))$.*

2.3 Thurston volume form on $\mathcal{ML}(S)$.

The space of measured laminations $\mathcal{ML}(S)$ has a piecewise linear integral structure defined using train track such that the integral points of $\mathcal{ML}(S)$ are in one to one correspondence with the set of integral multicurves on S.

Fix a a train track τ on S (See [HP]). Let $E(\tau)$ be the set of measures on a train track τ; more precisely, $u \in E(\tau)$ is an assignment of positive real numbers on the edges of the train track satisfying the switch condition,

$$\sum_{\text{incoming } e_i} u(e_i) = \sum_{\text{outgoing } e_j} u(e_j).$$

Recall that when a lamination λ carried by τ has an invariant measure μ, then the carrying map defines a counting measure $\mu(b)$ to each branch line b: $\mu(b)$ is just the transverse measure of the leaves of λ collapsed to a point on b. At a switch, the sum of the entering numbers equals the sum of the exiting numbers.

By work of Thurston [Th], for any maximal train track we have:

- $E(\tau)$ gives rise to an open set $U(\tau)$ in the space of measured laminations.
- For any train track τ, the integral points in $E(\tau)$ are in one to one correspondence with the set of integral multicurves in $U(\tau) \subset \mathcal{ML}(S)$
- The natural volume form on $E(\tau)$ defines a mapping class group invariant volume form μ_{Th} in the Lebesgue measure class on $\mathcal{ML}(S)$.
- This measure is quasi invariant under the action of $\mathbb{R}_+$; for any $U \subset \mathcal{ML}(S)$, and $t \in \mathbb{R}_+$, we have $\mu_{Th}(t \cdot U) = t^{6g(S)-6+2n(S)} \mu_{Th}(U)$.

The action of $\mathbb{R}_+$ on the set of multicurves extends continuously to $\mathcal{ML}(S)$. The quotient space $P\mathcal{ML}(S) = \mathcal{ML}(S)/\mathbb{R}_+$ is homeomorphic to $S^{6g(S)-7+2n(S)}$.

A measured lamination λ is *maximal* if it is filling and all the complementary polygons are triangles. Using train track coordinates one can show that:

Lemma 2.3. *With respect to the Lebesgue measure class on $\mathcal{ML}(S)$, almost every measured lamination is maximal; $\mu_{Th}(\mathcal{ML}(S) - \mathcal{G}(S)) = 0$.*

Moreover, up to scale μ_{Th} is the unique mapping class group invariant measure in the Lebesgue measure class [Mas1]:

Theorem 2.4. (Masur) μ_{Th} *is a locally finite* $\mathrm{Mod}(S)-$*invariant ergodic measure on $\mathcal{ML}(S)$.*

3 Invariant measures on the space of measured laminations

In section, we construct a family of locally finite $\mathrm{Mod}(S)-$invariant ergodic measures on $\mathcal{ML}(S)$ corresponding to subsurfaces of S.

Fix a multicurve γ on S, and consider the surface $S_{g,n}(\gamma)$ obtained by cutting $S_{g,n}$ along $\gamma_1, \ldots, \gamma_k$. Then $S_{g,n}(\gamma)$ is a (possibly disconnected) surface with $n+2k$ boundary components and $s = s(\gamma)$ connected components. Each connected component γ_i of γ gives rise to 2 boundary components, γ_i^1 and γ_i^2 on $S_{g,n}(\gamma)$, and we have

$$\partial(S_{g,n}(\gamma)) = \{\beta_1, \ldots, \beta_n\} \cup \{\gamma_1^1, \gamma_1^2, \ldots, \gamma_k^1, \gamma_k^2\}.$$

Given a subsurface $R \subset S$, there is a natural embedding

$$\mathcal{I}_R : \mathcal{ML}(R) \to \mathcal{ML}(S).$$

Let

$$\mathcal{G}_1^R(S) = \mathcal{I}_R(\mathcal{ML}(R)) \tag{3.1}$$

be the set of measured laminations supported on $R \subset S$. We say the multicurve γ is disjoint from the subsurface R if for any $\eta \in \mathcal{G}_1^R$, $i(\eta, \gamma) = 0$. If γ is disjoint from R, define

$$\mathcal{G}^{(R,\gamma)}(S) = \{\gamma + \lambda \mid \lambda \in \mathcal{G}_1^R(S)\}.$$

The map

$$\mathcal{I}^{(R,\gamma)} : \mathcal{ML}(R) \to \mathcal{G}^{(R,\gamma)}(S) \subset \mathcal{ML}(S)$$

given by

$$\mathcal{I}^{(R,\gamma)}(\lambda) = \mathcal{I}_R(\lambda) + \gamma$$

induces a measure $\mu_{Th}^{(R,\gamma)}$ on $\mathcal{G}^{(R,\gamma)}(S)$ which is invariant under $\mathrm{Stab}((R,\gamma)) \subset \mathrm{Mod}(S)$. Therefore it gives rise to a $\mathrm{Mod}(S)-$invariant measure $\mu_{Th}^{[(R,\gamma)]}$ supported on

$$\mathcal{G}^{[(R,\gamma)]}(S) = \bigcup_{g \in \mathrm{Mod}(S)} \mathcal{G}^{(g \cdot R, g \cdot \gamma)}(S). \tag{3.2}$$

A similar statement holds for the map $\mathcal{I}^R : \mathcal{ML}(R) \to \mathcal{G}_1^R(S)$. However these measures are not necessarily locally finite in general. Recall that the pair (R, γ) is complete iff R is a union of connected components of $S(\gamma)$.Then we have:

Lemma 3.1. *For any complete pair (R, γ) on S, the measure $\mu_{Th}^{[(R,\gamma)]}$ defines a locally finite ergodic* $\mathrm{Mod}(S)$*-invariant measure on $\mathcal{ML}(S)$ supported on $\mathcal{G}^{[(R,\gamma)]}(S)$.*

Proof. To prove the this claim, fix a hyperbolic surface $X \in \mathcal{T}(S)$. Then the geodesic length function defines a continuous function satisfying the following properties:

- $\ell_{t \cdot \lambda}(X) = t\, \ell_\lambda(X)$.
- if $i(\lambda_1, \lambda_2) = 0$, then $\ell_{\lambda_1 + \lambda_2}(X) = \ell_{\lambda_1}(X) + \ell_{\lambda_2}(X)$.
- Given $L > 0$, $|\{\alpha \in \mathcal{ML}(\mathbb{Z}) \,|\ell_\alpha(X) \leq L\}| < \infty$.

Define $B_L(X) = \{\lambda \in \mathcal{ML}(S), \ell_\lambda(X) \leq L\}$. For any L, $B_L(X)$ is compact, and $\mathcal{ML}(S) = \bigcup_{L>0} B_L(X)$. For any complete pair (R, γ),

$$|\frac{\mathrm{Stab}(\gamma)}{\mathrm{Stab}(R,\gamma)}| < \infty.$$

Hence, given a complete pair (R, γ)

$$|\{(R_1, \gamma_1) \in \operatorname{Mod}(S) \cdot (R, \gamma) |\, \mathcal{G}^{(R_1,\gamma_1)}(S) \cap B_L(X) \neq \emptyset\}| < \infty.$$

Therefore the measure on $\mathcal{G}^{[(R,\gamma)]}$ induced by the measures supported on $\mathcal{G}^{(g\cdot R, g\cdot\gamma)}$ is locally finite. Using Theorem 2.4, it is easy to check that this measure is $\operatorname{Mod}(S)$ ergodic. □

Remark. For any $R_1 \in \operatorname{Mod}(S) \cdot R$, $\mathcal{G}_1^{R_1}(S) \cap B_L(X) \neq \emptyset$. This suggests that the induced measure on $\mathcal{G}_1^{[R]}$ may not be locally finite. In §8 we show that there is no locally finite $\operatorname{Mod}(S)$ invariant measure supported on

$$\mathcal{G}_1^{[R]}(S) = \bigcup_{g \in \operatorname{Mod}(S)} \mathcal{G}_1^{g\cdot R}(S).$$

See Proposition 8.5 for the precise statement.

4 Moduli space of quadratic differentials

In this section, we investigate the relationship between measured laminations and holomorphic quadratic differentials. For more details see [Str], [Gd].

4.1 Moduli space of quadratic differentials.

The cotangent space of $\mathcal{T}(S)$ at a point X can be identified with the vector space $Q(X)$ of meromorphic quadratic differentials with simple poles at the punctures of X. Given $X \in \mathcal{T}(S)$, a quadratic differential $q = f(z)dz^2 \in Q(X)$ is a locally defined holomolphic function $f(z)$ with simple poles at n punctures $p_1 \dots p_n$ of X. Then the space $\mathcal{QT}(S) = \{(q, X) \mid X \in \mathcal{T}(S), q \in Q(X)\}$ is the cotangent space of $\mathcal{T}(S)$. Let $\pi : \mathcal{QT}(S) \to \mathcal{T}(S) : (X, q) \to X$ denote the projection map. Also let $\mathcal{QM}(S) \cong \mathcal{QT}(S)/\operatorname{Mod}(S)$.

Although the value of $q \in Q(X)$ at a point $x \in X$ depends on the local coordinates, the zero set of q is well defined. As a result, there is a natural stratification of the space $\mathcal{QM}(S)$ by the multiplicities of zeros of q. Define $\mathcal{QM}(S; a_1, \dots, a_m) \subset \mathcal{QM}(S)$ to be the subset consisting of pairs (X, q) of holomorphic quadratic differentials on X with m zeros with multiplicities $(a_1, \dots, a_m)$ and simple poles at the marked points of X. The Gauss-Bonnet formula implies that $4g(S) - 4 + n(S) = \sum_{i=1}^{m} a_i$. Then

$$\mathcal{QM}(S) = \bigsqcup_{(a_1,\dots,a_m)} \mathcal{QM}(S; a_1, \dots, a_k).$$

It is known that each $\mathcal{QM}(S; a_1, \dots, a_k)$ is an orbifold of dimension $4g(S) - 4 + 2k$. In particular $\dim(\mathcal{QM}(S; 1, \dots, 1)) = \dim(\mathcal{QM}(S))$.

One way to understand this moduli space is by studying the period coordinates. If $q = f(z)\ (dz)^2 \in Q(X)$, then on a neighborhood of $z_0 \in X$ the quadratic differential is given by $q = (dw)^2$, where $w(z) = \int\limits_{z_0}^{z} \sqrt{f(z)}dz$. We just need to choose the neighborhood small enough so that a single valued branch of the function $\sqrt{f}$ can be chosen. On a small chart which contains a singularity, coordinate w can be chosen in such way that $q = z^k dw^2$,where k is the order of the singularity. Here $k = -1$ if the point corresponding to $z = 0$ is one of the marked points; otherwise $k \geq 1$. A *saddle connection* is a geodesic segment which joins a pair of singular points without passing through one in its interior. In general, a geodesic segment e joining two zeros of a quadratic differential $q = \phi dz^2$ determines a complex number $\mathrm{hol}_q(e)$ (after choosing a branch of $\phi^{1/2}$ and an orientation of e) by

$$\mathrm{hol}_q(e) = \mathrm{Re}(\mathrm{hol}_q(e)) + \mathrm{Im}(\mathrm{hol}_q(e)),$$

where

$$\mathrm{Re}(\mathrm{hol}_q(e)) = \int\limits_e \mathrm{Re}(\phi^{1/2}),$$

and

$$\mathrm{Im}(\mathrm{hol}_q(e)) = \int\limits_e \mathrm{Im}(\phi^{1/2}).$$

The period coordinates gives $\mathcal{QT}(S; a_1, \ldots, a_m)$ the structure of a piecewise linear manifold as follows. For notational simplicity we discuss the case of $\mathcal{QT}(S; 1, \ldots, 1)$. Given $q_0 \in \mathcal{QT}(S; 1, \ldots, 1)$ there is a triangulation E of the underlying surface by saddle connections, $h = 6g(S) - 6 + n(S)$ directed edges $\delta_1, \ldots, \delta_h$ of E, and an open neighborhood $U_{q_0} \subset \mathcal{QT}(S; 1, \ldots, 1)$ of q_0 such that the map

$$\psi_{E,q_0} : \mathcal{QT}(S; 1, \ldots, 1) \to \mathbb{C}^{6g(S)-6+2n(S)}$$

by

$$\psi_{E,q_0}(q) = (\mathrm{hol}_q(\delta_i))_{i=1}^{h}$$

is a local homeomorphism. Also for any other geodesic triangulation $E^{'}$ the map the map $\psi_{E^{'},q_0} \circ \psi_{E,q_0}^{-1}$ is linear. For a discussion of these coordinates see [MS].

Note that the metric $|f(z)^{1/2}dz|$ defined by $q = fdz^2$ has zero curvature outside singular set of q. In terms of the volume form induced by this flat metric

$$\parallel q \parallel = \mathrm{Area}_q(S).$$

If q has at worst simple poles at the punctures of X, then $\parallel q \parallel < \infty$. See [Str] and for more details. Let $\mathcal{Q}^1\mathcal{T}(S)$ denote the Teichmüller space of unit area quadratic differentials on surfaces marked by S. Then

$$\mathcal{Q}^1\mathcal{M}(S) = \mathcal{Q}^1\mathcal{T}(S)/\operatorname{Mod}(S).$$

4.2 $\mathrm{SL}_2(\mathbb{R})$ action on the space of quadratic differentials.

A quadratic differential $q \in \mathcal{QT}(S)$ with simple poles at $p_1, \ldots, p_n$ and zeros at $x_1, \ldots x_k$ is determined by an atlas of charts $\{\phi_i\}$ mapping open subsets of $S - \{p_1, \ldots, p_n, x_1, \ldots, x_k\}$ to $\mathbb{R}^2$ such that the change of coordinates are of the form $v \to \pm v + c$. Therefore the group $\mathrm{SL}_2(\mathbb{R})$ acts naturally on $\mathcal{QM}(S)$ by acting on the corresponding atlas; given $A \in \mathrm{SL}_2(\mathbb{R})$, $A \cdot q \in \mathcal{QM}(S)$ is determined by the new atlas $\{A\phi_i\}$. The dynamics of the action of the following subgroups on $\mathcal{QM}(S)$ play an important role in this paper:

1. The action of the diagonal subgroup $g_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$ is the Teichmüller geodesic flow for the Teichmüller metric.
2. The action of the unipotent subgroup $u_t = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ is the Teichmüller unipotent flow.

Since $\| q \| = \| A \cdot q \|$, the group $\mathrm{SL}_2(\mathbb{R})$ acts naturally on $\mathcal{Q}^1\mathcal{M}(S)$. Moreover this action preserves $\mathcal{Q}^1\mathcal{M}(S; a_1, \ldots, a_k)$.

The Teichmüller unipotent flow has a simple form in the holonomy coordinates; for any quadratic differential q, $u_t(q)$ is determined by

$$\mathrm{Re}(\mathrm{hol}_{u_t q}(e)) = \mathrm{Re}(\mathrm{hol}_q(e)), \tag{4.1}$$

and

$$\mathrm{Im}(\mathrm{hol}_{u_t q}(e)) = \mathrm{Im}(\mathrm{hol}_q(e)) + t\ \mathrm{Re}(\mathrm{hol}_q(e)).$$

In holonomy coordinates the Teichmüller geodesic flow is given by

$$\mathrm{Re}(\mathrm{hol}_{g^t q}(e)) = e^{t/2}\,\mathrm{Re}(\mathrm{hol}_q(e)),\ \ \mathrm{Im}(\mathrm{hol}_{g^t q}(e)) = e^{-t/2}\,\mathrm{Im}(\mathrm{hol}_q(e)).$$

Hence for any $t, s \in \mathbb{R}$, we have $g^t \circ u_{se^{-t}} \circ g^{-t} = u_s$.
The following result plays an important role in this paper [V1],[Mas1]:

Theorem 4.1. *(***Veech-Masur***) The space $\mathcal{Q}^1\mathcal{M}(S)$ carries a natural measure μ_S in the Lebesgue measure class such that :*

- *the space $\mathcal{Q}^1\mathcal{M}(S)$ has finite measure;*
- *The action of $SL_2(\mathbb{R})$ is volume preserving and ergodic;*
- *Both the Teichmüller geodesic and unipotent flows are mixing.*

Remark. The measure μ_S is given by the Piecewise linear structure of $\mathcal{QM}(S)$. It also coincides with the measure defined by the Teichmüller norm on the unit cotangent bundle of $\mathcal{T}(S)$. For more details see [Mas2]. This measure is supported on $\mathcal{Q}^1\mathcal{M}(S; 1, 1, \ldots, 1))$; that is, $\mu_S(\mathcal{Q}^1\mathcal{M}(S) - \mathcal{Q}^1\mathcal{M}(S; 1, 1, \ldots, 1)) = 0$.

4.3 Hubbard-Masur map.

A holomorphic quadratic differential q on X determines two measured foliations $\mathrm{Re}(q)$ and $\mathrm{Im}(q)$ such that near a nonsingular point p with canonical coordinate $z = x + iy$, horizontal leaf segments are parallel to the x-axis and the transverse measure on $\mathrm{Re}(q)$ is defined by integration of $|dy|$, while vertical leaf segments are parallel to the y-axis with transverse measure defined by integrating $|dx|$. The foliations $\mathrm{Re}(q)$ and $\mathrm{Im}(q)$ have singularities of the same type at the zeros of q.

Recall that a *measured foliation* is a foliation of the surface with a transverse measure and only finitely many singularities similar to the singularities of holomorphic quadratic differentials. Let $\mathcal{MF}(S)$ be the set of equivalence classes of measured foliations on S with generalized saddle singularities (three prongs or more), where the equivalence relation is generated by isotopy and Whitehead moves (i.e. collapsing saddle connections) [FLP].

From our discussion we get a map

$$\mathcal{P}: \ \mathcal{QT}(S) \to \mathcal{MF}(S) \times \mathcal{MF}(S) - \Delta, \tag{4.2}$$
$$\mathcal{P}(q) = (F_+(q), F_-(q))$$

by $F_+(q) = \mathrm{Re}(q)$, and $F_-(q) = \mathrm{Im}(q)$, where $\Delta = \{(\lambda, \eta) \mid \text{there exists } \ \gamma; i(\gamma, \lambda) + i(\gamma, \eta) = 0\}$.

Then

$$\mathcal{P}(g^t(q)) = (e^{t/2} F_+(q), e^{-t/2} F_-(q)).$$
$$\mathrm{Re}(\mathcal{P}(u_t(q))) = \mathrm{Re}(\mathcal{P}(q)).$$

Theorem 4.2. (Hubbard-Masur, Gardiner) *The map $\mathcal{P}$ is a mapping class group equivariant homeomorphism.*

For the proof see [HM]. For the treatment of the case $n(S) > 0$ see [Gd], and [GM].

Remark on measured foliations. For any curve γ, $i(F, \gamma)$ is the transverse length of γ by $|q|$. Two measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are equivalent if $i(\mathcal{F}_1, \beta) = i(\mathcal{F}_2, \beta)$ for all classes β. The space $\mathcal{MF}(S)$ is a piecewise linear manifold of dimension $6g(S) - 6 + 2n(S)$. See [Th], [FLP].

In fact, it is easy to see that by straightening the leaves of a measured foliation one can obtain a measured lamination. Conversely, for $\lambda \in \mathcal{ML}(S)$, carried by a train track τ, we can define a measured foliation $\widetilde{\lambda}$ on a neighborhood of τ induced by λ. Then by collapsing each region of $X - \tau$ into a spine, we can extend this measured foliation. The measured foliation $\widetilde{\lambda}$ is well defined up to equivalent classes of $\mathcal{MF}(S)$. Also given $\lambda, \eta \in \mathcal{MF}(S)$, $i(\lambda, \eta) = i(\widetilde{\lambda}, \widetilde{\eta})$. Therefore, measured lamination and measured foliations are essentially the same [Le]. The Hubbard-Masur map gives up a map

$$\mathcal{QT}(S) \to \mathcal{ML}(S) \times \mathcal{ML}(S) - \Delta$$

which is denoted by the same latter $\mathcal{P}$. In this paper we only work with the language of measured laminations instead of measured foliations.

5 From $\mathcal{ML}(S)$ to $\mathcal{Q}^1\mathcal{M}(S)$

In this section, we construct measures on $\mathcal{Q}^1\mathcal{M}(S)$ induced by locally finite $\mathrm{Mod}(S)-$invariant measures on $\mathcal{ML}(S)$.

5.1 Quadratic differentials of norm one.

To study the moduli space of quadratic differentials of norm one, we modify the Hubbard-Masur map (§4.3) and obtain a bijection between $\mathcal{Q}^1\mathcal{T}(S)$ and $\mathcal{ML}(S) \times P\mathcal{ML}(S) - \Delta$ as follows. Consider the projection map sending $\eta \in \mathcal{ML}(S)$, to the corresponding projective measured lamination $[\eta] \in P\mathcal{ML}(S)$. Then the map $\mathcal{P}_1$ defined by

$$\mathcal{P}_1 : \mathcal{Q}^1\mathcal{T}(S) \to \mathcal{ML}(S) \times P\mathcal{ML}(S) - \Delta$$

$$q \to (\mathrm{Re}(q), [\mathrm{Im}(q)]),$$

is a homeomorphism. Here

$$\Delta = \{(\lambda, [\eta]) \mid \text{there exists } \ \gamma; i(\gamma, \lambda) + i(\gamma, \eta) = 0\}.$$

On the other hand given $(\lambda, [\eta]) \in \mathcal{ML}(S) \times P\mathcal{ML}(S) - \Delta$, there is a unique $q = (\lambda, \eta_1)$ such that

$$\eta_1 = \mathrm{Im}(q) \in [\eta] \ , \ \mathrm{Re}(q) = \lambda$$

and $\parallel q \parallel = 1$. Define $\tau^1 : \mathcal{ML}(S) \times P\mathcal{ML}(S) - \Delta \to \mathcal{Q}^1\mathcal{M}(S)$ by

$$\tau^1(\lambda, [\eta]) = \pi(\mathcal{P}_1^{-1}(\lambda, [\eta])) \tag{5.1}$$

where $\pi : \mathcal{Q}^1\mathcal{T}(S) \to \mathcal{Q}^1\mathcal{M}(S)$ is the projection map.

5.2 Construction of induced measures on $\mathcal{Q}^1\mathcal{M}(S)$.

Fix $\lambda \in \mathcal{ML}(S)$. Then the measure μ_{Th} defines a locally finite measure on $P\mathcal{ML}(\lambda)$ (see equation 2.1) by

$$\mu^{\lambda}_{Th}(U) = \mu_{Th}(\{\eta \mid [\eta] \in U \ , i(\lambda, \eta) \leq 1\}). \tag{5.2}$$

As a result, we get:

- $\{\mu^{\lambda}_{Th}\}_{\lambda \in \mathcal{ML}(S)}$ is Mod(S)-equivariant ; that is, for $g \in \mathrm{Mod}(S)$, we have $\mu^{g\cdot\lambda}_{Th}(g \cdot U) = \mu^{\lambda}_{Th}(U)$;
- for any $t \in \mathbb{R}$, $\mu^{e^t\cdot\lambda}_{Th}(U) = e^{-t\cdot h}\mu^{\lambda}_{Th}(U)$, where $h = 6g(S) - 6 + 2n(S)$;
- $\mu^{\lambda}_{Th}(P\mathcal{ML}(S)) = \infty$.

Given a locally finite mapping class group invariant measure μ, we construct a measure $\tilde{\mu}$ on $\mathcal{Q}^1\mathcal{M}(S)$ as follows.

Let $U_1 \times U_2 \subset \mathcal{ML}(S) \times P\mathcal{ML}(S) - \Delta$. By equation (5.2), any $\lambda \in U_1$ induces a measure μ_{Th}^{λ} on U_2. Now define $\tilde{\mu}$ by

$$\tilde{\mu}(U_1 \times U_2) = \int_{U_1} \mu_{Th}^{\lambda}(U_2) d\mu(\lambda). \tag{5.3}$$

Remark. For $\mu = \mu_{Th}$, the measure $\widetilde{\mu_{Th}}$ is the same as the measure μ_S introduced in §4.2.

Lemma 5.1. *Let μ be a locally finite* $\mathrm{Mod}(S)$*-invariant measure on $\mathcal{ML}(S)$. Then the induced measure $\widetilde{\mu}$ defined by equation (*5.3*) satisfies the following properties:*

1. *$\widetilde{\mu}$ is a locally finite,* $\mathrm{Mod}(S)$*-invariant measure on $\mathcal{Q}^1\mathcal{M}(S)$;*
2. *the measure $\widetilde{\mu}$ is supported on $\mathcal{Q}^1\mathcal{M}(S; 1, 1, \ldots, 1) \subset \mathcal{Q}^1\mathcal{M}(S)$, that is*

   $$\widetilde{\mu}(\mathcal{Q}^1\mathcal{M}(S) - \mathcal{Q}^1\mathcal{M}(S; 1, 1, \ldots, 1)) = 0;$$

3. *$\widetilde{\mu}$ is $\{u_t\}$-invariant;*
4. *Assume that μ is quasi invariant under the action of $\mathbb{R}_+$, so that $\mu(tU) = t^k\mu(U)$ for $U \subset \mathcal{ML}(S)$. Then for $V = U_1 \times U_2 \subset \mathcal{ML}(S) \times P\mathcal{ML}(S)$,*

   $$\frac{\tilde{\mu}(g^t(V))}{\tilde{\mu}(V)} = e^{t(k-h)/2},$$

   where $h = \dim(\mathcal{T}(S)) = 6g(S) - 6 + 2n(S)$.

Proof.
Part 1 is immediate from the definition.
Part 2): Note that if $\mathrm{Im}(q)$ is a maximal measured lamination, then $q \in \mathcal{Q}^1\mathcal{M}(S; 1, 1, \ldots, 1)$. Lemma 2.3 implies that for any $\lambda \in \mathcal{ML}(S)$, a μ_{Th}^{λ} typical point in $P\mathcal{ML}(\lambda)$ is maximal. Now the result is immediate by the definition (equation (5.3)).
Part 3): Fix $\lambda \in \mathcal{ML}(S)$, and a small open neighborhood $U_1 \subset P\mathcal{ML}(\lambda)$. Let

$$V = \{\eta \in \mathcal{ML}(S) \mid [\eta] \in U_1, i(\eta, \lambda) = 1\}.$$

As before, let (λ, η) be the quadratic differential q such that $\mathrm{Re}(q) = \lambda$, and $\mathrm{Im}(q) = \eta$. Define $u_t^{\lambda} : V \to V$ such that

$$u_t(\lambda, \eta) = (\lambda, u_t^{\lambda}(\eta))$$

Then we have

- By Theorem 2.1, $V \subset \mathcal{ML}(S)$ is a piecewise linear subspace of $\mathcal{ML}(S)$;
- Using the holonomy coordinates in §4.2 (equation (4.1)), it is easy to verify that for every $t \in \mathbb{R}$, the map u_t^{λ} is a translation.

Therefore for every $t \in \mathbb{R}$, $\mu^\lambda_{Th}(U_1) = \mu^\lambda_{Th}([u^\lambda_t(V)])$.
Part 4): By the definition of μ^λ_{Th}, we have

$$\mu^{e^t \cdot \lambda}_{Th}(U) = e^{-h \cdot t}\mu^\lambda_{Th}(U), \ \ g^t(U_1 \times U_2) = e^{t/2} \cdot U_1 \times U_2.$$

So we have

$$\tilde{\mu}(g^t(V)) = \tilde{\mu}(g^t(U_1 \times U_2)) = \int_{e^{t/2} \cdot U_1} \mu^\lambda_{Th}(U_2) d\mu(\lambda) =$$

$$= \int_{U_1} \mu^{e^{t/2} \cdot \lambda}_{Th}(U_2)(\frac{d\mu(e^{-t/2}\lambda)}{d\mu(\lambda)})d\mu(\lambda) = e^{-h \cdot t/2} \cdot \tilde{\mu}(V) \frac{\mu(U)}{\mu(e^{-t/2} \cdot U)} = e^{t(k-h)/2} \cdot \tilde{\mu}(V).$$

□

Remark. In fact, by the definition, the measure $\tilde{\mu}$ is invariant under the horospherical equivalence relation. The foliations

$$\mathcal{F}^+(q) = \{q_0 \in \mathcal{Q}^1\mathcal{T}(S) \mid \operatorname{Im}(q_0) = \operatorname{Im}(q)\}, \tag{5.4}$$

and

$$\mathcal{F}^-(q) = \{q_0 \in \mathcal{Q}^1\mathcal{T}(S) \mid \operatorname{Re}(q_0) = \operatorname{Re}(q)\}$$

of $\mathcal{Q}^1\mathcal{T}(S)$ play an important role in this paper. Note that $\mathcal{F}^+$ and $\mathcal{F}^-$ also give rise to foliations of $\mathcal{Q}^1\mathcal{M}(S)$ which will be denoted by the same letters.

There is a one to one correspondence between the space of leaves of the real foliation and $\mathcal{ML}(S)$ as follows:

$$\lambda \to \{q \in \mathcal{Q}^1\mathcal{T}(S) \mid \operatorname{Re}(q) = \lambda\}.$$

As a result of what we proved in this section, the measure μ_{Th} on $\mathcal{ML}(S)$ gives rise to a globally defined conditional measure μ_q on each $\mathcal{F}^+(q)$ such that

$$(g^t)_*\mu_q = e^{(6g(S)-6+2n(S))t}\mu_{g^t q}.$$

6 Non divergence of the Teichmüller unipotent flow

We say a quadratic differential q is *filling* if $\operatorname{Im}(q) \in \mathcal{ML}(S)$ is filling as a measured lamination. Define

$$\widetilde{\mathcal{G}}(S) = \{q \mid q \in \mathcal{Q}^1\mathcal{M}(S) \text{ is a filling quadratic differential}\}.$$

In this section we show that if for a locally finite $\operatorname{Mod}(S)$ invariant measure μ on $\mathcal{ML}(S)$, $\mu(\mathcal{G}) > 0$ then the measure of recurrent quadratic differentials with respect to the induced measure $\widetilde{\mu}$ on $\mathcal{Q}^1\mathcal{M}(S)$ is positive. Given $K \subset \mathcal{Q}^1\mathcal{M}(S)$, and $T \geq 0$, define

$$\operatorname{Ave}_{T,q}(K) = \frac{|\{t \mid t \in [0,T], \ u_t(q) \in K\ \}|}{T}.$$

First, using the ideas of [MW] we get the following result:

Theorem 6.1. *Given $\epsilon > 0$, there exists a compact set $K \subset \mathcal{Q}^1\mathcal{M}(S)$ such that for any $q \in \widetilde{\mathcal{G}}(S)$,*

$$\liminf_{T\to\infty} \mathrm{Ave}_{T,q}(K) \geq 1-\epsilon.$$

For the proof see the Appendix.

Remark. Using basic properties of quadratic differentials, one can show

- If a simple closed curve α is homotopic to a union of imaginary saddle connections on q, then α is homotopically non trivial.
- Given $q \in \mathcal{Q}^1\mathcal{M}(S)$, let $\mathcal{I}(q)$ denote the set of imaginary saddle connections on q. Note that $\mathcal{I}(q)$ consists of finitely many saddle connections meeting possibly at the singularities of q. If $q \notin \widetilde{\mathcal{G}}(S)$ then either there is a path in $\mathcal{I}(q)$ joining two poles of q, or $\mathcal{I}(q)$ contains a loop.

Corollary 6.2. *Let K be the compact subset of $\mathcal{Q}^1\mathcal{M}(S)$ defined as in the previous lemma. Let ν be an ergodic $\{u_t\}$-invariant probability measure on $\mathcal{Q}^1\mathcal{M}(S)$. If $\nu(\mathcal{Q}^1\mathcal{M}(S) - \widetilde{\mathcal{G}}(S)) = 0$, then $\nu(K) > 1-\epsilon$.*

Proof. By the pointwise ergodic theorem for ν almost every point q, we have

$$\nu(K) = \lim_{T\to\infty} \mathrm{Ave}_{T,q}(K).$$

Hence Theorem 6.1 implies that $\nu(K) \geq 1-\epsilon$. □

Corollary 6.3. *Let μ be locally finite ergodic* Mod$(S)-$*invariant measure on $\mathcal{ML}(S)$ such that $\mu(\mathcal{G}(S)) > 0$. Then the measure $\widetilde{\mu}$ induced by μ on $\mathcal{Q}^1\mathcal{M}(S)$ (§5.2) is finite; that is*

$$\widetilde{\mu}(\mathcal{Q}^1\mathcal{M}(S)) < \infty.$$

Moreover, for K defined in Theorem 6.1 we have

$$\widetilde{\mu}(K) \geq (1-\epsilon)\, \widetilde{\mu}(\mathcal{Q}^1\mathcal{M}(S)). \tag{6.1}$$

Proof. By the assumption $\widetilde{\mu}(\mathcal{Q}^1\mathcal{M}(S) - \widetilde{\mathcal{G}}(S)) = 0$. First we consider the ergodic decomposition of $\widetilde{\mu}$ for the $\{u_t\}$ flow

$$\widetilde{\mu} = \int_V \nu_s ds.$$

It is easy to see that for almost every $s \in V$, ν_s satisfies

$$\nu_s(\mathcal{Q}^1\mathcal{M}(S) - \widetilde{\mathcal{G}}(S)) = 0. \tag{6.2}$$

Theorem 6.1 implies that for every $s \in V$, ν_s is a finite measure (see Corollary 2.7 in [MW]) . Now it is enough to note that:

- if equation (6.2) holds then by Corollary 6.2, $(1-\epsilon)\ \nu_s(\mathcal{Q}^1(\mathcal{M}(S)) \leq \nu_s(K)$.

- since μ is locally finite, $\widetilde{\mu}(K) < \infty$. Hence $\int\limits_V \nu_s(K) ds < \infty$.

$\square$

Theorem 6.4. *There exists a compact set* $K_0 \subset \mathcal{Q}^1\mathcal{M}(S; 1, \ldots, 1)$*, and* $c_0 > 0$ *such that for any measure* $\widetilde{\mu}$ *as in Corollary* 6.3 *we have*

$$\widetilde{\mu}(K_0) \geq c_0\, \widetilde{\mu}(\mathcal{Q}^1\mathcal{M}(S)).$$

Proof. Fix the compact set $K \subset \mathcal{Q}^1\mathcal{M}(S)$ defined by Theorem 6.1 for $\epsilon = 1/2$. Consider the map $\tau^1 : \mathcal{ML}(S) \times P\mathcal{ML}(S) - \Delta \to \mathcal{Q}^1\mathcal{M}(S)$ (equation (5.1)). Note that if η is a maximal measured lamination, then for any $\lambda \in \mathcal{ML}(S)$, $\tau^1(\lambda, [\eta]) \in \mathcal{Q}^1\mathcal{M}(S; 1, \ldots, 1)$. Also we have:

- There exists a finite collection of bounded open sets $U_1, \ldots U_s \subset \mathcal{ML}(S)$ and $V_1, \ldots V_s \subset P\mathcal{ML}(S)$ such that

$$K \subset \bigcup_{i=1}^{s} \tau^1(U_i \times V_i).$$

- With respect to the Lebesgue measure class almost every point in $P\mathcal{ML}(S)$ is maximal. Therefore, one can find open sets $W_i \subset V_i$ such that every $\eta \in V_i - W_i$ is maximal, and for $\lambda \in U_i$, we have $\mu^{\lambda}_{Th}(W_i) < \frac{1}{2}\, \mu^{\lambda}_{Th}(V_i)$.

Hence for any measure $\widetilde{\mu}$ induced from a locally finite $\mathrm{Mod}(S)$ invariant measure μ on $\mathcal{ML}(S)$, we have

$$\widetilde{\mu}(\tau^1(U_i \times (V_i - W_i))) \geq \frac{1}{2}\, \widetilde{\mu}(\tau^1(U_i \times V_i)).$$

Now the set K_0 defined by

$$K_0 = \bigcup_{i=1}^{s} \tau^1(\overline{U_i} \times (\overline{V_i} - W_i)) \subset \mathcal{Q}^1\mathcal{M}(S; 1, \ldots, 1)$$

is compact. Let $c_0 = \frac{1}{4s}$. For any $\widetilde{\mu}$ (as in Corollary 6.3), by equation (6.1) we have

$$\widetilde{\mu}(K_0) \geq \frac{1}{s} \sum_{i=1}^{s} \widetilde{\mu}(\tau^1(\overline{U_i} \times (\overline{V_i} - W_i))) \geq \frac{1}{2s} \sum_{i=1}^{s} \widetilde{\mu}(\tau^1(U_i \times V_i)) \geq \frac{1}{2s} \widetilde{\mu}(K) \geq$$

$$\geq \frac{1}{4s} \widetilde{\mu}(\mathcal{Q}^1\mathcal{M}(S)).$$

$\square$

6.1 Recurrent quadratic differentials.

As a result of Theorem 6.4, we show that the set of geodesic recurrent points in $\mathcal{Q}^1\mathcal{M}(S;1,\ldots,1)$ has positive $\widetilde{\mu}$ measure:

Corollary 6.5. *Let K_0 be the compact set of $\mathcal{Q}^1\mathcal{M}(S;1,\ldots,1)$ defined in Theorem 6.4. Then for any $\widetilde{\mu}$ induced by a locally finite* Mod(S)*−invariant ergodic measure μ on $\mathcal{ML}(S)$ with $\mu(\mathcal{G}(S))>0$ we have*

$$\widetilde{\mu}(\{q\in\mathcal{Q}^1\mathcal{M}(S)\mid g^T(q)\in K_0 \text{ for arbitrary large } T>0\})>0.$$

Proof. Given $t\in\mathbb{R}_+$, define $\widetilde{\mu}_t$ by $\widetilde{\mu}_t(U)=\widetilde{\mu}(g^tU)$. One can check that in terms of the notation of §5.2, $\widetilde{\mu}_t$ is induced by the measure μ_t on $\mathcal{ML}(S)$ satisfying $\mu_t(U)=\mu(e^{t/2}U)$. By Corollary 6.3, $\widetilde{\mu}$ is a finite measure. Without loss of generality we can assume that $\widetilde{\mu}$ is a probability measure. Hence Theorem 6.4 implies that $\{\widetilde{\mu}_t\}_t$ is a sequence of probability measures on $\mathcal{Q}^1\mathcal{M}(S)$ satisfying

$$\lim_{i\to+\infty}\widetilde{\mu}_{t_i}(K_0)>c_0>0.$$

For $r\in\mathbb{R}_+$ define

$$A_r=\{q\mid g^T(q)\notin K_0,\ \ \text{for } T\geq r\}.$$

By the definition, $\widetilde{\mu}_{t_i}(K_0)>c_0$ implies that $\widetilde{\mu}(\{q\mid g^{t_i}(q)\notin K_0\})<1-c_0$. Therefore $\widetilde{\mu}(A_{t_i})<1-c_0$. On the other hand, for $r\leq s$, $A_r\subset A_s$. Hence

$$\widetilde{\mu}(\bigcup_{n=1}^{\infty}A_n)<1-c_0.$$

Therefore

$$\tilde{\mu}(\{q\in\mathcal{Q}^1\mathcal{M}(S)\mid g^T(q)\in K_0 \text{ for arbitrary big } T>0\})\geq c_0>0$$

which implies the result. □

Also the same argument implies that:

Theorem 6.6. *There exists a compact set $K_0\subset\mathcal{Q}^1\mathcal{M}(S;1,\ldots,1)$ such that for any filling quadratic differential q, there is a sequence $\{q_i\}_i$ of quadratic differentials such that for every $i\geq 1$*

1. $q_i\in\mathcal{F}^-(q)$.
2. $g^i(q_i)\in K_0$.

Sketch of the proof. We can choose K_0 as in the proof of Theorem 6.4. Now for $n\in\mathbb{N}$, there is $1\leq k\leq s$ such that $g^n(q)=\tau^1(\lambda,[\eta])$ such that $\lambda\in U_k$, and $[\eta]\in V_k$. If $g^n(q)\notin K_0$, then $[\eta]\in W_k$. Since $V_k-W_k\neq\emptyset$, we can choose $[\eta_0]\in V_k-W_k$. Now let $q_n=g^{-n}(\tau^1(\lambda,[\eta_0]))$. By the definition, $q_n\in\mathcal{F}^-(q)$ and $g^n(q_n)\in\tau^1(U_k\times(V_k-W_k))\subset K_0$. □

Remark. Using the same method, one can show that for any locally finite u_t ergodic measure ν on $\mathcal{Q}^1\mathcal{M}(S)$, ν positive set of points are backward recurrent in $\mathcal{Q}^1\mathcal{M}(S)$. However, in the case where ν is supported on the subset of quadratic differentials with imaginary saddle connections, for almost every q , g^tq is divergent in $\mathcal{Q}^1\mathcal{M}(S;1,1,\ldots,1)$.

7 Invariant measures supported on filling laminations

In this section we study the set of ergodic measures for the action of the mapping class group supported on the locus $\mathcal{G}$ of filling measured laminations, and obtain the following theorem:

Theorem 7.1. *Let μ be a locally finite* $\mathrm{Mod}(S)$ *invariant ergodic measure on $\mathcal{ML}(S)$ such that $\mu(\mathcal{G}) > 0$. Then μ is a constant multiple of the Thurston measure μ_{Th}.*

A subset $B \subset \mathcal{Q}^1\mathcal{M}(S)$ will be called a *box* if $B = \tau^1(U_1 \times U_2)$ (see equation (5.1)) for open subsets $U_1 \subset \mathcal{ML}(S)$ and $U_2 \subset P\mathcal{ML}(S)$ satisfying (i) U_1 is a convex subset of a single train-track chart (see §2.3) and similarly for U_2, (ii) the map $\tau^1|_{U_1 \times U_2}$ is a homeomorphism $U_1 \times U_2 \to B$, (iii) There is some holonomy coordinate chart ψ_{E,q_0} around some $q_0 \in \mathcal{Q}^1\mathcal{M}$ which is defined on $\overline{B}$. Note that it follows from (i)-(iii) above that the map $\psi_{E,q_0} \circ \tau^1$ satisfies that

$$\operatorname{Re} \psi_{E,q_0} \circ \tau^1(\eta, [\lambda']) = \operatorname{Re} \psi_{E,q_0} \circ \tau^1(\eta, [\lambda]) \qquad \text{for all } \eta \in U_1, [\lambda], [\lambda'] \in U_2 \tag{7.1}$$

$$\operatorname{Im} \psi_{E,q_0} \circ \tau^1(\eta', [\lambda]) = \frac{i(\eta', \lambda)}{i(\eta, \lambda)} \operatorname{Im} \psi_{E,q_0} \circ \tau^1(\eta, [\lambda]) \qquad \text{for all } \eta, \eta' \in U_1, [\lambda] \in U_2. \tag{7.2}$$

Note the normalization in (7.2) which arises because we are restricting ourselves to quadratic differentials of area one. Note that for any $\eta \in U_1, [\lambda], [\lambda'] \in U_2$ the points $\tau^1(\eta, [\lambda])$ and $\tau^1(\eta, [\lambda'])$ are on the same leaf of the foliation $\mathcal{F}^-$. One the other hand, for $\eta, \eta' \in U_1, [\lambda] \in U_2$ the points $\tau^1(\eta, [\lambda])$ and $\tau^1(\eta', [\lambda])$ are not necessarily one the same $\mathcal{F}^+$-leaf; this is true, however, if (and only if) in addition $i(\eta, \lambda) = i(\eta', \lambda)$. For any $[\lambda] \in U_2$ and $\eta \in U_1$ we let

$$\mathcal{F}^+_B(\eta; [\lambda]) = \left\{ \tau^1(\eta', [\lambda]) : \eta' \in U_1 \text{ with } i(\eta, \lambda) = i(\eta', \lambda) \right\}$$

this is an open piece of the $\mathcal{F}^+$-leaf through $q = \tau^1(\eta, [\lambda])$. We will also use the analogous notation

$$\mathcal{F}^{0,+}_B(\eta; [\lambda]) = \left\{ \tau^1(\eta', [\lambda]) : \eta' \in U_1 \right\}.$$

Let $B = \tau^1(U_1 \times U_2)$ be a box. A (finite) measure ν on B will be said to be *admissible* if it is of the form

$$\tau^1_* \left(\int_{U_1} \mu^\lambda_{Th}|_{U_2} \, d\nu_1(\lambda) \right) \qquad \text{for some finite measure } \nu_1 \text{ on } U_1. \tag{7.3}$$

Denote the class of admissible measures on B by $\mathcal{A}_B$. In particular, if μ is a locally finite $\mathrm{Mod}(S)$ invariant measure on $\mathcal{ML}(S)$ and $\tilde{\mu}$ the corresponding measure on $\mathcal{Q}^1\mathcal{M}$ as constructed in Section 5 then for any box B, $\tilde{\mu}|_B \in \mathcal{A}_B$. For the special case of $\tilde{\mu} = \mu_S$, (7.3) becomes

$$\mu_S|_B = \tau^1_* \left(\int_{U_1} \mu^{\lambda}_{Th}|_{U_2} \, d\mu_{Th}(\lambda) \right). \tag{7.4}$$

Let d be an arbitrary smooth metric on $\mathcal{Q}^1\mathcal{M}(S)$; for any $Q \subset \mathcal{Q}^1\mathcal{M}(S)$, we let $\operatorname{diam} Q = \sup_{q,q' \in Q} d(q, q')$.

Following is our basic lemma:

Lemma 7.2. *Let f be a compactly supported continuous function on $\mathcal{Q}^1\mathcal{M}(S)$, and $B \subset \mathcal{Q}^1\mathcal{M}(S)$ a box. Then*

$$\sup_{\nu \in \mathcal{A}_B} \left| \frac{1}{\nu(B)} \int_{g^{-T}B} f \, d(g^{-T})_* \nu - \int_{\mathcal{Q}^1\mathcal{M}(S)} f \, d\mu_S \right| \to 0 \qquad \text{as } T \to \infty. \tag{7.5}$$

The proof of this lemma combines two ingredients: the mixing properties of the Teichmüller geodesic flow (see Section 4.2) and the following result of Veech establishing nonuniform hyperbolicity for the Teichmüller geodesic flow on $\mathcal{Q}^1\mathcal{M}(S)$ and identifies the stable/unstable manifolds for this action (see also [For]). Let d be an arbitrary smooth metric on $\mathcal{Q}^1\mathcal{M}(S)$; for any $Q \subset \mathcal{Q}^1\mathcal{M}(S)$, we let $\operatorname{diam} Q = \sup_{q,q' \in Q} d(q, q')$.

Theorem 7.3 (Veech [V1, Thm 5.1]). *Let $B = \tau^1(U_1 \times U_2)$ be a box as above, and let $K_0 \subset \mathcal{Q}^1\mathcal{M}(S; 1, \ldots, 1)$ be compact. There is some $c_0 > 0$ so that for μ_S-almost every $q = \tau^1(\eta, [\lambda]) \in B$,*

$$\operatorname{diam}\left(g^{-t}\mathcal{F}^+_B(\eta; [\lambda])\right) < \exp(-c_0 t) \qquad \text{for all } t > t_0(q) \text{ with } g^t q \in K_0. \tag{7.6}$$

Veech' result is somewhat more precise as it is stated in terms of a specific metric D on $\mathcal{Q}^1\mathcal{M}(S; 1, \ldots, 1)$ in which case the requirement that $g^t q \in K_0$ is not needed; a different choice of metric is used by Forni in [For] which is even better for some purposes. For us, however, the crude form of these results given above is (much) more than sufficient.

Proof of Lemma 7.2. Let $f \in C_c(\mathcal{Q}^1\mathcal{M}(S))$, $\epsilon > 0$ be given. Then one can find open subsets $V_1^{(i)} \subset U_1$, $V_2^{(i)} \subset U_2$ for $i = 1, \ldots, M$ (for some M) be such that

(V-1) $\mu_S(\tau^1(V_1^{(i)} \times V_2^{(i)})) > 0$ and $(V_1^{(i)} \times V_2^{(i)}) \cap (V_1^{(j)} \times V_2^{(j)})$ for $i \neq j$,

(V-2) $\sum_{i=1}^M \mu_S(\tau^1(V_1^{(i)} \times V_2^{(i)})) > (1 - \epsilon)\mu_S(B)$

(V-3) $\sum_{i=1}^M \nu(\tau^1(V_1^{(i)} \times V_2^{(i)})) > (1 - \epsilon)\nu(B)$

(V-4) $\forall i$, for any $[\lambda] \in V_2^{(i)}$ and any $\eta, \eta' \in V_1^{(i)}$

$$1 - \epsilon < \frac{i(\eta, \lambda)}{i(\eta', \lambda)} < 1 + \epsilon.$$

(V-5) $\forall i,\ (1-10\epsilon)V_1^{(i)}, (1+10\epsilon)V_1^{(i)} \subset U_1$.

Let $B^{(i)} = \tau^1(V_1^{(i)} \times V_2^{(i)})$. If we show that for every i,

$$\limsup_{T\to\infty} \left| \frac{1}{\nu(B^{(i)})} \int_{g^{-T}B^{(i)}} f\, d(g^{-T})_*\nu - \int_{\mathcal{Q}^1\mathcal{M}(S)} f\, d\mu_S \right| < \alpha(\epsilon) \tag{7.7}$$

with $\alpha(\epsilon) \to 0$ as $\epsilon \to 0$ then by (V-3)

$$\limsup_{T\to\infty} \left| \frac{1}{\nu(B)} \int_{g^{-T}B} f\, d(g^{-T})_*\nu - \int_{\mathcal{Q}^1\mathcal{M}(S)} f\, d\mu_S \right| < \epsilon \sup_q |f(q)| + \alpha(\epsilon),$$

and the lemma follows by taking $\epsilon \to 0$.

Since in the rest of the proof i is fixed we set $V_1 = V_1^{(i)}$, $V_2 = V_1^{(i)}$, and $B' = B^{(i)}$. Let $\epsilon_0 = \epsilon/\mu_S(B')$. We set K_0 to be a compact subset of $\mathcal{Q}^1\mathcal{M}(S; 1, \dots, 1)$ with $\mu_S(K_0^{\complement}) < \epsilon_0$, and take T_0 be such that (7.6) is satisfied (for this choice of K_0 and some $c_0 > 0$) for every $T > T_0$ outside a set Y_0 with of measure $\mu(Y_0) < \epsilon_0$.

By mixing

$$\frac{1}{\mu_S(B')} \int_{g^{-T}B'} f\, d\mu_S \to \int_{\mathcal{Q}^1\mathcal{M}(S)} f\, d\mu_S \qquad \text{as } T \to \infty.$$

Recall that by definition of μ_{Th}^{η} and (7.3)

$$\int_{g^{-T}B'} f\, d(g^{-T})_*\nu = \int_{B'} f(g^{-T}q)\, d\nu(q)$$
$$\int_{\{(\eta,\lambda):\eta\in V_1, [\lambda]\in V_2, i(\eta,\lambda)<1\}} f(g^{-T}\tau^1(\eta,[\lambda]))\, d\nu_1(\eta)\, d\mu_{Th}(\lambda).$$

For any $\eta \in V_1$ let

$$\tilde{V}_2^{(\eta)} = \{\lambda : [\lambda] \in V_2, i(\eta,\lambda) < 1\},$$

and set $\tilde{V}_2 = (1-\epsilon)\tilde{V}_2^{(\eta_0)}$ with η_0 some arbitrary fixed element of V_1. By (V-4) for every $\eta \in V_1$

$$\tilde{V}_2 \subset \tilde{V}_2^{(\eta)} \subset (1+3\epsilon)\tilde{V}_2$$

and hence $\mu_{Th}(\tilde{V}_2^{(\eta)} \setminus \tilde{E}_2)/\mu_{Th}(\tilde{V}_2) < c\epsilon$. It follows that

$$\left| \int_{B'} f(g^{-T}q)\, d\nu(q) - \int_{V_1} \int_{\tilde{V}_2} f(g^{-T}\tau^1(\eta,[\lambda])\, d\nu_1(\eta)\, d\mu_{Th}(\lambda) \right| < c\epsilon\nu(B') \sup_q |f(q)|, \tag{7.8}$$

and similarly for μ_S with μ_{Th} replacing ν_1. It follows also that

$$\left| \frac{\nu(B')}{\mu_S(B')} - \frac{\nu_1(V_1)}{\mu_{Th}(V_1)} \right| < c\epsilon. \tag{7.9}$$

It follows from (7.8) and (7.9) that

$$(7.7) \leq c'\epsilon \sup_q |f(q)| + \limsup_{T\to\infty} \int_{\tilde{V}_2} \frac{d\mu_{Th}(\lambda)}{\mu_{Th}(\tilde{V}_2)} \left| \int_{V_1} f(g^{-T}\tau^1(\eta,[\lambda])) \frac{d\nu_1(\eta)}{\nu_1(V_1)} - \right.$$

$$\left. - \int_{V_1} f(g^{-T}\tau^1(\eta,[\lambda])) \frac{d\mu_{Th}(\eta)}{\mu_{Th}(V_1)} \right|, \tag{7.10}$$

and it remains to estimate the second term on the r.h.s. of (7.10).

Let $T > T_0$ and $q_0 = \tau^1(\eta_0,[\lambda_0]) \in B' \cap g^T K_0 \cap Y_0^{\complement}$. Then for any $\eta \in V_1$ by (V-5)

$$\frac{i(\eta_0,\lambda_0)}{i(\eta,\lambda_0)} V_1 \subset U_1.$$

Set

$$t(\eta;\eta_0,\lambda_0) = 2\log\frac{i(\eta_0,\lambda_0)}{i(\eta,\lambda_0)} \in (-10\epsilon, 10\epsilon).$$

It follows that for any $\eta \in V_1$

$$g^{t(\eta;\eta_0,\lambda_0)}\tau^1(\eta,[\lambda_0]) \in \mathcal{F}_B^+(q_0)$$

hence by (7.6) and $q_0 \notin Y_0$

$$d\left(g^{-T+t(\eta;\eta_0,\lambda_0)}\tau^1(\eta,[\lambda_0]), g^{-T}q_0\right) < e^{-c_0 T}.$$

Therefore

$$\int_{\tilde{V}_2} \frac{d\mu_{Th}(\lambda)}{\mu_{Th}(\tilde{V}_2)} \left| \int_{V_1} f(g^{-T}\tau^1(\eta,[\lambda])) \frac{d\nu_1(\eta)}{\nu_1(V_1)} - \int_{V_1} f(g^{-T}\tau^1(\eta,[\lambda])) \frac{d\mu_{Th}(\eta)}{\mu_{Th}(V_1)} \right|$$

$$\leq \frac{2\mu_S(Y_0 \cup g^T K_0)}{\mu_S(B')} + \sup_{d(q,q')<e^{-c_0 T}} |f(q) - f(q')| + \sup_{q,t\in(-10\epsilon,10\epsilon)} \left|f(q) - f(g^t q)\right|.$$

Since $\frac{2\mu_S(Y_0\cup g^T K_0)}{\mu_S(B')} < 4\epsilon$ by assumption (and invariance of μ_S under g^t), the r.h.s. $\to 0$ as first $T \to \infty$ and then $\epsilon \to 0$, establishing (7.7) and the lemma follows. □

Proof of Theorem 7.1. Let $\widetilde{\mu}$ be the measure induced by μ on $\mathcal{Q}^1\mathcal{M}(S)$ as in §5. We show that there exists $C > 0$ such that for any compactly supported positive continuous function f on $\mathcal{Q}^1\mathcal{M}(S;1,\ldots,1)$,

$$\int_{\mathcal{Q}^1\mathcal{M}(S)} f \; d\widetilde{\mu} \geq C \int_{\mathcal{Q}^1\mathcal{M}(S)} f d\mu_S. \tag{7.11}$$

This implies that μ_S is absolutely continuous with respect to $\widetilde{\mu}$. Hence μ is absolutely continuous with respect to μ_{Th}. But both μ and μ_{Th} are ergodic with respect to the action of $\mathrm{Mod}(S)$, hence there exists $c \in \mathbb{R}_+$ such that $\mu = c\,\mu_{Th}$.

To prove equation (7.11), we consider the sequence of measure $\widetilde{\mu}_t = g_t\widetilde{\mu}$. Then by Theorem 6.4, there exists $K_0 \subset \mathcal{Q}^1\mathcal{M}(S; 1, \ldots, 1)$ and $c_0 > 0$ such that for every $t \in \mathbb{R}$

$$\widetilde{\mu}_t(K_0) > c_0.$$

Therefore, there exists a box B and $\epsilon > 0$ such that for a sequence $t_i \to \infty$

$$\widetilde{\mu}_{t_i}(B) > \epsilon.$$

Now we apply Lemma 7.2 for $\nu_t = \widetilde{\mu}_t|_B$. Note that $\nu_t \in \mathcal{A}_B$, and $g^{-t}\nu_t = \widetilde{\mu}$. As a result, equation (7.5) implies that for large t_i

$$\frac{1}{\widetilde{\mu}_{t_i}(B)} \int_{g^{-t_i}B} f d\widetilde{\mu} \geq \frac{1}{2} \int_{\mathcal{Q}^1\mathcal{M}(S)} f d\mu_S.$$

On the other hand,

$$\int_{\mathcal{Q}^1\mathcal{M}(S)} f d\widetilde{\mu} \geq \frac{\epsilon}{\widetilde{\mu}_{t_i}(B)} \int_{g^{-t_i}B} f d\widetilde{\mu}$$

which implies equation (7.11) for $C = \epsilon/2$. □

8 Classifying invariant ergodic measures

In this section, we classify all locally finite ergodic measures for the action of the mapping class group on $\mathcal{ML}(S)$:

Theorem 8.1. *Let μ be a locally finite* Mod(S) *invariant ergodic measure on $\mathcal{ML}(S)$. Then exactly one of the following holds:*

1. *μ almost every point of $\mathcal{ML}(S)$ is filling; in this case, μ is a constant multiple of the Thurston measure μ_{Th}, or*

2. *μ almost every point of $\mathcal{ML}(S)$ has a multicurve γ (with positive mass) in its support. In this case μ is a constant multiple of $\mu_{Th}^{[(R,\gamma)]}$ (see §3) for a complete pair (R, γ).*

For $R = \emptyset$ the measure $\mu_{Th}^{[(R,\gamma)]}$ is the discrete counting measure supported on Mod$(S) \cdot \gamma \subset \mathcal{ML}(S)$.

8.1 $\mathbb{R}_+$ Quasi-invariant measures on $\mathcal{ML}(S)$.

In this section, we apply results of §6.1 to get a bound on the exponent of a Mod(S) invariant measure which is quasi invariant under the action of $\mathbb{R}_+$:

Proposition 8.2. *Let μ be a locally finite* Mod$(S)-$ *invariant measure on $\mathcal{ML}(S)$ such that $\mu(t \cdot U) = t^k\mu(U)$, then we have $k \geq \dim(\mathcal{ML}(S)) = 6g(S) - 6 + 2n(S)$.*

The proof of this proposition is based on the following lemma:

Lemma 8.3. *Given $\epsilon > 0$, there exists a compact set $K \subset \mathcal{Q}^1\mathcal{M}(S)$ such that for every u_t-invariant ergodic measure ν on $\mathcal{Q}^1\mathcal{M}(S)$, we have*

$$\nu(\bigcup_{T\geq 1} g^T(K)) \geq (1-\epsilon)\,\nu(\mathcal{Q}^1\mathcal{M}(S)). \tag{8.1}$$

Remark. In [MW, Cor. 2.7] show that if ν is a (locally finite) measure u_t-invariant and ergodic on $\mathcal{Q}^1\mathcal{M}(S)$ it is in fact finite (and hence so is the r.h.s. of (8.1)). This follows easily e.g. from Theorem A.4 below, in conjunction with the Hurewicz ratio ergodic theorem.

Proof. By Theorem 6.3 of [MW], given $\epsilon > 0$, there exists $\epsilon_0 > 0$ and $K \subset \mathcal{Q}^1\mathcal{M}(S)$ such that the following holds for any $q \in \mathcal{Q}^1\mathcal{M}(S)$. If all imaginary saddle connections of q have length at least ϵ_0, then

$$\liminf_{t\to\infty} \mathrm{Ave}_{t,q} > (1-\epsilon).$$

(This also follows from Thoerem A.4 below.) On the other hand, given $q \in \mathcal{Q}^1\mathcal{M}(S)$, there exists T (depending on q) such that $g^{-T}q$ does not have any imaginary saddle connections of length less than ϵ_0. Therefore

$$\liminf_{t\to\infty} \mathrm{Ave}_{t,g^{-T}q}(K) > 1-\epsilon \implies \liminf_{t\to\infty} \mathrm{Ave}_{t,q}(\bigcup_{T\geq 1} g^T K) > 1-\epsilon.$$

By applying the pointwise ergodic theorem for a ν generic point q

$$\nu(\bigcup_{T\geq 1} g^T(K)) \geq (1-\epsilon)\nu(\mathcal{Q}^1\mathcal{M}(S)).$$

□

Proof of Proposition 8.2**.** Assume that $s = 6g(S) - 6 + 2n(S) - k > 0$. Let $\widetilde{\mu}$ denote the corresponding locally finite measure on $\mathcal{Q}^1\mathcal{M}(S)$ defined in §5. By Lemma 5.1 (part 3), for any open set $V \subset \mathcal{Q}^1\mathcal{M}(S)$, we have

$$\widetilde{\mu}(g^t(V)) = e^{-t\cdot s}\widetilde{\mu}(V). \tag{8.2}$$

As a result

$$\widetilde{\mu}(\mathcal{Q}^1\mathcal{M}(S)) = \infty.$$

Let $\widetilde{\mu} = \int_V \nu_s ds$ be the ergodic decomposition of $\widetilde{\mu}$ for the $\{u_t\}$ flow. Lemma 8.3 implies that there is a compact set $K \subset \mathcal{Q}^1\mathcal{M}(S)$ such that for every $s \in V$

$$\nu_s(\bigcup_{T=1}^{\infty} g^T(K)) \geq \frac{1}{2}\nu_s(\mathcal{Q}^1\mathcal{M}(S)).$$

Therefore,

$$\widetilde{\mu}(\bigcup_{T=1}^{\infty} g^T(K)) = \infty. \tag{8.3}$$

On the other hand, since $\widetilde{\mu}$ is locally finite $\widetilde{\mu}(K) < \infty$, and equation (8.2) implies that

$$\widetilde{\mu}(\bigcup_{T=1}^{\infty} g^T(K)) \leq \int_{t\geq 1} \widetilde{\mu}(g^t(K))dt \leq \int_{t\geq 1} e^{-ts}dt < \infty.$$

Therefore

$$\widetilde{\mu}(\mathcal{Q}^1\mathcal{M}(S)) < \infty$$

which contradicts equation (8.3). □

8.2 A lemma about product actions

In order to be able to deal with surfaces with several connected components, the following lemma will be useful:

Lemma 8.4. *Let X_1, X_2 be two locally compact second countable metric spaces, and let Γ_1, Γ_2 countable groups with Γ_i acting continuously on X_i. Then any $\Gamma_1 \times \Gamma_2$-invariant and ergodic locally finite measure μ on $X_1 \times X_2$ is of the form $\mu = \mu_1 \times \mu_2$ with each μ_i a locally finite Γ_i-invariant and ergodic measure on X_i.*

Proof. Decompose μ as

$$\mu = \int_{X_1} d\mu_2^x d\nu_1(x) \tag{8.4}$$

with ν_1 a locally finite measure on X_1 and for every $x \in X_1$ μ_2^x is a locally finite measure on X_2. This determines the μ_2^x up to a scalar, and the measure class of ν_1 (in general there need not be a canonical way to normalize the μ_2^x).

Since Γ_1 fixes μ the uniqueness properties of the decomposition (8.4) imply that for every $g \in \Gamma_1$ for ν_1-a.e. x,

$$\mu_2^x \propto \mu_2^{g.x},$$

It follows that for every continuous compactly supported functions f_1, f_2 on X_2 the function

$$(x, y) \mapsto \frac{\int f_1 \, d\mu_2^x}{\int f_2 \, d\mu_2^x}$$

is $\Gamma_1 \times \Gamma_2$-invariant, hence a.e. constant. Since $C_c(X_2)$ is separable it follows that there is a locally finite measure μ_2 on X_2 so that for ν-a.e. x

$$\mu_2^x = c_x \mu_2.$$

Since μ is locally finite, equation (8.4) implies that the measure μ_1 given by $d\mu_1 = c_x \, d\nu_1$ is locally finite, and (8.4) simplifies to $\mu = \mu_1 \times \mu_2$; and invariance of μ clearly implies that for $i = 1$ and 2, μ_i is Γ_i-invariant. Also if e.g. μ_1 was not ergodic, then taking B to be a Γ_1-invariant set which is neither null nor co-null then $B \times X_2$ is a $\Gamma_1 \times \Gamma_2$-invariant set on $X_1 \times X_2$ which is neither null nor co-null — in contradiction to the ergodicity of μ. □

8.3 Proof of the measure classification theorem.

Let μ be a locally finite Mod(S)-ergodic invariant measure on $\mathcal{ML}(S)$. Recall the definition $\mathcal{G}_1^R(S) = \mathcal{I}_R(\mathcal{ML}(R))$, from Section 3 and let $\mathcal{G}_1^{[R]}(S) = \bigcup_{h \in \mathrm{Mod}(S)} \mathcal{G}^{h \cdot R}(S)$. We let $\mathcal{Y}(S) \subset \mathcal{ML}(S)$ the set of measured laminations with no closed curve in their support, $\mathcal{Y}^R(S) = \mathcal{I}_R(\mathcal{Y}(R))$, and $\mathcal{Y}^{[R]}(S) = \bigcup_{h \in \mathrm{Mod}(S)} \mathcal{Y}^{h \cdot R}(S)$.

We prove by induction simultaneously Theorem 8.1 and the following result:

Proposition 8.5. *Let μ be a locally finite* Mod(S)*-invariant measure on $\mathcal{ML}(S)$. Then for every subsurface $R \subsetneq S$*

$$\mu(\mathcal{Y}^{[R]}(S)) = 0.$$

By the ergodic decomposition, Proposition 8.5 reduces to the case where μ is Mod(S)-ergodic, in which case it follows easily from Theorem 8.1. Our inductive scheme, however, works the opposite way.

We prove the theorem by induction on $N(S) = \dim \mathcal{ML}(S) = 2g(S) - 2 + n(S)$. More generally, if $R = \bigsqcup R_i$ with each R_i a hyperbolic connected components of R, then hyperbolicity means $2g(R_i) - 2 + n(R_i) > 0$ and

$$N(S) = \sum_i N(R_i) = \sum_i 6g(S_i) - 6 + 2n(S_i). \tag{8.5}$$

We note that for any surface S and subsurface R $N(R) < N(S)$. This is an easy consequence of (8.5); an even easier way to see this is to observe that (as piecewise linear spaces) $\mathcal{ML}(R) \cong \mathcal{G}_1^R$, and any $\lambda \in \mathcal{G}_1^R$ satisfies the nontrivial linear equations $i(\lambda, \gamma_i) = 0$ for any γ_i bounding R.

We prove by induction the following two statements:

A_N. Theorem 8.1 holds for all S with $N(S) \leq N$.

B_N. Proposition 8.5 holds for all S with $N(S) \leq N$.

We will show that $A_N \Longrightarrow B_{N+1} \Longrightarrow A_{N+1}$. The base of the induction is a case of $N = 0$, i.e. S is a pair of pants, in which case $\mathcal{ML}(S)$ is null, and both (A_0) and (B_0) are satisfied vacuously.

Lemma 8.6. $A_N \Longrightarrow B_{N+1}$.

Proof. Let μ be a locally finite Mod(S)-invariant and ergodic measure on $\mathcal{ML}(S)$ and let R be a proper subsurface of S such that $\mu(\mathcal{Y}^R(S)) > 0$. The map $\mathcal{I}_R$ is a piecewise linear isomorphism between $\mathcal{ML}(R)$ and $\mathcal{G}_1^R$. We can view Mod(R) as a subgroup of Mod(S); it preserves both $\mathcal{Y}^R(S)$ and $\mathcal{G}_1^R$. The action of Mod(R) on $\mathcal{Y}^R(S)$ commutes with $\mathcal{I}_R$. This allows us to identify $\mu|_{\mathcal{G}_1^R}$ with a locally finite Mod(R)-invariant measure μ_R supported on $\mathcal{Y}(R)$.

Suppose $R = R_1 \sqcup \cdots \sqcup R_k$ is the decomposition of R into connected components. Without loss of generality we may assume that R is minimal with this property (i.e. $\mu(\mathcal{Y}^R(S)) > 0$), and hence μ-almost every λ in $\mathcal{Y}^R(S)$ actually meets each component R_i, or in other words that μ_R is supported on $\prod_i \mathcal{Y}(R_i) \subset \prod_i \mathcal{ML}(R_i)$. Let μ'_R be any $\mathrm{Mod}(R) = \prod_i \mathrm{Mod}(R_i)$-invariant measure appearing in the ergodic decomposition of μ_R. By Lemma 8.4 we can write $\mu'_R = \prod_i \mu_{R_i}$ with each μ_{R_i} a $\mathrm{Mod}(R_i)$-invariant locally finite measure on $\mathcal{ML}(R_i)$ supported on $\mathcal{Y}(R_i)$. Since for every i, $N(R_i) \leq N(R) < N(S)$ we may apply Theorem 8.1. It follows from the measure classification of Theorem 8.1 that the only $\mathrm{Mod}(R_i)$-invariant locally finite measure on $\mathcal{ML}(R_i)$ supported on $\mathcal{Y}(R_i)$ is (a constant multiple of) $\mu_{Th}^{R_i}$ (note that we have added a superscript to denote which surface we are using). It follows that $\mu|_{\mathcal{Y}^R}$ is a constant multiple of Thurston measure on R, or more precisely its push forward $(\mathcal{I}_R)_*(\mu_{Th}^R)$.

Let M_t denote the map $\lambda \mapsto t\lambda$ on $\mathcal{ML}(S)$. Since M_t commutes with the action of the mapping class group, and μ is $\mathrm{Mod}(S)$-ergodic, for every $t \in \mathbb{R}_+$ either $\mu \propto (M_t)_*\mu$ or these measures are mutually singular. On $\mathcal{Y}^R(S)$ the measures μ and $[\mathcal{I}_R]_*(\mu_{Th}^R)$ agree (up to a scalar). We know that $\mu_{Th}^R(tU) = t^{N(R)}\mu_{Th}^R(U)$ for every U, or $(M_t)_*\mu_{Th}^R = t^{-N(R)}\mu_{Th}^R$, so $(M_t)_*\mu = t^{-N(R)}\mu$.

But as we have already noted, $N(R) < N(S)$. Therefore, this behavior of μ under M_t is in contradiction to Proposition 8.2. $\square$

Lemma 8.7. $B_N \Longrightarrow A_N$.

Proof. Let $\mathcal{Z}(S)$ be the set of pairs $(R, \boldsymbol{\alpha})$ where R is a subsurface of S and $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$ is a finite subset of disjoint, essential non peripheral simple closed curves[2] which contains in particular all boundary components of R. If $\gamma = \sum_{i=1}^k c_i\gamma_i$ is a multicurve and $\boldsymbol{\gamma} = \{\gamma_1, \ldots, \gamma_k\}$ then $(R, \boldsymbol{\gamma}) \in \mathcal{Z}(S)$ iff (R, γ) is a complete pair. $\mathrm{Mod}(S)$ acts on $\mathcal{Z}(S)$, and note that $|\mathcal{Z}(S)/\mathrm{Mod}(S)| < \infty$.

There are three cases:

(I) There is no (essential, non-peripheral) closed curve γ so that

$$\mu\{\eta \in \mathcal{ML}(S) : i(\eta, \gamma) = 0\} > 0. \tag{8.6}$$

In this case, since there are only countably many closed curves up to homotopy, μ is supported on the set of filling laminations $\mathcal{G}(S)$, in which case by Theorem 7.1 we conclude that μ is a constant multiple of μ_{Th}.

(II) There is at least one closed curve γ_1 satisfying (8.6), but there is no γ for which

$$\mu\{\eta \in \mathcal{ML}(S) : \gamma \in \operatorname{supp}\eta\} > 0. \tag{8.7}$$

In this case let R be the subsurface obtained by removing γ_1. Then $\mu(\mathcal{Y}^R(S)) > 0$ — in contradiction to B_N.

(III) There is at least one closed curve γ satisfying (8.7).

Let $(R, \boldsymbol{\alpha}) \in \mathcal{Z}(S)$ with $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_k\}$ be such that

[2] As elsewhere in this paper we implicitly identify homotopic curves.

(a) the set $Z^{R,\boldsymbol{\alpha}} = \{\eta \in \mathcal{ML}(S) : \forall i, \alpha_i \in \mathrm{supp}(\eta) \text{ and } \mathrm{supp}(\eta) \subset R \cup \bigcup_i \alpha_i\}$ has positive μ-measure.

(b) k is maximal with this property, and there is no subsurface $R' \subset R$ with $(R', \boldsymbol{\alpha}) \in \mathcal{Z}(S)$ satisfying $\mu(Z^{R',\boldsymbol{\alpha}}) > 0$.

Every $\eta \in Z^{R,\boldsymbol{\alpha}}$ can be written in a unique way as $\sum_i c_i\alpha_i + \lambda$ for some $c_i > 0$ and $\lambda \in \mathcal{G}_1^R$. The map $\eta \mapsto \{c_1, \ldots, c_k\}$ can be extended in a Mod(S)-invariant way to $Z^{[R,\boldsymbol{\alpha}]} = \bigcup_{g\in \mathrm{Mod}(S)} Z^{g\cdot R, g\cdot\boldsymbol{\alpha}}$. Since μ is ergodic, this map is a.e. constant, hence there is some $\alpha = \sum_i c_i\alpha_i$ for which

$$\mu(\mathcal{G}^{R,\alpha}) > 0.$$

The map $\lambda \mapsto \mathcal{I}_R(\lambda) + \alpha$ is a piecewise affine isomorphism between $\mathcal{ML}(R)$ and $\mathcal{G}^{R,\alpha}$ commuting with the action of Mod(R). Let μ^R be the locally finite measure on $\mathcal{ML}(R)$ corresponding to $\mu|_{\mathcal{G}^{R,\alpha}}$. Let $R = \bigsqcup_i R_i$ be the decomposition of R into connected components. Our assumption (b) regarding minimality of R implies that μ^R is supported on $\prod_i \mathcal{ML}(R_i)$. Our assumption that $\boldsymbol{\alpha}$ is a maximal assures us that in fact μ_R is supported on $\prod_i \mathcal{Y}(R_i)$. Arguing as before using Lemma 8.4 and applying B_N on each component separately we get that μ_R is proportional to μ^R_{Th}.

It follows that μ and $\mu_{Th}^{[R,\alpha]}$ agree (up to scalar) on a set of positive measure with respect to both, namely $\mathcal{G}^{R,\alpha}$. Since both measures are ergodic, μ is a constant multiple of $\mu_{Th}^{[R,\alpha]}$. □

Lemma 8.6 and Lemma 8.7 complete the induction, and Theorem 8.1 follows. □

8.4 Classifying orbit closures.

Lemma 7.2 also shows that the mapping class group orbit of any filling measured lamination is dense. This gives rise to the classification of orbit closures of the action of the mapping class group on $\mathcal{ML}(S)$ as in Theorem 1.2.

Theorem 8.8. *If $\lambda \in \mathcal{G}(S)$ then $\overline{\mathrm{Mod}(S)\cdot\lambda} = \mathcal{ML}(S)$.*

Proof. Let U be an open subset of $\mathcal{ML}(S)$. We show that there exists $g \in$ Mod(S) such that $g \cdot \lambda \in U$. First, we choose a small open set $U_0 \subset U$, and $V_0 \subset P\mathcal{ML}(S)$ such that $B_0 = \tau^1(U_0 \times V_0)$ is a box in $\mathcal{Q}^1\mathcal{M}(S; 1, \ldots, 1)$ as in §7.

Let $q \in \mathcal{Q}^1\mathcal{T}(S)$ be such that $\lambda = \mathrm{Re}(q)$. It is enough to show that $\pi(\mathcal{F}^-(q))$ meets $\pi(B_0)$; the argument shows that $\pi(\mathcal{F}^-(q)$ is dense in $\mathcal{Q}^1\mathcal{M}(S)$. By Theorem 6.6, one can find a sequence $\{q_i\}$ and a box B in $\mathcal{Q}^1\mathcal{M}(S)$ such that $g^n(q_n) \in B$. Then we can use Lemma 7.2 for the measure ν_n supported on $\mathcal{F}^-(g^n(q_n)) \cap B$, and a non negative continuous function f supported on B_0. Therefore since B_0 has positive measure with respect to μ_S,

$$a_n(f, B) = |\int_{g^{-n}B} f\, d(g^{-n})_*\nu_n|$$

is bounded away from 0 as $n \to \infty$. On the other hand, if $a_n(f, B) > 0$, then $\pi(\mathcal{F}^-(q)) \cap B_0 \neq \emptyset$. As a result there exists $g \in \mathrm{Mod}(S)$ and $q_0 \in \mathcal{F}^-(q)$ such that $g \cdot q \in B_0$ which means that $\mathrm{Mod}(S) \cdot \lambda \cap U_0 \neq \emptyset$. □

Given a measured lamination λ, we can write $\lambda = \gamma + \sum_{i=1}^{k} \eta_i$ where γ is a multicurve and η_i's are minimal components of λ without simple closed curves in their support. Define

$$\mathcal{R}_\lambda = (R, \gamma),$$

where $R = \bigcup_{i=1}^{k} R_i$ is the union of connected components of $S(\gamma)$ containing $\eta_1, \ldots \eta_k$.

Then in terms of the notation used in §3:

Theorem 8.9. *For $\lambda \in \mathcal{ML}(S)$ we have*

$$\overline{\mathrm{Mod}(S) \cdot \lambda} = \mathcal{G}^{[\mathcal{R}_\lambda]}(S).$$

Proof. Let $C(\lambda) = \overline{\mathrm{Mod}(S) \cdot \lambda} \subset \mathcal{ML}(S)$. Assume that $\lambda \in \mathcal{ML}(S)$ does not contain any simple closed curves in its support. We show that $C(\lambda) = \mathcal{ML}(S)$. Note that in this case, λ is filling in a subsurface $R \subset S$. Hence $\lambda = \mathcal{I}_R(\lambda_0) \in \mathcal{G}_1^R(S)$ (see §3), where $\lambda_0 \in \mathcal{ML}(R)$ is filling. Therefore by Theorem 8.8, $C(\lambda_0) = \mathcal{ML}(R)$. As a result for any $t \in \mathbb{R}_+$, $t \cdot \lambda \in C(\lambda)$, and $t \cdot C(\lambda) = C(\lambda)$. On the other hand every $\mathrm{Mod}(S)$ orbit in $P\mathcal{ML}(S)$ is dense. Hence $C(\lambda) = \mathcal{ML}(S)$. This means that if λ does not contain any simple closed curves in its support then it has a dense orbit in $\mathcal{ML}(S)$.

In general for $\lambda = \gamma + \sum\limits_{i=1}^{k} \eta_i$, by the definition $\eta_i \subset R_i$ does not contain any simple closed curves. Also $R_\lambda = (R, \gamma)$ is a complete pair. So by the same argument used in the proof earlier $\mathcal{G}^{[(R,\gamma)]} \subset C(\lambda)$. Since $\mathcal{G}^{[(R,\gamma)]}$ is closed we get $C(\lambda) = \mathcal{G}^{[(R,\gamma)]}$. □

A Appendix: Quantative nondivergence for quadratic differentials

Let S be a surface of genus g and with $n \geq 0$ punctures (we assume that $g \geq 2$ or more generally that S is of hyperbolic type, i.e can be given a hyperbolic metric).

In this appendix we prove the following:

Theorem A.1. *Given $\epsilon_0 > 0$, there exists a compact set $K \subset \mathcal{Q}^1\mathcal{M}(S)$ such that for any $q \in \widetilde{\mathcal{G}}(S)$,*

$$\liminf_{T\to\infty} \mathrm{Ave}_{T,q}(K) \geq 1 - \epsilon_0. \tag{A.1}$$

This theorem is closely related to the results obtained by Minsky and Weiss in [MW], and our proof is based on theirs (mainly because of personal preferences and to make the writing of this appendix more interesting for us we have used variants of their argument at several places). The proofs of Minsky and Weiss in turn rely on ideas from other works, specifically [V2, KMS, KM].

In [MW, Thm. H2] Minsky and Weiss prove that there is a compact subset $K \subset \mathcal{Q}^1\mathcal{M}$ so that (A.1) holds for every $q \in \mathcal{Q}^1\mathcal{M}$ which does not have an imaginary saddle connection. This is more restrictive than our assumption $q \in \tilde{\mathcal{G}}(S)$ — i.e. that there is no closed loop of imaginary saddle connections for q, nor is there a path consisting of imaginary saddle connections connecting two punctures.

As in [MW] for simplicity, we reduce first to the case that there are no punctures (this is not strictly essential, but makes the sequel slightly easier to write).

A.1 Reduction to the case of $n = 0$

To pass from surfaces S with n punctures to a surface without punctures we simply take an appropriate branched cover $\beta : \tilde{S} \to S$ as per the following well-known construction (cf. [MW, Lem. 4.9]):

Let $\hat{S}$ denote the compact surface obtained by plugging the punctures in S. If S has an even number of punctures we divide these punctures into pairs, connecting the pairs by disjoint segments, cutting along the segments and gluing three copies to obtain a three-fold branched cover $\beta : \tilde{S} \to \hat{S}$ which has degree 3 over each puncture of S (and unramified everywhere else). This induces a map $\beta^* : \mathcal{Q}^1\mathcal{M}(S) \to \mathcal{Q}^1\mathcal{M}(\tilde{S})$ (note that we use β^* to denote the *normalized* pullback map). If $q \in \mathcal{Q}^1\mathcal{M}(S)$ and $p \in \hat{S}$ a puncture then the total angle in the locally Euclidean structure corresponding to q around the puncture p is π, and so the total angle in the locally Euclidean structure corresponding to $\beta^* q$ around the unique preimage $\beta^{-1}(p)$ is 3π — so $\beta^* q$ is a holomorphic quadratic differential with a simple zero at the preimage of every puncture of S.

If S has an odd number of punctures, we first take some double cover of S, then apply the previous construction. The map β^* commutes with the u_t-flow on $\mathcal{Q}^1\mathcal{M}(S)$ and $\mathcal{Q}^1\mathcal{M}(\tilde{S})$, and we only need to verify the relation between both the assumption and the conclusion of Theorem A.1 for S and $\tilde{S}$.

We now introduce some terminology which will also be used in the next subsection. We first introduce some terminology:

A *saddle connection complex* is a collection of saddle connections with disjoint segments (note that the saddle connections in a saddle connection complex are allowed to have common endpoints). Let $\mathcal{E}$ denote the set of all saddle connection complexes on S. The number M of saddle connections that can be put together in a saddle connection complex is bounded above in terms of g and n: specifically, $M \leq 3(6g-6+2n)$.

A special kind of saddle connection complex is a *saddle connection loop* which is a sequence of saddle connections which together from a simple closed polygonal curve on S. Recall that a simple closed curve on S is *essential* if it is not homotopic to a point. For any saddle connection complex E and $q \in \mathcal{Q}^1\mathcal{M}$ we let $\ell(q,E) = \max_{\delta\in E}\ell(q,\delta)$.

For any $\epsilon > 0$ let $K(\epsilon, S) \subset \mathcal{Q}^1\mathcal{M}(S)$ be the set of $q \in \mathcal{Q}^1\mathcal{M}(S)$ so that for every E which is either (i) an essential simple loop of saddle connections or (ii) a path E of saddle connections connecting two punctures, we have that $\ell(q,E) \geq \epsilon$. These are compact subsets of $\mathcal{Q}^1\mathcal{M}(S)$, and every compact subset $K \subset \mathcal{Q}^1\mathcal{M}(S)$ is contained in some $K(\epsilon, S)$.

Clearly, to show that the conclusion (A.1) of Theorem A.1 for $\tilde{S}$ and β^*q implies it for S and q it is enough to show that the map β^* is compact, i.e. $(\beta^*)^-1(K)$ is compact for every compact K. In fact we have:

Lemma A.2. *Let S be a surface with $n > 0$ punctures and $\beta : \tilde{S} \to \hat{S}$ a branched cover as above. Then for any ϵ*

$$\beta_*^{-1}(K(\epsilon,\tilde{S})) \subset K(\epsilon, S)$$

The lemma is essentially obvious, the only observation needed is that if E is a path of saddle connections for $q \in \mathcal{Q}^1\mathcal{M}(S)$ connecting two punctures of S then $\beta^{-1}(E)$ contains an essential saddle connection loop for $\beta^*(q)$, and that if δ is a saddle connections for q then for any saddle connection $\delta' \in \beta^{-1}(\delta)$ we have that $\ell(\beta^*q,\delta') = \deg(\beta)^{-1/2}\ell(q,\delta)$ (the factor $\deg(\beta)$ arising from the normalization according to area).

We also need to observe the following:

Lemma A.3. *If $q \in \tilde{\mathcal{G}}(S)$ then $\beta^*(q) \in \tilde{\mathcal{G}}(\tilde{S})$.*

Again this is almost obvious. For suppose $\beta^*(q) \notin \tilde{\mathcal{G}}(\tilde{S})$. Then there is some essential loop in $\tilde{S}$ consisting of imaginary saddle connections for $\beta^*(q)$ (since $\tilde{S}$ has no punctures). Then the image $\beta(E)$ of E in S consists of imaginary saddle connections for q, and one only needs to observe that since these saddle connections are all parallel (and change direction only at punctures) this collection of saddle connections contains either a simple essential loop or a path connecting two punctures.

A.2 Proof of Theorems A.1 for surfaces with no punctures

Let us now assume that S is a surface of genus g with no punctures.

We deduce Theorem A.1 from the following more quantitative result (which of course also holds with on surfaces with punctures up to the obvious modification of considering paths between two punctures in addition to loops):

Theorem A.4. *There is a ρ_0 depending on g so that the following holds. Let $q \in \mathcal{Q}^1\mathcal{M}$, $T > 0$, and $\rho \le \rho_0$ satisfy that*

$$\max_{0 \le t \le T} \ell(u_t.q, E) \ge \rho \qquad \text{for every essential saddle connection loop } E. \tag{A.2}$$

Then for every $\epsilon < \rho$,

$$|\{t \in [0,T] : u_t.q \notin K(\epsilon, S)\}| < C \left(\frac{\epsilon}{\rho}\right)^{\alpha} T. \tag{A.3}$$

Proof of Theorem A.1 assuming Theorem A.4. Let $q \in \tilde{\mathcal{G}}(S)$. There are only finitely many essential saddle connection loops $E_1, \ldots, E_l$ with $\ell(q, E_i) < \rho_0$, and by assumption each one of them contains at least one saddle connection which is not imaginary. Therefore if we choose T_0 large enough $\ell(u_{T_0}q, E_i) > \rho_0$ for all i, and (A.2) is satisfied for every $T \ge T_0$ and $\rho = \rho_0$.

Let ϵ be such that $C(\epsilon/\rho)^\alpha < \epsilon_0$. Let $K = K(\epsilon, S)$. Then (A.3) implies that $\mathrm{Ave}_{T,q}(K) \ge 1 - \epsilon_0$ for every $T \ge T_0$. □

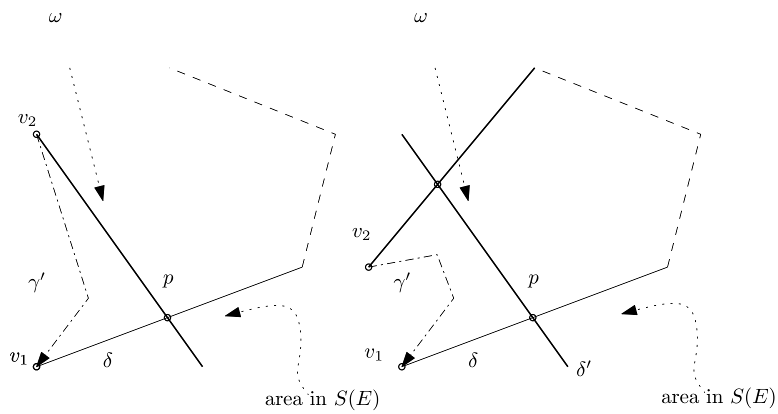
Since $g > 0$, for any nonessential saddle connection loop E, there is precisely one simply connected component of $S \backslash E$, which we will call the interior of E. For any saddle connection complex E we set $S(E) \subset S$ to be the union of the points of all saddle connections $\delta \in E$ as well as the interior of all nonessential saddle connection loops contained in E.[3] This definition has the nice property that if E_1, E_2 are two saddle connection complexes and $E_2 \subseteq S(E_1)$ then $S(E_2) \subseteq S(E_1)$.

The proof of Theorem A.4 hinges on the following two basic geometric lemmas which are pretty simple to prove (see [KMS, §3], [V2] or [MW])

Proposition A.5 (Cf. [MW, Prop. 6.1]). *There exists ρ_0 (depending only on n, g) such that for every $q \in \mathcal{Q}^1\mathcal{M}$ and every $E \in \mathcal{E}$ with $\ell(q, E) < \rho_0$, we have that $S(E) \subsetneq S$.*

Sketch of proof. The number of saddle connections in E is bounded above by a function $M = M(g)$ of g, and hence the number of simple saddle connection loops in E is also bounded from above by a function of g, say $F(g)$. The area of the interior of a nonessential saddle connection loop of total length L is bounded from above by CL^2, and hence if $\ell(q, E) < \rho_0$ the total area of $S(E)$ (in the

[3] This is slightly bigger than the set $S(E)$ defined in [MW], where $S(E)$ was defined as the union of all simply connected connected components of $S \setminus \bigcup_{\delta \in E} \delta$ and presumably all the saddle connections in E. Correspondingly, the boundary $\partial S(E)$ we get is a subset of the boundary as defined in [MW].



Case 1: ω ends in a saddle. Case 2: ω ends on $\partial S(E)$

Figure 1: δ, δ' and $\partial S(E)$

flat metric corresponding to q) is at most $CF(g)M(g)^2\rho_0^2$. Since $q \in \mathcal{Q}^1\mathcal{M}$, i.e. the area of S in the flat metric corresponding to q is one, we have that if ρ_0 is sufficiently small (depending only on g), $S(E) \neq S$. □

The following is a slight variant on [MW, Lem. 6.2] (the proof given by Minsky and Weiss yields this statement without any modification).

Lemma A.6 (Cf. [MW, (*) on p. 30]). *Let $q \in \mathcal{Q}^1\mathcal{M}$ and E a complex of saddle connections. Let δ be a saddle connection on $\partial S(E)$. Assume that $\ell(q,E) < \theta/3$ and that for every $E' \supsetneq E$ as $\ell(q,E') \geq \theta$. Then for any saddle connection $\delta' \neq \delta$ properly intersecting δ it holds that $\ell(q,\delta') \geq 2\theta/3$.*

The proof is by an elegant but elementary argument (cf. also [KMS], [V2])

Sketch of proof. first one shows that any component Ω of $S \setminus E$ whose boundary $\partial\Omega$ contains at least one saddle where the internal angle is $< \pi$ is contained in $S(E)$.

Now assume that $\delta \in \partial S(E)$, and $\delta' \not\subset S(E)$ intersects it with $\ell(q,\delta') < 2\theta/3$. Let p be a point in $\delta \cap \theta'$, v_1 be the endpoint of δ closer to p, and ω a maximal segment in $\delta' \cap \overline{S \setminus S(E)}$ one of whose endpoint is p (see Figure 1. There are two cases: either the other endpoint of ω is a saddle or it is on a saddle collection in $\partial S(E)$; in either case we get a polygonal path γ connecting v_1 to a saddle v_2 containing ω of total length $< \theta$. Let γ' be the length minimizing path in the homotopy class of γ rel. v_1, v_2 in $\overline{S \setminus S(E)}$. Then γ' is a chain of saddle connections, each of which has length at most θ. It follows from the conditions on E in the lemma that all the saddle connections composing γ' have to be in $\partial S(E)$. But then γ and γ' bound a simply connected domain Ω in $S \setminus S(E)$, and the internal angle at every vertex of the boundary $\partial\Omega$ except possibly the endpoints of ω is $> \pi$ — which is in contradiction to the Gauss-Bonnet theorem (as this domain is locally Euclidean with possibly some exceptional points inside where the angle is $\geq 3\pi$). $\square$

We add to these two results another simple observation:

Lemma A.7. *Let E be a saddle connection complex with $S(E) \subsetneq S$. Suppose that $\partial S(E)$ contains no essential saddle connection loop. Then $S(E)$ contains no essential saddle connection loop.*

Proof. $\partial \overline{\operatorname{int} S(E)}$ is a finite union of saddle connection loops, which by assumption are nonessential. It follows that $\operatorname{int} S(E)$ is a finite union of disjoint contractible sets. Let V be one of these open components. Any path connecting two points in ∂V is homotopic in $\overline{V}$ to a path in ∂V. It follows that any saddle connection loop contained in $S(E)$ is homotopic to a saddle connection loop contained in $\partial S(E)$, hence nonessential. $\square$

Let I be an interval. A function $f : I \to \mathbb{R}$ is said to be *of order α* (with constant C) if for every sub interval $I' \subset I$ and any $\epsilon > 0$

$$|\{t \in I' : |f(t)| < \epsilon\}| < C(\epsilon / \|f\|_I)^\alpha \, |I'|$$

where $\|f\|_I = \sup_{t \in I} |f(t)|$. (following [KM], Minsky and Weiss call such functions (C, α)-good in [MW][4]). For any saddle connection δ, quadratic differential $q \in \mathcal{Q}^1\mathcal{M}$ and interval I, the function $t \mapsto \ell(u_t.q, \delta)$ is of order 1 (the constant being in this case 2).

Two saddle connection complexes $E_1, E_2 \in \mathcal{E}$ are said to be *weakly compatible* if $E_1 \cup E_2$ is a saddle connection complex. They are *strongly compatible* if in addition $E_1 \cap E_2 = \emptyset$. A saddle connection δ is compatible with a saddle connection compact E if $\{\delta\}$ and E are strongly compatible.

Slightly modifying the formalism of [KM] to fit our particular needs, we will use the following definitions:

Definition A.8. *We say that $q \in \mathcal{Q}^1\mathcal{M}$ is ϵ, ρ-*marked for loops *if there is some $E_1 \in \mathcal{E}$ so that*

1. *$\ell(q, \delta) \leq \rho$ for every $\delta \in E_1$*
2. *$\ell(u_t.q, \delta) \geq 3\rho$ for every saddle connection δ compatible with E_1*
3. *$\ell(q, E) \geq \epsilon$ for every saddle connection loop $E \subset E_1$*

*For any $E_0 \in \mathcal{E}$, we say that $q \in \mathcal{Q}^1\mathcal{M}$ is ϵ, ρ-*marked relative to E_0 *if there is a saddle connection complex $E_1 \subset \mathcal{E}$ containing E_0 satisfying 1 and 2 above as well as*

$3'$. *$\ell(q, \delta) \geq \epsilon$ for every $\delta \in E_1 \setminus E_0$*

[4] Paraphrasing Hamlet, no valid mathematical notion is either good or bad, though thinking may make it so!

Recall that we denote by M to be the maximal cardinality of a saddle connection complex.

Proposition A.9. *Let $q \in \mathcal{Q}^1\mathcal{M}$, $T > 0$ and $\rho < 3^{-M}\rho_0$ satisfy (A.2). For any $\epsilon < \rho$ let*

$$A^{\text{loop}}_{\epsilon,\rho,T} = \left\{t \in [0,T] : u_t.q \text{ is } \epsilon, \rho'\text{-marked for loops for some } \rho \leq \rho' \leq 3^M \rho\right\}$$

then we have that

$$|[0,T] \setminus A_{\epsilon,\rho,T}| < C(\epsilon/\rho)T.$$

By taking E_0 to be a maximal saddle connection complex with the property that for every $\delta \in E_0$ and every $t \in [0,T]$ we have that $\ell(u_t.q,\delta) \leq \rho$ (which by (A.2) does not contain any short saddle connection loop), Proposition A.9 is an immediate corollary of the following lemma, which we prove by induction.

Lemma A.10. *Let $q \in \mathcal{Q}^1\mathcal{M}$, $I_0 \subset \mathbb{R}$ an interval, $\rho > 0$ and $E_0 \in \mathcal{E}$ be given so that*

$$\max_{t\in I_0} \ell(u_t.q,\delta) \geq \rho \qquad \forall \text{ saddle connection } \delta \text{ compatible with } E_0 \tag{A.4}$$

$$\max_{t\in I_0} \ell(u_t.q,\delta) \leq \rho \qquad \forall \text{ saddle connection } \delta \in E_0. \tag{A.5}$$

Let $k = |E_0|$. For any $\epsilon < \rho$ let

$$A^{E_0}_{\epsilon,\rho,I_0} = \left\{t \in I_0 : u_t.q \textit{ is } \epsilon, \rho'\textit{-marked rel. } E_0 \textit{ for some } \rho \leq \rho' \leq 3^{M-k}\rho\right\}$$

then we have that

$$\left|I_0 \setminus A^{E_0}_{\epsilon,\rho,I_0}\right| < C_k(\epsilon/\rho)\,|I_0|\,.$$

Proof. The proof is by induction on $M - k$. There are three cases:
Case I. There is no saddle connection δ compatible with E_0 for which

$$\min_{t\in I_0} \ell(u_t.q,\delta) < 3\rho.$$

Note that this case in particular includes the base of the induction, i.e. when $M = k$ and there are no saddle connections of any length of compatible with E_0. In this case by definition for every $t \in I_0$ the quadratic differential $u_t.q$ is ϵ,ρ-marked relative to E_0 (for any $\epsilon > 0$).
Case II. There is a saddle connection δ compatible with E_0 for which

$$\max_{t\in I_0} \ell(u_t.q,\delta) \leq 3\rho.$$

Since that can only be finitely many saddle connections with this property we may choose δ to be the one for which $\max_{t\in I_0} \ell(u_t.q,\delta)$ is minimal. In that case $E_0' = E_0 \cup \{\delta\}$, $I_0' = I_0$ and $\rho' = \max_{t\in I_0} \ell(u_t.q,\delta)$ satisfy the conditions of the lemma with $|E_0'| = k+1$, and hence we know that

$$\left|I_0 \setminus A^{E_0'}_{\epsilon,\rho',I_0}\right| < C_{k+1}(\epsilon/\rho')\,|I_0|\,.$$

But if $t \in A^{E_0'}_{\epsilon,\rho',I_0}$, i.e. there is some saddle connection complex $E_1 \ni E_0 \cup \{\delta\}$ and $\rho'' \in [\rho', 3^{M-k-1}\rho']$ so that $u_t.q$ is (ϵ, ρ'')-marked by E_1 relative to $E_0' = E_0 \cup \{\delta\}$. By definition, $\rho' \in [\rho, 3\rho)$ so $\rho'' \in [\rho, 3^{M-k}\rho]$. Furthermore, if $\ell(u_t.q) \geq \epsilon$ then $u_t.q$ is also (ϵ, ρ'')-marked by E_1 relative to E_0, hence

$$A^{E_0}_{\epsilon,\rho,I_0} \supset A^{E_0'}_{\epsilon,\rho',I_0} \setminus \{t \in I_0 : \ell(u_t.q, \delta) < \epsilon\}\,.$$

Since the function $t \mapsto \ell(u_t.q, \delta)$ is of order one (with the constant being 2) we have that

$$|\{t \in I_0 : \ell(u_t.q, \delta) < \epsilon\}| \leq 2(\epsilon/\rho')\,|I_0|$$

hence (since $\rho \leq \rho'$)

$$\left| I_0 \setminus A^{E_0}_{\epsilon,\rho,I_0} \right| \leq (C_{k+1} + 2)(\epsilon/\rho)\,|I_0|\,.$$

Case III. For every saddle connection δ compatible with E_0, $\max_{t\in I} \ell(u_t.q, \delta) \geq 3\rho$ but there is some δ compatible with E_0 for which $\min_{t\in I} \ell(u_t.q, \delta) < 3\rho$.

In this case we cover the set

$$J = \{t \in I_0 : \exists \delta \text{ compatible with } E_0 \text{ s.t. } \ell(u_t.q, \delta) < 3\rho\}$$

by finitely many intervals $\{I_j\}_{j=1}^m$ such that

1. for every j there is some saddle connection δ_j compatible with E_0 such that
$$I_j = \{t \in I_0 : \ell(u_t.q, \delta_j) < 3\rho\}\,;$$
2. each I_j is maximal in the sense that there is no other saddle connection δ' compatible with E_0 such that
$$I_j \subsetneq \{t \in I_0 : \ell(u_t.q, \delta') < 3\rho\}\,;$$
3. $J \subset \bigcup_j I_j$
4. for every j we have that $I_j \not\subset \bigcup_{r\neq j} I_r$.

Such a covering always exists, and moreover has multiplicity at most 2 (i.e. the intersection over every 3 distinct members of $\{I_j\}$ is empty) — indeed, chose I_1 to be the largest subinterval of I_0 of the form given in 1., I_2 the largest such subinterval not contained in I_1, I_3 the largest not contained in $I_1 \cup I_2$ etc. As there are only finitely many saddle connections for which $\min_{t\in I_0} \ell(u_t.q, \delta) < 3\rho$ this process will terminate after finitely many steps and posibly after eliminating some intervals we get a collection satisfying 1–4 above.

One each such interval I_j apply this lemma with $\rho' = 3\rho$, $E_j = E_0 \cup \{\delta_i\}$ (since $|E_j| = k+1$ this is permissible, while 1. and 2. above imply that this choice of parameters does indeed satisfy the conditions of this lemma). Clearly $J \subset A^{E_0}_{\epsilon,\rho,I_0}$ and for every j,

$$A^{E_j}_{\epsilon,3\rho,I_j} \cap \{t \in I_j : \ell(u_t.q, \delta_j) \geq \epsilon\} \subset A^{E_0}_{\epsilon,\rho,I_0}$$

hence

$$\left|I_0 \setminus A^{E_0}_{\epsilon,\rho,I_0}\right| \leq \sum_j \left|I_j \setminus A^{E_j}_{\epsilon,3\rho,I_j}\right| + \sum_j |\{t \in I_j : \ell(u_t.q, \delta_j) \geq \epsilon\}|$$
$$\leq \sum_j (2 + C_{k+1})(\epsilon/3\rho)\,|I_j| \leq (2 + C_{k+1})(\epsilon/\rho)\,|I_0|\,.$$

This completes the inductive proof.

□

Lemma A.11. *Suppose that $t_0 \in [0, T]$ is ϵ_i, ρ_i-marked for loops by E_i for $i = 1, 2$, with $\rho_2 < \epsilon_1 < \rho_1 < \rho_0$. Then $S(E_2) \subseteq S(E_1)$. If $S(E_2) = S(E_1)$ then $S(E_1)$ contains no essential loops.*

Proof. By definition of ϵ_1, ρ_1-marked for loops, E_1 and $u_t.q$ satisfy the conditions of Lemma A.6 with $\theta = 3\rho_1$. Therefore any saddle connection δ intersecting $\partial S(e_1)$ satisfies $\ell(u_t.q, \delta) \geq \ell(u_t.q, E_1) > \rho_1$: so it cannot be in E_2. Similarly, since by definition of ϵ_1, ρ_1-marked, any δ with $\ell(u_t.q, \delta) \leq \ell(u_t.q, E_1)$ has to intersect some saddle connection of E_1. We conclude that every $\delta \in E_2$ is in $S(E_1)$ and therefore $S(E_2) \subset S(E_1)$.

Assume that $S(E_1)$ contains an essential saddle connection loop. Since (for appropriate choice of ρ_0) Proposition A.5 implies that $S(E_1) \subsetneq S$, by Lemma A.7 we have that $\partial S(E_1)$ contains an essential saddle connection loop, say E_1'.

Since E_1 gives an ϵ_1, ρ_1-marking for loops, one of the saddle connections in E_1', say δ' has $\ell(u_t.q, \delta') > \epsilon_1$. Since $\epsilon_1 > \rho_2$ this saddle connection $\delta' \in \partial S(E_1)$ is not in E_2 and it follows that $S(E_2) \subsetneq S(E_1)$. □

Proof of Theorem A.4. Suppose that $q \in \mathcal{Q}^1\mathcal{M}$, $T > 0$ and $\epsilon < \rho < \rho_0$ satisfy (A.2), with ρ_0 as in Proposition A.5. By appropriate choice of constant in (A.3), this equation can be made to be clearly true for $\epsilon \geq 3^{-M(M+1)}\rho$, so we may as well assume that $\epsilon < 3^{-M(M+1)}\rho$.

Let $\eta = (\epsilon/\rho)^{1/M+1}$, M being as before the maximal cardinality of a saddle connection complex on S.

For every $k = 0, \ldots, M$ let $\rho_{(k)} = 3^{-M}\eta^k\rho$ and $\epsilon_{(k)} = \eta^{k+1}\rho$. We apply Proposition A.9 for each pair $\epsilon_{(k)}, \rho_{(k)}$ and deduce that there is a set $A^{\text{loop}}_{\text{total}} \subset [0, T]$ with $\left|[0, T] \setminus A^{\text{loop}}_{\text{total}}\right| < C'\eta T$ so that for every $t \in A^{\text{loop}}_{\text{total}}$, there is a $\epsilon_{(k)}, \rho'_{(k)}$-marking of $u_t.q$ by a saddle connection complex $E^{(k)}_t$, with $\rho_{(k)} \leq \rho'_{(k)} \leq 3^M\rho_{(k)}$. Note that for any $k \geq 1$, we have $3^M\rho_{(k)} < \epsilon_{k-1}$.

By Lemma A.11 and Proposition A.5,

$$S \supset S(E^{(0)}_t) \supset S(E^{(1)}_t) \supset \cdots \supset S(E^{(M)}_t).$$

Since there can be at most M compatible saddle connections on S at least one of these inclusions must actually be equality, and hence (again invoking

Lemma A.11) for some k_0, $S(E_t^{(k_0)})$ does not contain any essential saddle connection loops.

Suppose now that E is an essential saddle connection loop with $\ell(u_t.q, E) < \epsilon$. Lemma A.6 implies that for any $\delta \in E$ intersecting $S(E_t^{(k_0)})$ $\ell(u_t, \delta) \geq \rho_{(k_0)}$; the maximallity of $S(E_t^{(k_0)})$ implies a similar inequality if δ is compatible with $E_t^{(k_0)})$, and so the only possibility is $E \subset S(E_t^{(k_0)})$. But this is a contradiction because $S(E^{(k)})$ contains no essential saddle connection loops.

It follows that for every $t \in A_{\text{total}}^{\text{loop}}$ there is no no essential saddle connection loop E with $\ell(u_t.q, E) < \epsilon$. □

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