ON SETS INVARIANT UNDER THE ACTION OF THE DIAGONAL GROUP

ELON LINDENSTRAUSS AND BARAK WEISS

1. Introduction

Starting with Hillel Furstenberg's seminal paper [F], it has been noted that many natural actions of $\mathbb{R}^k$, $\mathbb{Z}^k$ or the semigroup $\mathbb{Z}^k_+$ for $k \geq 2$ (say on a space X) display remarkable rigidity properties. In particular, in stark contrast to hyperbolic actions of $\mathbb{Z}$ or $\mathbb{R}$, hyperbolic-like actions of these k-dimensional groups display a scarcity both of closed invariant subsets of X and of invariant measures on X.

In this paper we consider the action of A, the positive diagonal matrices in $G = \mathrm{SL}(n, \mathbb{R})$ on G/Γ, where Γ is a lattice in G. This action is defined by $a(g\Gamma) = ag\Gamma$. For $n = 2$, i.e., when $\dim A = 1$, it is known that this action has many irregular invariant subsets — in fact for any $\alpha \in [1, 3]$ there is a point $x \in G/\Gamma$ such that $\overline{Ax}$ has Hausdorff dimension α (this is an unpublished result of Furstenberg and Benjamin Weiss).

For $n \geq 3$ it was conjectured by Gregory A. Margulis (see [M1], p. 203) that typically an orbit-closure for A is an orbit of a group containing A, i.e.,

$$\overline{Ax} = Hx$$

for some Lie subgroup $H < G$. Some care must be taken in this conjecture to rule out some irregular A-orbits arising from rank-1 actions which appear as factors. For a recent formulation of Margulis' Conjecture see [M3, Conjecture 1.1]. Related positive results restricting the A-invariant measures on G/Γ were established by Anatole Katok and Ralf Spatzier in [KS] (see [KS] also for more on conjectures regarding invariant measures).

In this paper we consider the rather special case of A-orbits whose closures contain compact orbits. For general lattices, we prove the following theorem:

Theorem 1.1. *Let $G = \mathrm{SL}(n, \mathbb{R}), n \geq 3$, let A, Γ be as above and let $y \in G/\Gamma$. Assume that $F = \overline{Ay}$ contains a point x satisfying:*

1. *Ax is compact;*

2. *For every* $1 \le i < j \le n$, $N_{ij} = \{\mathrm{diag}(a_1, \dots, a_n) \in A : a_i = a_j\}$ *acts ergodically on* Ax *(since* N_{ij} *is of co-dimension* 1 *in* A*, this simply means that* $N_{ij}x$ *is not compact).*

Then there exists a reductive subgroup H*, containing* A*, such that* $F = Hy$ *and* F *carries an* H*-invariant probability measure.*

The proof we present for Theorem 1.1 is valid not only for $\mathrm{SL}(n, \mathbb{R})$ but also for products of $\mathrm{SL}(n_i, \mathbb{R})$. To simplify the notation we restrict ourselves to $\mathrm{SL}(n, \mathbb{R})$. Interestingly, it does not work when G is any other noncompact simple Lie group (see Remark following the proof of Lemma 4.2); this might be seen as an indication of the subtlety of the questions considered here.

A result similar to Theorem 1.1 was obtained in the S-arithmetic context by Shahar Mozes in [Mo] for the case $G = \mathrm{PGL}(\mathbb{Q}_p) \times \mathrm{PGL}(\mathbb{Q}_\ell)$ and Γ an irreducible lattice.

In unpublished work, Mary Rees has given an example of a co-compact lattice in $\mathrm{SL}(3, \mathbb{R})$ for which the assumption 2 is not satisfied, and irregular orbits for the action of A do arise. Using her example as a model we prove:

Theorem 1.2. *Let* G, A, Γ *be as above. If* $x \in G/\Gamma$ *is such that* Ax *is compact but there are some* $i < j$ *such that* N_{ij} *does not act ergodically on* Ax *then there is* $y \in G/\Gamma$ *with* $Ax \subset \overline{Ay}$ *but* $\overline{Ay}$ *is not of the form* Hy *for any Lie subgroup of* G.

For the case $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, i.e., for the action on the space of lattices in $\mathbb{R}^n$, we are able to weaken our assumptions and strengthen our conclusions.

Theorem 1.3. *Let* G *and* A *be as in Theorem 1.1. Let* $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, $y \in G/\Gamma$, $F = \overline{Ay}$. *Assume that* F *contains a compact* A*-orbit. Then there are integers* k *and* d *with* $n = kd$ *and a permutation matrix* P *such that* $F = Hy$, *where*

$$H = \left\{ P \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & B_d \end{pmatrix} P^{-1} : B_i \in \mathrm{GL}(k, \mathbb{R}) \right\} \cap G,$$

(here the 0*'s stand for* 0 *matrices in* $M_k(\mathbb{R})$*). Moreover, if* $F \neq Ay$ *then* F *is not compact.*

Corollary 1.4. *Assume in addition to the hypotheses of Theorem 1.3 that* n *is prime. Then* Ay *is either compact or dense.*

When n is not a prime, there are orbits whose closure contain a compact orbit that are neither dense nor closed. We give an explicit description of all such possible orbit-closures. These orbit-closures correspond to pairs of number fields K and K' where $K' < K$ and $[K\colon \mathbb{Q}] = n$.

From Corollary 1.4 we draw the following strengthening of an isolation theorem due to Cassels and Swinnerton-Dyer (see [CaS-D], Theorem 2 and also [M2], §2.1). Recall that $\{v_1, \dots, v_m\} \subset \mathbb{Z}^n$, $m < n$ is said to be *primitive* if it can be completed to a set of n generators of $\mathbb{Z}^n$.

Corollary 1.5. *Let $n \geq 3$ be prime, and let $f = L_1 \cdot \ldots \cdot L_n$ be a product of n independent linear forms on $\mathbb{R}^n$. Assume that the coefficients of f are integers, and that f does not represent 0 nontrivially over $\mathbb{Q}$, that is*

$$f(q) \neq 0 \qquad \text{for all nonzero } q \in \mathbb{Q}^n.$$

Then for any open $V \subset \mathbb{R}^{n-1}$ there is a neighborhood U of f (in the space of products of n linear forms) such that for every $h \in U$ which is not a multiple of f there exists a primitive set $v_1, \dots, v_{n-1} \in \mathbb{Z}^n$ such that

$$(h(v_1), \dots, h(v_{n-1})) \in V.$$

The argument deducing Corollary 1.5 from Corollary 1.4 follows along the lines of an argument given by Armand Borel and Gopal Prasad in [BoPr].

There is a number–theoretic motivation for the isolation theorem proved by Cassels and Swinnerton-Dyer, namely the following elementary conjecture due to Littlewood:

Conjecture 1 (Littlewood)**.** *For every $\alpha, \beta \in \mathbb{R}$, we have*

$$\liminf_{n \to \infty} n\{n\alpha\}\{n\beta\} = 0$$

(where $\{x\}$ is the distance from x to the nearest integer).

Using their isolation theorem, Cassels and Swinnerton-Dyer (albeit in a different 'language', see [M2] §2.1), showed that Littlewood's Conjecture follows from the statement that any compact A-invariant subset of $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$ contains a compact orbit.

Finally, we wish to mention some of the progress made on problems related to rigidity of hyperbolic actions of higher-rank abelian groups. In [KS], Katok and Spatzier addressed, inter alia, the problem of describing the ergodic invariant measures for the action of A on G/Γ. Their results place severe restrictions on the invariant measures which have positive entropy with respect to any one parameter subgroup of A.

An analogue to the action of A on G/Γ is the action of a (large enough) semigroup of commuting endomorphisms on tori. For such actions the topological questions regarding the closed invariant subsets have been completely solved by Furstenberg [F] for the one dimensional torus $\mathbb{T}^1$ and by Daniel Berend (see [B]) for $\mathbb{T}^d$, $d \geq 2$.

A key part of Furstenberg's and Berend's argument was proving that any closed invariant subset containing a non–isolated periodic point is the complete torus. They use this result twice. First, in conjunction with a general disjointness argument due to Furstenberg (for which, unfortunately, there is no clear analogue in the case of G/Γ), to show that any minimal closed invariant subset of the torus is a periodic orbit. Then, in view of the fact that any closed invariant subset contains a minimal one, to show that any closed invariant subset of the torus is either a periodic orbit or the whole torus. Our result can be seen as the analogue of this key ingredient in Furstenberg's and Berend's proof. In many cases it reduces the problem of describing a general orbit–closure for the action of the diagonal group to that of *classifying the minimal compact invariant subsets* for the action.

Overview: After some notations and preliminaries, we prove Theorem 1.1 in §4. A key component of the proof is Lemma 4.2, which shows in particular that if $\overline{Ay}$ contains a compact orbit Ax then there is a unipotent subgroup U such that $Ux \subset \overline{Ay}$. This argument was used implicitly in [CaS-D]. We then use Marina Ratner's orbit closure theorem [R] to show that there is some closed orbit $Hx \subset \overline{Ay}$ where H is a reductive Lie subgroup of G (Lemma 4.1). If $Hx \neq \overline{Ay}$ we then find a higher dimensional unipotent subgroup U' such that $U'x \subset \overline{Ay}$ (this is what is proved in steps 4.7–4.9).

We discuss Rees' example and prove Theorem 1.2 in §5. In §6 we consider the case of $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, and prove Theorem 1.3. We also provide a general construction of orbit closures that contain a compact orbit but are neither a compact orbit nor the whole space. We prove Corollary 1.5 in §7. The appendix contains the proof of an auxilliary result due to Nimish Shah, which has been included for the reader's convenience.

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2. Notation and Definitions

Throughout this paper $G = \mathrm{SL}(n, \mathbb{R})$ with $n \geq 3$. Let k be a subfield of $\mathbb{R}$. By a k-subgroup of G we mean a subgroup of matrices which satisfy a set of polynomial equations with coefficients in k, where the variables of the polynomials are the n^2 matrix entries. If H is a k-subgroup and R is a ring in k then $H(k)$ (respectively, $H(R)$) denotes the matrices in H with entries in k (respectively, R). A field automorphism σ of k acts on a matrix $M = (m_{ij})$ with entries in k by

$$\sigma(M) = M^\sigma = (\sigma(m_{ij})),$$

and on a polynomial $P(x) = \sum a_i x^i$ with coefficients in k by

$$P^\sigma(x) = \sum \sigma(a_i) x^i.$$

If H is a k-subgroup of G then H^σ is the group obtained by acting on the polynomials defining H. The above actions are compatible in the sense that

$$M \in H(k) \iff M^\sigma \in H^\sigma(k).$$

For a subgroup H of G, $\mathrm{Lie}(H)$ denotes the Lie algebra of H, $N_G(H)$ denotes the normalizer of H in G, and

$$N_G^1(H) = \{g \in N_G(H) : |\det(\mathrm{Ad}(g)|_{\mathrm{Lie}(H)})| = 1\}.$$

For any subset $S \subset G$, $C_G(S)$ denotes the centralizer of S in G. For any $x \in G/\Gamma$,

$$H_x = \{h \in H : hx = x\}.$$

Let U^+ (respectively U^-) denote the group of upper-triangular (resp., lower triangular) matrices with all diagonal elements equal to 1. For $1 \leq i \neq j \leq n$ let λ_{ij} be the linear functional on $\mathrm{Lie}(A) = \{\mathrm{diag}(a_1, \dots, a_n) : \sum a_i = 0\}$ given by

$$\lambda_{ij} : \mathrm{diag}(a_1, \dots, a_n) \mapsto a_i - a_j$$

and let U_{ij} be the one-parameter subgroup such that $[a, u] = \lambda_{ij}(a)u$ for every $a \in \mathrm{Lie}(A), u \in \mathrm{Lie}(U_{ij})$. Thus $\Phi = \{\lambda_{ij}\}$ is the set of roots and $\mathrm{Lie}(U_{ij}) = \mathfrak{G}_{ij}$ are the root subspaces of $\mathrm{Lie}(G)$. We will sometimes also let Φ denote the set $\{(i,j) : 1 \leq i \neq j \leq n\}$.

Throughout this paper $u^+, u^-, \mathbf{u}^+, \mathbf{u}^-, u_{ij}, \mathbf{u}_{ij}, a, \mathbf{a}$ denote elements of U^+, U^-, $\mathrm{Lie}(U^+)$, $\mathrm{Lie}(U^-)$, U_{ij}, $\mathrm{Lie}(U_{ij})$, A, and $\mathrm{Lie}(A)$ respectively. Fix nonzero elements $\mathbf{w}_{ij} \in \mathrm{Lie}(U_{ij})$. Then each $\mathbf{w} \in \mathrm{Lie}(G)$ can be written uniquely as $\mathbf{a} + \sum c_{ij}\mathbf{w}_{ij}$, and we denote the projection $w \mapsto c_{ij}$ by P_{ij}. We choose some norm $\|\cdot\|_A$ on $\mathrm{Lie}(A)$ and define a norm on $\mathrm{Lie}(G)$ by

$$\|\mathbf{w}\| = \max\{\|\mathbf{a}\|_A, |c_{ij}|\}.$$

For every $z \in G/\Gamma$, we can find neighborhoods V^0, V^+, V^- of the identity in $\mathrm{Lie}(A), \mathrm{Lie}(U^+), \mathrm{Lie}(U^-)$ respectively so that the maps

$$\Psi : V^+ \times V^- \to G/\Gamma, \ \Psi(\mathbf{u}^+, \mathbf{u}^-) = \exp(\mathbf{u}^+)\exp(\mathbf{u}^-)z$$

and

$$\hat{\Psi} : V^0 \times V^+ \times V^- \to G/\Gamma, \ \hat{\Psi}(\mathbf{a}, \mathbf{u}^+, \mathbf{u}^-) = \exp(\mathbf{a})\exp(\mathbf{u}^+)\exp(\mathbf{u}^-)z$$

are diffeomorphisms onto their images, and the image of $\hat{\Psi}$ is a neighborhood of z in G/Γ. We call Ψ and $\hat{\Psi}$ *box maps* for z.

Recall that two lattices Γ_1 and Γ_2 are *commensurable* if their intersection is of finite index in both, and that the commensurator of Γ is

$$\mathrm{Comm}_G(\Gamma) = \{g \in G : \Gamma \text{ and } g\Gamma g^{-1} \text{ are commensurable}\}.$$

3. Preliminaries

In this section we collect some of the results we will need.

The following is probably well–known:

Proposition 3.1. *Let H be a closed connected subgroup of G normalized by A. Then*

$$\mathrm{Lie}(H) = \mathrm{Lie}(A \cap H) \oplus \bigoplus_{\lambda_{ij} \in \Delta} \mathfrak{G}_{ij}$$

for some $\Delta \subset \Phi$. If H is reductive then H is in 'block form', i.e., there is a partition $\{1, \dots, n\} = B_1 \sqcup \ldots \sqcup B_k$ such that

$$\mathrm{Lie}(H) = \mathrm{Lie}(A \cap H) \oplus \bigoplus_{l=1}^{k} \Big(\bigoplus_{i,j \in B_l, i \neq j} \mathfrak{G}_{ij} \Big), \tag{1}$$

and H is equal to the connected component of the identity in $C_G(A \cap C_G(H))$.

Proof: The subalgebras $\mathfrak{G}_{ij}$, which are the eigenspaces for the action of $\mathrm{Ad}(A)$ on $\mathrm{Lie}(G)$, are one-dimensional. Thus an $\mathrm{Ad}(A)$-invariant subspace of $\mathrm{Lie}(G)$ is a sum of the $\mathfrak{G}_{ij}$. This proves the first assertion.

For the second assertion, define a relation R on $\{1, \dots, n\}$ by

$$iRj \iff i = j \text{ or } \mathrm{Lie}(U_{ij}) \subset \mathrm{Lie}(H).$$

Transitivity of R follows from the fact that H is closed under the Lie bracket, and symmetry follows from the fact that H is reductive and normalized by A. Thus R is an equivalence relation, and the desired partition of $\{1, \dots, n\}$ is given by the equivalence classes for R.

The third statement follows easily from the second.

$\square$

We will need the following ergodicity condition:

Theorem 3.2. Moore's Ergodicity Theorem: *Let H be a connected semisimple Lie group with no compact factors and finite center, and let Δ be a lattice in H. Let A be a subgroup of H such that $A \cap N$ is noncompact for each nontrivial normal subgroup N of H. Then A acts ergodically on H/Δ (with respect to the H-invariant measure).*

Proof: In case Δ is irreducible, this is a well–known theorem of C. C. Moore (see [Z], Theorem 2.2.6). The general case follows by reduction to this case: By [Ra], Theorem 5.22, we can write $H = H_1 \cdot \ldots \cdot H_k$ with $H_i \cap H_j$ discrete and central for $i \neq j$, and find $\Delta_i \subset H_i \cap \Delta$, such that Δ_i is an irreducible lattice in H_i, and $[\Delta : \Delta_1 \cdot \ldots \cdot \Delta_k] < \infty$. Since $A \cap H_i$ is noncompact, it acts ergodically on H_i/Δ_i and therefore A acts ergodically on $H_1/\Delta_1 \times \cdots \times H_k/\Delta_k$. Therefore the factor action on H/Δ is also ergodic.

□

Our results depend essentially on the following result, which is a special case of Theorems A and B in [R]:

Theorem 3.3. Ratner's Orbit Closure Theorem: *Let U be a connected Lie subgroup of G generated by unipotent elements, and let $x \in G/\Gamma$. Then there is a closed connected subgroup H containing U such that $\overline{Ux} = Hx$ and H_x is a lattice in H. Moreover, if $U = \{u(t) : t \in \mathbb{R}\}$ is a one-parameter subgroup, the positive semi–orbit $\{u(t)x : t \geq 0\}$ is dense in Hx.*

We will need some additional properties of the group H appearing in the conclusion of Theorem 3.3:

Theorem 3.4. *Let the notation be as in Theorem 3.3 (recall that $G = \mathrm{SL}(n, \mathbb{R})$ and $n \geq 3$). Then:*

1. *H is an $\mathbb{R}$-subgroup of G.*
2. *H_x is a Zariski dense lattice in H.*
3. *The unipotent radical of H is equal to the radical of H.*
4. *$\mathrm{Comm}_H(H_x)$ is dense in H.*

Proof: This was proved by Nimish Shah in section 3 of [Sh], but not stated explicitly. For the reader's convenience, we show in Appendix A how to deduce this result from Shah's results.

□

We also need the following ([Sh], Lemma 2.2):

Proposition 3.5. *If H_1 and H_2 are two closed subgroups of G such that $H_i z$ is closed for some $z \in G/\Gamma$, then $(H_1 \cap H_2)z$ is also closed.*

Proposition 3.6. *Let $z \in G/\Gamma$. Then for any finite subset $\Lambda \in G_z$, $C_G(\Lambda)z$ is closed.*

Proof: This is Lemma 1.14 of [Ra].

4. Invariant Sets for General Lattices

In this section we will prove Theorem 1.1. We will need some lemmas.

Lemma 4.1. *Let V be a connected unipotent subgroup normalized by A, and let $z \in G/\Gamma$ be such that Az is compact. Then $\overline{Vz} = Hz$, where H is connected, semisimple, and $A \subset N^1_G(H)$.*

Proof: From Theorem 3.3 it follows that there exists a connected closed subgroup H such that $\overline{Vz} = Hz$ and H_z is a lattice in H.

Let $a \in A_z$. We have

$$aHa^{-1}z = aHz = a\overline{Vz} = \overline{aVz} = \overline{Vaz} = \overline{Vz} = Hz,$$

and since Γ is discrete this implies that $\mathrm{Ad}(a)(\mathrm{Lie}(H)) = \mathrm{Lie}(H)$. Since H is connected, $a \in N_G(H)$, and since A_z is Zariski dense in A, A normalizes H. Conjugation by elements of A_z preserves H_z and therefore preserves H/H_z, which has finite volume. H is unimodular since it contains the lattice H_z, and therefore the volume of H/H_z is just the haar measure of any fundamental domain. The Jacobian for the volume change when conjugating by a is $|\det(\mathrm{Ad}(a)|_{\mathrm{Lie}(H)})|$ and this implies that for $a \in A_z$, $\det(\mathrm{Ad}(a)|_{\mathrm{Lie}(H)})^2 = 1$. Since A_z is Zariski dense we obtain that $A \subset N^1_G(H)$.

Let R denote the unipotent radical of H. By Theorem 3.4, R is also the radical of H. It is invariant under automorphisms of H and therefore A normalizes R. By Lie's theorem, the group AR can be brought into triangular form, and thus if R is nontrivial then $a \mapsto \det(\mathrm{Ad}(a)|_{\mathrm{Lie}(R)})$ is a nontrivial character on A. Thus to show that R is trivial it will suffice to show that

$$|\det(\mathrm{Ad}(a)|_{\mathrm{Lie}(H)})| = |\det(\mathrm{Ad}(a)|_{\mathrm{Lie}(R)})|. \tag{2}$$

To see this, notice that a induces an automorphism of H/R, that is it acts as a bundle automorphism of the bundle $R \to H \to H/R$. Since H/R is semisimple it is unimodular and has a finite group of outer automorphisms, thus an automorphism of H/R does not affect haar measure on H/R. Now considering H as the total space of the bundle $R \to H \to H/R$ and disintegrating the haar measure on H into the haar measure on R and H/R yields (2).

Thus R must be trivial and therefore H is semisimple.

$\square$

The following lemma is the core of our argument.

Lemma 4.2. *Suppose $z \in F$ is such that Az is compact and for every $1 \le i \ne j \le n$, N_{ij} acts ergodically on Az. Suppose H is a semisimple subgroup of G, normalized by A, such that Hz is closed, H_z is a lattice in H, and AHz is properly contained in F. Then there exists $1 \le i \ne j \le n$ such that $U_{ij} \not\subset H$ and $U_{ij}z \subset F$.*

Proof: The proof of the lemma will be divided into steps.

Step 4.1. *Let $A_0 = A \cap C_G(H)$. Then A_0z is compact.*

Since Az is compact, by Proposition 3.5 it suffices to show that $C_G(H)z$ is closed. By Theorem 3.4 H_z is Zariski dense in H, hence $C_G(H) = C_G(H_z)$ and by the descending chain condition for algebraic groups, there is a finite subset $\Lambda \subset H_z$ such that $C_G(H) = C_G(\Lambda)$. Now we may use Proposition 3.6.

Step 4.2. *Let*

$$\Delta = \{(i,j) : 1 \le i \ne j \le n, U_{ij} \subset H \cup U^-\}.$$

With no loss of generality we may assume that there is there is an element $a = \exp(\mathbf{a})$ such that $az = z$ and

(3)
$$\lambda_{ij}(\mathbf{a}) > 0 \text{ whenever } (i,j) \notin \Delta, \text{ and } \lambda_{ij}(\mathbf{a}) \le 0 \text{ whenever } (i,j) \in \Delta.$$

First notice that neither the conditions nor the conclusions of the Lemma are affected if we replace z by $P_\sigma z$, H by $P_\sigma H P_\sigma^{-1}$ and F by $P_\sigma F$, where P_σ is a permutation matrix corresponding to some $\sigma \in S_n$. So using Proposition 3.1 we may assume that H is in 'block form' and moreover that the B_i in (1) all consist of consecutive integers. Under this assumption $\mathrm{Lie}(A_0)$ consists of traceless diagonal matrices $\mathrm{diag}(a_1, \dots, a_n)$ with $a_j = a_k$ whenever $j, k \in B_i$ for some i, and by Proposition 3.1 H is of finite index in $C_G(A_0)$ and hence $\mathrm{Ad}(A_0)$ acts nontrivially on $\mathrm{Lie}(U_{ij})$ whenever $U_{ij} \not\subset H$. This implies that there is $a \in A_0, a = \exp(\mathbf{a})$ such that (3) is satisfied. Since $(A_0)_z$ is cocompact in A_0 by Step 4.1, we can choose such an a so that in addition $az = z$.

Step 4.3. *Let $\Psi, \hat{\Psi}$ be box maps for z. There is a sequence of elements $(\mathbf{u}_k^+, \mathbf{u}_k^-) \in V^+ \times V^-$, $k = 1, 2, \dots$, such that at least one of $\mathbf{u}_k^+, \mathbf{u}_k^-$ is not in $\mathrm{Lie}(H)$ and such that $\Psi(\mathbf{u}_k^+, \mathbf{u}_k^-) \in F$ and $(\mathbf{u}_k^+, \mathbf{u}_k^-) \to 0$.*

If $y_k \in F - AHz$ satisfy $y_k \to z$, we can assume that for all k, y_k is in the image of $\hat{\Psi}$, and write

$$\hat{\Psi}(0) = z \leftarrow y_k = \hat{\Psi}(\mathbf{a}_k, \mathbf{u}_k^+, \mathbf{u}_k^-).$$

Then $(\mathbf{u}_k^+, \mathbf{u}_k^-)$ are nontrivial, at least one of them is not in $\mathrm{Lie}(H)$, they converge to 0, and $\Psi(\mathbf{u}_k^+, \mathbf{u}_k^-) \in F$ since $y_k \in F$ and F is A-invariant.

Step 4.4. *There is a nontrivial* $\mathbf{u} \in \mathrm{Lie}(U^+) \cup \mathrm{Lie}(U^-)$ *such that* $\exp(\mathbf{u})z \in F$ *and*

$$U_{ij} \subset H \Rightarrow P_{ij}(\mathbf{u}) = 0.$$

By Step 4.3, there are elements of the form $u_k^+ u_k^- z$ in F, with $u_k^+ \to e$ and $u_k^- \to e$, and at least one of u_k^+, u_k^- is not in H. Assume with no loss of generality that for all k, $u_k^+ \notin H$, and that both u_k^+ and u_k^- are nontrivial for all k (otherwise there is nothing to prove). Let

$$\Delta = \{(i,j) : 1 \leq i \neq j \leq n, U_{ij} \subset H \cup U^-\}.$$

It will suffice to find $R > 1$ such that for each $\epsilon > 0$, there is an element of F in

$$V_{\epsilon,R,H} = \{\Psi(\mathbf{u}^+, \mathbf{u}^-) : 1 \leq \|\mathbf{u}^+ + \mathbf{u}^-\| \leq R, (i,j) \in \Delta \Rightarrow |P_{ij}(\mathbf{u}^+ + \mathbf{u}^-)| < \epsilon\}.$$

Let $a = \exp(\mathbf{a})$ be as in step 4.2. Let

$$R_1 = \min\{e^{\lambda_{ij}(\mathbf{a})} : (i,j) \notin \Delta\}$$

and

$$R_2 = \max\{e^{\lambda_{ij}(\mathbf{a})} : (i,j) \notin \Delta\}.$$

We have $1 < R_1 \leq R_2$. Since

$$au^+u^-z = au^+a^{-1}\ au^-a^{-1}z,$$

we get that for all $(\mathbf{u}^+, \mathbf{u}^-) \in V^+ \times V^-$, $a\Psi(\mathbf{u}^+, \mathbf{u}^-) = \Psi(\mathbf{u}_1^+, \mathbf{u}_1^-)$ with

$$R_1|P_{ij}(\mathbf{u}^+ + \mathbf{u}^-)| \leq |P_{ij}(\mathbf{u}_1^+ + \mathbf{u}_1^-)| \leq R_2|P_{ij}(\mathbf{u}^+ + \mathbf{u}^-)| \text{ whenever } (i,j) \notin \Delta$$

and

$$|P_{ij}(\mathbf{u}_1^+ + \mathbf{u}_1^-)| \leq |P_{ij}(\mathbf{u}^+ + \mathbf{u}^-)| \text{ whenever } (i,j) \in \Delta.$$

Therefore if we choose $(\mathbf{u}^+, \mathbf{u}^-)$ such that $\Psi(\mathbf{u}^+, \mathbf{u}^-) \in F$, $\mathbf{u}^+ + \mathbf{u}^- \notin \mathrm{Lie}(H)$ and $\|\mathbf{u}^+ + \mathbf{u}^-\| < \epsilon$, for some natural k we will have $a^k\Psi(\mathbf{u}^+, \mathbf{u}^-) \in V_{\epsilon,R_2,H}$.

Step 4.5. *For some* $1 \leq i \neq j \leq n$, *there is* $\mathbf{u}_{ij} \neq 0$ *such that* $\exp(\mathbf{u}_{ij})z \in F$ *and* $\exp(\mathbf{u}_{ij}) \notin H$.

Let $\mathbf{u}$ be an element as in Step 4.4. Define

$$\Delta' = \{\lambda_{ij} : P_{ij}(\mathbf{u}) \neq 0\}.$$

By Step 4.4,

$$\lambda_{ij} \in \Delta' \Rightarrow U_{ij} \not\subset H.$$

Let $\lambda = \lambda_{rs} \in \Delta'$ be extremal in the sense that it is not in the convex cone over $\Delta' - \{\lambda\}$ (in $\mathrm{Lie}(A)^*$). Then we can find $\mathbf{a} \in \mathrm{Lie}(A) \cong \mathrm{Lie}(A)^{**}$ such that $\lambda(\mathbf{a}) = 0$ and $\alpha(\mathbf{a}) < 0$ for every $\alpha \in \Delta' - \{\lambda\}$. Since Az is a torus, the orbit of z under any element of A is recurrent and therefore there is a subsequence $k_\ell \to \infty$ such that $a^{k_\ell} z \to z$, where $a = \exp(\mathbf{a})$. Thus:

$$\begin{aligned}
F \ni & \, a^{k_\ell} \exp(\mathbf{u}) z \\
&= a^{k_\ell} \exp\Big(\sum_{\lambda_{ij} \in \Delta'} \mathbf{u}_{ij}\Big) a^{-k_\ell} a^{k_\ell} z \\
&= \exp\Big(\mathrm{Ad}(a)^{k_\ell}\Big(\sum_{\lambda_{ij} \in \Delta'} \mathbf{u}_{ij}\Big)\Big) a^{k_\ell} z \\
&= \exp\Big(\sum_{\lambda_{ij} \in \Delta'} e^{k_\ell \lambda_{ij}(\mathbf{a})} \mathbf{u}_{ij}\Big) a^{k_\ell} z \\
&\longrightarrow_{\ell \to \infty} \exp(\mathbf{u}_{rs}) z.
\end{aligned}$$

Thus (r, s) satisfies the required conclusions.

Step 4.6. *Let $1 \leq i \neq j \leq n$ be as in 4.5. Then $U_{ij} z \subset F$.*

Let $\lambda = \lambda_{ij}$,

$$\mathrm{Lie}(A)_z = \{\mathbf{a} \in \mathrm{Lie}(A) : \exp(\mathbf{a}) z = z\},$$

and

$$Q = \{\lambda(\mathbf{a}) : \mathbf{a} \in \mathrm{Lie}(A)_z\} \subset \mathbb{R}.$$

Since $\mathrm{Lie}(A)_z$ is a lattice in $\mathrm{Lie}(A)$, Q is either discrete or dense. But if Q is discrete then $N_{ij} \cap A_z$ is a lattice in N_{ij}, and this contradicts the assumption that N_{ij} acts ergodically on Az. Therefore Q is dense.

Now we have for $q \in Q$:

$$\begin{aligned}
\exp(e^q \mathbf{u}_{ij}) z &= \exp(e^{\lambda_{ij}(\mathbf{a})} \mathbf{u}_{ij}) z \\
&= \exp(\mathrm{Ad}(a) \mathbf{u}_{ij}) z \\
&= a \exp(\mathbf{u}_{ij}) a^{-1} z \\
&= a u a^{-1} z \\
&= a u z \in F,
\end{aligned}$$

where $a = \exp(\mathbf{a}) \in A_z, u = \exp(\mathbf{u}_{ij})$.

Therefore by continuity, $\exp(t\mathbf{u}_{ij}) z \in F$ for all $t \geq 0$, and now it follows from the last assertion in Theorem 3.3 that $\exp(t\mathbf{u}_{ij}) z \in F$ for all $t \in \mathbb{R}$.

$\square$

Remark: The main obstacle to extending the results of this paper to noncompact simple groups G other than $\mathrm{SL}(n, \mathbb{R})$ is the fact that the statement that H is the connected component of $C_G(A \cap C_G(H))$ holds for $G = \mathrm{SL}(n, \mathbb{R})$ but not for other G. This was used in an in the proof of Step 4.2 of the preceding Lemma.

Proof of Theorem 1.1: Let V be a connected subgroup of G of maximal possible dimension satisfying the following:

1. V is unipotent.
2. $Vx \subset F$.
3. V is normalized by A.

By Ratner's theorem 3.3 there is a connected closed subgroup H of G such that $\overline{Vx} = Hx$ and H_x is a lattice in H. By Lemma 4.1, H is semisimple and normalized by A. Let $H' = AH$. Then H' is reductive. From Proposition 3.1 we see that $A = (A \cap H) \cdot A_0$, where $A_0 = A \cap C_G(H)$, hence $H' = A_0 \cdot H$. From Lemma 4.2, step 4.1 it follows that $A_0 x$ is compact. It follows from this that $H'x$ is closed; indeed, if $a_k h_k x \to z$ where $a_k \in A_0$ and $h_k \in H$, let $a_k x = a'_k x$ where a'_k are in a compact subset of A_0 and $a'_k \to a_0$. Then

$$z \leftarrow a_k h_k x = h_k a_k x = h_k a'_k x = a'_k h_k x,$$

hence $a_0 h_k x \to z$ so $z \in H'x$. Also, $H'x$ supports the finite H'-invariant measure obtained by pushing forward the $A_0 \times H$-invariant measure on $A_0 x \times Hx$ via the map $(a_0 x, hx) \mapsto a_0 hx$.

Thus the theorem will be proved if we show that $F = H'x$. Suppose the contrary is true; we will reach a contradiction by finding a connected subgroup V' of G satisfying conditions 1,2,3 above and such that $\dim V' > \dim V$. Once again we divide our argument into steps.

Step 4.7. *The set*

$$D = \{z \in Hx : Az \text{ is compact}, N_{ij} \text{ acts ergodically on } Az\}$$

is dense in Hx.

By Theorem 3.4, $\mathrm{Comm}_H(H_x)$ is dense in H. For each $q \in \mathrm{Comm}_H(H_x)$, $A_{qx} \cap A_x$ is of finite index in A_x and is therefore cocompact in A, and similarly, for $1 \le i \ne j \le n$, $(N_{ij})_{qx}$ is not a lattice in N_{ij}. Thus $qx \in D$ for every $q \in \mathrm{Comm}_H(H_x)$.

Step 4.8. *There exist $1 \le i \ne j \le n$ such that $U_{ij} \not\subset H$ and $U_{ij} Hx \subset F$.*

For each $1 \le i \ne j \le n$ such that $U_{ij} \not\subset H$, let

$$D_{ij} = \{z \in Hx : U_{ij} z \subset F\}.$$

Then D_{ij} is closed. By step 4.7, there is a dense set of points in Hx for which the hypotheses of Lemma 4.2 are satisfied. Therefore $Hx \subset \bigcup D_{ij}$. By the Baire category theorem, one of the D_{ij}'s contains an open subset of Hx. Each of the D_{ij} is $A \cap H$-invariant and by Moore's Theorem 3.2 $A \cap H$ acts topologically transitively on Hx; thus there is $1 \le i \ne j \le n$ for which $Hx = D_{ij}$, proving the claim.

Suppose with no loss of generality that $U_{ij} \subset U^+$. Let us also replace V with $H \cap U^+$, which we may since $\dim H \cap U^+ = \dim V$. Denote $\dim V$ by d. Write

$$U_{ij} = \{u_{ij}(t) : t \in \mathbb{R}\}.$$

By step 4.8, for all t,

$$u_{ij}(t) V u_{ij}(t)^{-1} u_{ij}(t) x \subset F.$$

By Ratner's theorem there is a sequence $t_k \to \infty$ such that $u_{ij}(t_k)x \to_{k\to\infty} x$. This implies that if g is a limit point of

$$\{u_{ij}(t_k) H u_{ij}(t_k)^{-1} : k = 1, 2, \ldots\},$$

then $gx \in F$. Therefore we can increase d, and conclude the proof, by taking $V' = U_{ij} V_0$ where V_0 is as in the following

Step 4.9. *The limit* $\lim_{t\to\infty} \mathrm{Ad}(u_{ij}(t))(\mathrm{Lie}(V))$ *exists in the Grassmannian manifold* $\mathcal{G}$ *of* d*-dimensional subspaces of* $\mathrm{Lie}(G)$*, and is the Lie subalgebra of a unipotent subgroup* V_0 *of* G *not containing* U_{ij}*, normalized by* A *and by* U_{ij}*.*

The map $t \mapsto \mathrm{Ad}(u_{ij}(t))(\mathrm{Lie}(V))$ is a polynomial map from $\mathbb{R}$ to the projective variety $\mathcal{G}$; i.e., considering $\mathcal{G}$ as a subset of projective space $\mathbb{P}^M$ for suitable M, this map is the composition of a polynomial $\phi : \mathbb{R} \to \mathbb{R}^{M+1}$ with the natural projection $\mathbb{R}^{M+1} \to \mathbb{P}^M$. Let N be the highest degree of any of the polynomials determining the coordinates of ϕ, and let $\mathbf{p}$ be the point in $\mathbb{P}^M$ corresponding to the coefficients of the monomials of degree N. Then it is clear that $\mathbf{p} = \lim_{t\to\infty} \mathrm{Ad}(u_{ij}(t))(\mathrm{Lie}(V))$.

Since the Lie subalgebras form a closed subset of $\mathcal{G}$, $\mathbf{p}$ corresponds to a Lie subalgebra of $\mathrm{Lie}(G)$. Let V_0 be the corresponding connected subgroup of U^+. A normalizes V_0 and by definition it is clear that U_{ij} normalizes V_0. Since $V_0 \subset U^+$, V_0 is unipotent.

It remains to show that $U_{ij} \not\subset V_0$. Write $\mathfrak{U}^0 = \mathrm{Lie}(U^+)$, $\mathfrak{U}^{i+1} = [\mathfrak{U}^0, \mathfrak{U}^i]$, and let k be the largest integer for which $\mathrm{Lie}(U_{ij}) \subset \mathfrak{U}^k$. Let $\mathfrak{V} = \mathrm{Lie}(V) + \mathfrak{U}^{k+1}$. Then $\mathfrak{V}$ is invariant under $\mathrm{Ad}(U_{ij})$ and does not contain U_{ij}. For every t, $\mathrm{Ad}(u_{ij}(t))(\mathrm{Lie}(V)) \subset \mathfrak{V}$, and therefore $\mathbf{p} \in \mathfrak{V}$. Therefore $\mathrm{Lie}(U_{ij}) \not\subset \mathbf{p} = \mathrm{Lie}(V_0)$. □

5. Rees' Example and Related Constructions

In an unpublished manuscript Rees showed how to construct uniform lattices Γ in $G = \mathrm{SL}(3, \mathbb{R})$ such that $\Gamma_i = \Gamma \cap H_i$ is a lattice in H_i, where

$$H_1 = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$$

and

$$H_2 = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

From this construction Rees obtained A-invariant closed subsets of G/Γ which are contained in H_1/Γ_1 and are circle bundles over an arbitrary orbit-closure for $A_2 = A \cap H_2$ in H_2/Γ_2. Note that H_2 is isomorphic to $\mathrm{SL}(2, \mathbb{R})$. Thus any irregular orbit-closure for the action of A_2 on H_2/Γ_2 – that is, for the geodesic flow on the unit tangent bundle to a finite volume Riemann surface – can appear as the base of a bundle which is an orbit-closure for A in G/Γ. As is well-known, for these one-parameter flows many irregular orbit-closures can appear –e.g., orbit-closures which have fractional Hausdorff dimensions, orbit-closures which contain but are not equal to periodic orbits, etc.

In this section we show that a construction similar to Rees' is possible for the action of the diagonal matrices on $\mathrm{SL}(n, \mathbb{R})/\Gamma$ whenever assumption 1 but not assumption 2 of Theorem 1.1 holds. That is, we show

Proposition 5.1. *Let G, A, Γ be as in Theorem 1.1. Let $x \in G/\Gamma$ such that Ax is compact and N_{ij} does not act ergodically on Ax. Then there exists a subgroup H_2 of G which is isomorphic to* $\mathrm{SL}(2, \mathbb{R})$ *and normalized by A such that:*

1. *H_2x is closed and admits an H_2-invariant finite measure.*
2. *Let $H_1 = AH_2$. Then H_1x is closed.*
3. *Let $A_0 = A \cap C_G(H_2)$ and $A_2 = A \cap H_2$. Then for any $y \in H_2x$, A_0y is compact and $\overline{Ay} = A_0\overline{A_2y}$.*

Proof: Since Ax is compact and N_{ij} does not act ergodically on Ax, $(N_{ij})_x$ is a cocompact lattice in N_{ij}. By Zariski density, we obtain the existence of $\gamma \in (N_{ij})_x$ such that $\gamma = \mathrm{diag}(a_1, \ldots, a_n)$ with $a_i = a_j$ and $a_k \neq a_l$ whenever $\{k, l\} \neq \{i, j\}$. Let H_1 be the connected component of the identity in $C_G(\gamma)$, and let H_2 be the subgroup generated by U_{ij} and U_{ji}. Then obviously H_2 is isomorphic to $\mathrm{SL}(2, \mathbb{R})$, A normalizes H_2 and $AH_2 \subset H_1$. By considering dimensions we see that $H_1 = AH_2$.

By Proposition 3.6, H_1x is closed. By Theorem 3.3, $\overline{U_{ij}x} = Hx$ for some subgroup H of G containing U_{ij}, and there is a finite H-invariant measure on H_x. Since H_1x is closed we must have that $H \subset H_1$, and since, by Lemma 4.1, H is semisimple, we must have that $H = H_2$.

Arguing as in Step 4.1 in the proof of Lemma 4.2 we see that A_0x is compact. Therefore for any $y \in H_2x$, say $y = h_2x$ for $h_2 \in H_2$, we get that

$$A_0y = A_0h_2x = h_2A_0x$$

is compact. Now let $a_k y \to z$, where $a_k \in A$. Write $a_k = a_k^2 a_k^0$ with $a_k^2 \in A_2$ and $a_k^0 \in A_0$. Then for each k there is $a_k^1 \in A_0$ such that $a_k^1 y = a_k^0 y$ and $a_k^1 \to a_0$. Thus

$$z \leftarrow a_k^2 a_k^0 y = a_k^2 a_k^1 y = a_k^1 a_k^2 y$$

and therefore

$$a_k^2 y \to a_0^{-1} z.$$

This implies that $z \in A_0 \overline{A_2 y}$.

□

Proposition 5.1 shows that any irregular orbit for the geodesic flow on $\mathrm{SL}(2,\mathbb{R})/\Lambda$ (where Λ is an appropriate lattice) yields an irregular orbit for the action of A on G/Γ. For instance, Theorem 1.2 now follows from Proposition 5.1 and the following simple

Lemma 5.2. *Let Λ be a lattice in $\mathrm{SL}(2,\mathbb{R})$ and $D = \{\mathrm{diag}(e^t, e^{-t}) : t \in \mathbb{R}\}$. Then for any $x \in \mathrm{SL}(2,\mathbb{R})/\Lambda$ such that Dx is a periodic orbit, there exists $y \in \mathrm{SL}(2,\mathbb{R})/\Lambda$ such that Dy is open in $\overline{Dy}$ and $x \in \overline{Dy}$.*

Proof: Let U^+ (respectively, U^-) be the subgroup of $\mathrm{SL}(2,\mathbb{R})$ consisting of upper (respectively lower) triangular unipotent matrices. Then $U^- D U^+$ is a dense open subset of $\mathrm{SL}(2,\mathbb{R})$. Since the periodic D-orbits in $\mathrm{SL}(2,\mathbb{R})/\Lambda$ are dense (see [A]), there is $z \in \mathrm{SL}(2,\mathbb{R})/\Lambda$, $z = u^- d u^+ x$, such that Dz is periodic and $Dz \neq Dx$. Let $y = u^+ x$. From the commutation relations for D, U^+, U^- it is then obvious that $\overline{Dy} = Dy \cup Dz \cup Dx$.

□

6. Orbit-Closures for $\Gamma = \mathrm{SL}(n,\mathbb{Z})$

In this section we examine the case $\Gamma = \mathrm{SL}(n,\mathbb{Z})$ in more detail. First we show how to explicitly construct orbit-closures for A in G/Γ that contain a compact A-orbit, and are not equal to an A-orbit or to G/Γ. We then prove Theorem 1.3. In the proof of Theorem 1.3 it develops that all orbit-closures containing a compact orbit are like the ones in the example.

Example:

First let us construct some compact orbits. These orbits are closely related to the norm tori of [PlRa, page 54]. Let K be a totally real number field, with $[K : \mathbb{Q}] = n$, and let L be the splitting field for K. We take $\mathcal{O}(K)$ to be the ring of algebraic integers of K, and we recall that $\mathcal{O}(K)$ considered as an additive group is isomorphic to $\mathbb{Z}^n$. Now take $R < \mathcal{O}(K)$ an additive subgroup of finite index. Let $\alpha_1, \ldots,$ α_n be generators for R. We further set $\Lambda = \mathrm{Gal}(L/\mathbb{Q})$, and Λ_K the subgroup of Λ that acts trivially on K. Recall that $[\Lambda : \Lambda_K] = n$. Let $\{\sigma_1 = \mathrm{id}, \sigma_2, \dots, \sigma_n\}$ be a set of representatives of the cosets of Λ/Λ_K. For any unit $\theta \in \mathcal{O}(K)^*$, take

$$a(\theta) = \mathrm{diag}(\sigma_1(\theta), \dots, \sigma_n(\theta)) \in A.$$

We now take $g \in \mathrm{GL}(n, \mathbb{R})$ to be

$$g = (\sigma_i(\alpha_j))_{i,j=1\dots n}. \tag{4}$$

and

$$\bar{g} = \det(g)^{-1/n} g\Gamma.$$

We note that $\det(g)$ is fixed by Λ, so $\det(g) \in \mathbb{Q} \cap \mathcal{O}(L) = \mathbb{Z}$.

$A_0 = g^{-1}Ag$ is a norm torus in the sense of [PlRa, page 54]. There it is shown that A_0 is defined over $\mathbb{Q}$ and has no $\mathbb{Q}$-characters. By [BoH], $A_0 \cap \Gamma$ is a lattice in A_0, so $A\bar{g}$ is compact.

Suppose that $a = \mathrm{diag}(a_1, \dots, a_n) \in A_0 \cap \Gamma$. Then $ag = g\gamma$ for some $\gamma \in \Gamma$. In particular (carrying out the matrix multiplication in the first row),

$$a_1 \alpha_j = \sum_{i=1}^{n} \alpha_i \gamma_{ij},$$

so $a_1 = \alpha_j^{-1} \sum_{i=1}^{n} \alpha_i \gamma_{ij} \in K$, and also $a_1 R \subset R$ so $a_1 \in \mathcal{O}(K)$.

Computing similarly in the ℓth row we get

$$a_1 \sigma_\ell(\alpha_j) = \sum_{i=1}^{n} \sigma_\ell(\alpha_i)\gamma_{ij},$$

and hence

$$a_\ell = \sigma_\ell(\alpha_j^{-1}) \sum_{i=1}^{n} \sigma_\ell(\alpha_i)\gamma_{ij} = \sigma_\ell(\alpha_j^{-1} \sum_{i=1}^{n} \alpha_i \gamma_{ij}) = \sigma_\ell(a_1).$$

Therefore $\prod_{\ell=1}^{n} \sigma_\ell(a_1) = \det a = 1$, and we have $a_1 \in \mathcal{O}(K)^*$. Thus $a = a(a_1) \in a(\mathcal{O}(K)^*)$.

Note that by Dirichlet's Unit Theorem [Sa, Theorem 4.4.1], $a(\mathcal{O}(K)^*)$ is also a lattice in A, so $A_{\bar{g}} = g(A_0 \cap \Gamma)g^{-1}$ is a subgroup of finite index in $a(\mathcal{O}(K)^*)$.

Now let us construct a closed A-invariant set containing such a compact orbit. Assume there is some intermediate field $\mathbb{Q} < K' < K$. Set

$$d = [K' : \mathbb{Q}]$$
$$k = [K : K'].$$

Since $A_{\bar{g}}$ is of finite index in $a(\mathcal{O}(K)^*) \cong \mathcal{O}(K)^*$, there is

$$\theta' \in A^{-1}\mathcal{O}(K') \cap A_{\bar{g}}$$

such that $\mathbb{Q}(\theta') = K'$. Let $H = C_G(a(\theta'))$. Since $a(\theta') \in A_{\bar{g}}$, we know that $a(\theta')\bar{g} = \bar{g}$, hence by Proposition 3.6 $H\bar{g}$ is closed, and in addition since $A \subset H$ it contains the compact orbit $A\bar{g}$. Since $\mathbb{Q}(\theta') = K'$, θ' has exactly d conjugates, and so the n-tuple

$$\sigma_1(\theta'), \sigma_2(\theta'), \dots, \sigma_n(\theta')$$

consists of d different elements of L each appearing with multiplicity k. This shows that for some permutation matrix P,

(5)

$$H = \left\{ P \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & B_d \end{pmatrix} P^{-1} : B_1, \dots, B_d \in \mathrm{GL}(k, \mathbb{R}) \right\} \cap G.$$

We now show that $H\bar{g}$ is an orbit–closure of some $y \in G/\Gamma$.

Proposition 6.1. *$H/H_{\bar{g}}$ admits a finite H-invariant measure, and A acts ergodically $H/H_{\bar{g}} \cong H\bar{g}$. In particular, there is a dense orbit in $H\bar{g}$.*

Remark: We remark that even though $H/H_{\bar{g}}$ has finite volume as we shall see later it is not compact.

Proof: Let $H_0 = g^{-1}Hg = C_G(g^{-1}a(\theta')g)$ and $A_0 = g^{-1}Ag$ where g is as in (4), and let $H_0' = [H_0, H_0]$. By our choice of θ', we have $g^{-1}a(\theta')g \in \Gamma$ and hence H_0 is defined over $\mathbb{Q}$. Therefore H_0 is also defined over $\mathbb{Q}$. It follows from (5) that $H_0 = A_0 \cdot H_0'$, thus any character on H_0 whose restriction to A_0 and H_0' is trivial must be trivial. Thus, since A_0 has no $\mathbb{Q}$-characters, and H_0' has no characters at all, H_0 does not have $\mathbb{Q}$-characters.

Now using a Theorem of Borel and Harish-Chandra ([BoH]) we obtain that $H_0/(H_0 \cap \Gamma)$ (respectively, $H_0'/(H_0' \cap \Gamma)$) admits a finite H_0-(respectively, H_0'-) invariant measure. From Moore's Theorem 3.2 we then conclude that $A_0' = A_0 \cap H_0'$ acts ergodically on $H_0'/(H_0' \cap \Gamma)$, and since $H_0 = A_0 \cdot H_0'$, we obtain that A_0 acts ergodically on $H_0/(H_0 \cap \Gamma)$ and hence that A acts ergodically on $H\bar{g}$. $\square$

We now prove Theorem 1.3. In the course of the proof it will develop that any orbit-closure for A containing a compact orbit arises as in the example above. We will first need the following result, which is probably well–known, and gives a more explicit description of all compact orbits in G/Γ:

Lemma 6.2. *Let $\bar{g} = g\Gamma \in G/\Gamma$ be such that $A\bar{g}$ is compact. Let $A_0 = g^{-1}Ag$. Then there is a Galois number field L and a matrix*

$g_0 \in \mathrm{GL}(n, L)$ such that $g_0 g$ is a diagonal matrix, $A_0 = g_0^{-1} A g_0$, and $\mathrm{Gal}(L/\mathbb{Q})$ permutes the rows of g_0 transitively. In particular, if $a_1, \dots,$ $a_n \in \mathbb{Q}$ then the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ acts on $g_0^{-1}\mathrm{diag}(a_1, \dots, a_n) g_0 \in$ $A_0(L)$, by permuting the a_i's transitively.

Proof: Since $\Gamma_0 = \Gamma \cap A_0$ is Zariski dense in A_0, there is $\gamma \in \Gamma_0$ with distinct eigenvalues. Let L be a splitting field for the minimal polynomial P_γ of γ. Let us show that P_γ is irreducible over $\mathbb{Q}$. If not, there is a nontrivial γ-invariant proper $\mathbb{Q}$-subspace $V \subset \mathbb{R}^n$ (see e.g. [HK], Theorem 6.12). Since γ has distinct eigenvalues and $A_0 = C_G(\gamma)$, V is also an A_0-invariant subspace. The character $a \mapsto \det(a|_V)$ is then a nontrivial $\mathbb{Q}$-character and since it is trivial on integer matrices and Γ_0 is Zariski-dense, it must be trivial on A_0. This however implies that $\dim A_0 \le n-2$, which is impossible.

As a consequence $\Lambda = \mathrm{Gal}(L/\mathbb{Q})$ acts transitively on the eigenvalues of γ.

We know that there are $\theta_1, \dots, \theta_n \in L$ such that

$$g\gamma = \mathrm{diag}(\theta_1, \dots, \theta_n) g.$$

Thus the rows of g are left eigenvectors for γ. On the other hand, since $\gamma \in \mathrm{SL}(n, \mathbb{Z})$ there is a row vector $v_1 \in \mathbb{Q}(\theta_1)^n$ satisfying

$$v_1 \gamma_1 = \theta_1 v_1.$$

The group $\mathrm{Gal}(L/\mathbb{Q})$ permutes the eigenvalues of γ, and so the orbit of v_1 under $\mathrm{Gal}(L/\mathbb{Q})$ are n linearly independent eigenvectors: v_1 with eigenvalue θ_1, v_2 with eigenvalue θ_2, etc. Let

$$g_0 = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

i.e., the v_i's are the rows of the matrix g_0. It is clear that $g_0 \in \mathrm{GL}(n, L)$. Also, since each row of g_0 spans a one-dimensional eigenspace for γ, and the eigenvalues are permuted transitively, so are the eigenspaces. In particular, the ith row of g is the same as v_i up to multiplication by a scalar, and so $g_0 g^{-1}$ is a diagonal matrix. This implies that $g_0^{-1} A g_0 =$ $g^{-1} A g$. $\square$

Remark: Lemma 6.2 shows in particular that any compact orbit arises as in the example, with $K = \mathbb{Q}(\alpha)$ for some eigenvalue α of γ.

We are now ready for the

Proof of Theorem 1.3: Our first step is show that the assumptions of Theorem 1.1 are satisfied automatically:

Step 6.1. *If Ax is compact then for any $1 \le i \ne j \le n$, $N_{ij}x$ is not compact.*

Since Ax is a torus, it is enough to find some $z \in Ax$ such that $N_{ij}z$ is not compact. Thus we can replace x with $z = \hat{g}_0\Gamma$, where g_0 is as in Lemma 6.2, and

$$\hat{g}_0 = (\frac{1}{\det g_0})^{1/n} g_0.$$

If $N_{ij}z$ is compact then $N_0 = g_0^{-1} N_{ij} g_0$ intersects Γ co-compactly and is therefore defined over $\mathbb{Q}$. Therefore the action of Λ leaves $N_0(L)$ invariant. By Lemma 6.2, Λ permutes the eigenvalues of elements of the dense subset $g_0^{-1}A(\mathbb{Q})g_0$. Therefore for any $\sigma \in \Lambda$ there is a permutation (also denoted by σ) of $\{1, \dots, n\}$ such that $N_0^\sigma = g_0^{-1} N_{\sigma(i)\sigma(j)} g_0$. That is, for every $\sigma \in \Lambda$, $\{\sigma(i), \sigma(j)\} = \{i, j\}$. This contradicts the fact that the action of Λ on $\{1, \dots, n\}$ is transitive and $n \ge 3$.

Step 6.2. *Let H be as in the conclusion of theorem 1.1. Then H is in 'equal blocks form', i.e., there is a partition*

$$\{1, \dots, n\} = B_1 \sqcup \ldots \sqcup B_d, \tag{6}$$

where all the B_ℓ are of the same cardinality, such that

$$\mathrm{Lie}(H) = \mathrm{Lie}(A) \oplus \bigoplus_{\ell=1}^{d} (\bigoplus_{i,j \in B_\ell} \mathfrak{G}_{ij}). \tag{7}$$

The proof of Theorem 1.1 shows that $H = A \cdot S$ where S is semisimple, normalized by A, and S_x is a lattice in S. S consists of those block matrices in H such that the determinant of each block is 1. Moreover, if $x = \hat{g}_0\Gamma$ where $\hat{g}_0 = (\det g_0)^{-1/n} g_0$, g_0 as in Lemma 6.2, then Theorem 3.4 shows that $S_0 = g_0^{-1} S g_0$ is defined over $\mathbb{Q}$ and $S_0(\mathbb{Z})$ is a Zariski dense lattice in S_0. Using Lemma 6.2 we can also assume that there is a Galois field L over $\mathbb{Q}$ such that $\Lambda = \mathrm{Gal}(L/\mathbb{Q})$ acts on $g_0^{-1}S(\mathbb{Q})g_0$ by permuting the rows of g_0. Thus for each $\sigma \in \Lambda$ there is an associated permutation (also denoted by σ) of $\{1, \dots, n\}$ and a corresponding permutation matrix P_σ, such that for

$$h \in g_0^{-1} S_0(\mathbb{Q}) g_0, \ h = g_0^{-1} h' g_0$$

we have

$$h^\sigma = g_0^{-1} P_\sigma h' P_\sigma^{-1} g_0.$$

Thus the action of σ on $\{1, \dots, n\}$ preserves the partition (6). Since Λ is transitive on $\{1, \dots, n\}$, for every $1 \le \ell_1 \ne \ell_2 \le d$ there is $\sigma \in \Lambda$ such that $B_{\ell_1} \cap \sigma(B_{\ell_2}) \ne \emptyset$ and thus $\sigma(B_{\ell_1}) = B_{\ell_2}$. Therefore all the B_ℓ have the same cardinality.

Step 6.3. *If $H \neq A$ then Hx is noncompact.*

If $H \neq A$ there is some $1 \leq i \neq j \leq n$ such that $U_{ij} \subset H$. Thus it will suffice to prove that $AU_{ij}z\Gamma$ is an unbounded subset of G/Γ for any $z \in G$. For this we adapt the argument of [CaS-D].

We claim first that there are $z' \in AU_{ij}z$, $v \in \mathbb{Z}^n - \{0\}$, and some $1 \leq \ell \leq n$ such that $< z'_\ell, v >= 0$, where z'_i is the ith row of z' and $< \cdot, \cdot >$ is the standard inner product on $\mathbb{R}^n$. Indeed if for z there are no such v then for any $v \in \mathbb{Z}^n - \{0\}$ we can let $t = -\frac{<z_j,v>}{<z_i,v>}$ and $z' = (1 + tE_{ij})z$. Then $< z'_i, v >=< z_j + tz_i, v >= 0$.

Now we claim that $Az'\Gamma$ is unbounded. Recall that by Mahler's compactness criterion (see [Ra], Corollary 10.9) it is enough to find $a_k \in A$ and $v_k \in \mathbb{Z}^n - \{0\}$ such that $\|a_k z' v_k\| \to_{k\to\infty} 0$. This is satisfied by $v_k = v$ and any $a_k = \mathrm{diag}(a_1^k, \ldots, a_n^k) \in A$ such that for all $i \neq \ell$, $a_i^k \to_{k\to\infty} 0$.

□

Remarks:

1. The proof of Theorem 1.3 shows that any orbit-closure for A arises as in the example, with $K' = \mathbb{Q}(\alpha)$, where α is an eigenvalue of a generic matrix in $C_G(H) \cap A_x$.
2. If one is only interested in a proof of Corollary 1.4, the following shorter proof is available. Suppose $\overline{Ay} \neq Ay$. Then from Lemma 4.2 (which is easier in this case since $H = \{e\}$) there is U such that $\overline{Ux} \subset \overline{Ay}$. Then $\overline{Ux} = Hx$ by Ratner's theorem, and by Lemma 4.1 and Step 6.2, $H = G$.

7. An Isolation Result

In this section $n \geq 3$ is prime and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. We first state a corollary similar to Corollary 1.4:

Corollary 7.1. *Let F be a closed A-invariant subset of G/Γ containing a compact orbit Ax, and suppose that $F - Ax$ is not closed. Then $F = G/\Gamma$.*

Proof: Arguing as in the proof of Lemma 4.2, we obtain a unipotent subgroup U such that $Ux \subset F$. By Ratner's theorem $\overline{Ux} = Hx$ and by Lemma 4.1 and Step 6.2, $H = G$.

□

Now we turn to

Proof of Corollary 1.5: Any product h of n linearly independent forms can be represented by n vectors in $\mathbb{R}^n$. If we place these vectors as rows in a matrix, which we denote by $\tilde{h}$, we get an element of $\mathrm{GL}(n, \mathbb{R})$. Note that $\tilde{h}$ is not uniquely defined by h, but this will be of no importance. For our argument it will suffice to restrict attention to those $\tilde{h} \in G$.

Let

$$R(h) = \{(h(v_1), \dots , h(v_{n-1})) : \{v_1, \dots , v_{n-1}\} \text{ is primitive}\}.$$

Note that for every $a \in A$ and every $\gamma \in \Gamma$, $R(\tilde{h}) = R(a\tilde{h}\gamma)$. If the assertion is untrue, there is an open $V \subset \mathbb{R}^{n-1}$ and a sequence of forms h_k such that for each k, $V \cap R(\tilde{h_k}) = \emptyset$, $h_k \to f$ in the space of forms, and the h_k are not multiples of f. By multiplying $\tilde{h}_i$ and $\tilde{f}$ with the inverse of its determinant, we may assume that $\tilde{h}_i \in G$ for all $i \in \mathbb{N}$ and $\tilde{f} \in G$. Let

$$F = \overline{A\{\tilde{h}_k : k = 1, 2, \dots \}\Gamma}.$$

Our assumption imples that $V \cap R(\tilde{h}) = \emptyset$ for every $\tilde{h} \in F$. The projection of F to G/Γ is an A-invariant closed set. It contains $\tilde{f}\Gamma$. We will show momentarily that $A\tilde{f}\Gamma$ is compact. This will complete the proof since $F - A\tilde{f}\Gamma$ is not closed and then by Corollary 7.1, $F = G$, contradicting the easily established fact there exists $g \in G$ such that $V \cap R(\tilde{g}) \neq \emptyset$.

Let us show that $A\tilde{f}\Gamma$ is closed, or equivalently that $\tilde{A}\Gamma$ is closed, where $\tilde{A} = \tilde{f}^{-1}A\tilde{f}$. Since f is a product of n linearly independent form, $\tilde{A}$ is the connected component of A_f the stabilizer of f in G, where G acts on the right by $f^g(v) = f(gv)$. Since $\tilde{A}$ is normal and of finite index in A_f, to prove that $\tilde{A}\Gamma$ is closed it is enough to prove that $A_f\Gamma$ is closed. So let $a_k \in A_f, \gamma_k \in \Gamma$ such that $a_k\gamma_k \to z$. Then

$$f^z \leftarrow f^{a_k\gamma_k} = f^{\gamma_k}.$$

For each k the form f^{γ_k} also has integer coefficients and is therefore contained in a discrete subset of the set of products of n forms. Hence there is γ_0 such that $f^z = f^{\gamma_0}$, and therefore $z\gamma_0^{-1}$ stabilizes f. Thus $z \in A_f\Gamma$.

Now suppose $A\tilde{f}\Gamma$ is unbounded. By Mahler's compactness criterion (see [Ra], Corollary 10.9) this implies that there are $a_k \in A$ and $v_k \in \mathbb{Z}^n - \{0\}$ such that $\|a_k\tilde{f}v_k\| \to 0$. If we write $a_k = \operatorname{diag}(a_1^k, \dots , a_n^k)$ and $f = L_1 \cdot \ldots \cdot L_n$, where $L_i =< w_i, \cdot >$, then we get

$$\max_i |a_i^k L_i(v_k)| \to_{k\to\infty} 0$$

and therefore

$$\mathbb{Z} - \{0\} \ni f(v_k) = \prod_{i=1}^{n} a_i^k L_i(v_k) \to_{k\to\infty} 0.$$

$\square$

Appendix A. Proof of Theorem 3.4

In [Sh], Shah proves this theorem uncer the hypotheses that G is an algebraic group defined over $\mathbb{Q}$, $\Gamma = G(\mathbb{Z})$ and x is the coset Γ. More precisely, assertions 1 and 3 of the proposition are proved in [Sh], Proposition 3.2. Assertion 4 follows since

$$H(\mathbb{Q}) \subset \mathrm{Comm}_H(H(\mathbb{Z})),$$

and assertion 2 is Corollary 2.13 of [Sh]. We are interested in the case $G = \mathrm{SL}(n, \mathbb{R}),\ n \geq 3$. This case can be easily reduced to Shah's results using Margulis' arithmeticity theorem (see [Z], Chapter 6), as follows:

Suppose $x = g\Gamma$. Replacing H with $g^{-1}Hg$ and U with $g^{-1}Ug$ shows that there is no loss of generality in assuming that $g = e$. Recall that the arithmeticity theorem states that there is an algebraic group G' defined over $\mathbb{Q}$ and a surjective homomorphism $\rho : G' \to G$ such that $K' = \ker \rho$ is compact and Γ is commensurable with $\rho(G'(\mathbb{Z}))$. Let $U' \subset G'$ be a connected unipotent subgroup such that $\rho(U') = U$ (such a group exists since there is an embedding $\mathrm{Lie}(G) \subset \mathrm{Lie}(G')$). Let Γ' be a subgroup of finite index in $G'(\mathbb{Z})$ such that $\rho(\Gamma')$ is a finite index subgroup of Γ, and let x' be the coset Γ'. Then the map $\bar{\rho} : G'/\Gamma' \to G/\Gamma$ defined by $\bar{\rho}(g'\Gamma') = \rho(g')\Gamma$ is a proper map.

Let H' be a connected subgroup containing U' such that $\overline{U'G'(\mathbb{Z})} = H'G'(\mathbb{Z})$. By [Sh], H' satisfies the conclusions of the Theorem. Let us show that also $\overline{U'x'} = H'x'$. By Ratner's orbit-closure theorem, $\overline{U'x'} = H''x'$. Since $H'G'(\mathbb{Z})$ is closed and the map $G'/\Gamma' \to G'/G'(\mathbb{Z})$ is continuous and equivariant, $H'x'$ is closed and contains $U'x'$, and hence $H'' \subset H'$. Also

$$H' \subset \overline{U'G(\mathbb{Z})} = \overline{\bigcup_\gamma U'\Gamma'\gamma} = \bigcup_\gamma \overline{U'\Gamma'}\gamma = \bigcup_\gamma H''\Gamma'\gamma,$$

where the union is over a finite set of representatives of cosets in $\Gamma' \backslash G'(\mathbb{Z})$. Therefore we must have $\dim H' = \dim H''$ and hence by connectedness $H' = H''$.

By Ratner's orbit-closure theorem there is a connected group H such that $\overline{Ux} = Hx$. We claim that $H = \rho(H')$. Indeed, since $\bar{\rho}$ is proper, $\overline{Ux} \subset \rho(H')x$ and since $U'x'$ is dense in $H'x'$, its image $\bar{\rho}(U'x') = Ux$ is dense in $\bar{\rho}(H'x') = \rho(H)x$. Thus $Hx = \bar{\rho}(H')x$ and since Γ is discrete and H and $\rho(H')$ are connected, $H = \rho(H')$. Thus the assertions that H is an $\mathbb{R}$-subgroup and that the unipotent radical of H is equal to its radical follow from the corresponding ones for H'.

Now we claim that $\rho(H'_{x'})$ is of finite index in H_x. It is obvious that $\rho(H'_{x'}) \subset H_x$ and since both are lattices in H, the assertion follows.

Thus the remaining assertions about H_x follow from the corresponding ones for $H'_{x'}$.

$\square$

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