Measurable Distal and Topological Distal Systems

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Abstract

In this paper we prove that any ergodic measurably distal system can be realized as a minimal topologically distal system with an invariant Borel measure of full support. The proof depends upon a theorem stating that every measurable function from a measurable system with its base space being a compact metric space to a connected compact group is cohomologous to a continuous function.

§1. Introduction

A topological distal dynamical system is a dynamical system (X, T) (where X is a compact metric space and $T: X \to X$ a continuous invertible transformation) such that for every $a \neq b \in X$,

$$\inf_{i \in \mathbf{Z}} d(T^i a, T^i b) > 0.$$

A minimal dynamical system is a dynamical system with no closed invariant proper subsets. One of the basic facts about distal systems is that they decompose into the disjoint union of minimal systems.

In [3], H. Furstenberg proved a structure theorem for minimal topologically distal systems. If a dynamical system is topologically distal then any isometric topological extension of it is topologically distal, where an isometric topological extension is an extension of the original system (in the category of topological dynamic systems) such that each fiber is isometric to a homogeneous compact metric space (see the next section for more detailed definitions). H. Furstenberg proved that every minimal topologically distal system is the transfinite inverse limit of a tower of isometric topological extensions.

An analogous notion in the category of measurable systems is the notion of measurably distal. It was introduced by W. Parry in [6], and played a major role in H. Furstenbergs proof of Szemeredi's theorem[4]. The definition of measurably distal in [4] is particularly suitable to our needs. A measurable system is measurably distal if it is the transfinite inverse limit of a tower of isometric measurable extensions.

What we wish to prove is that every ergodic measurably distal system is isomorphic (as a measurable system) to a minimal topologically distal system equipped with a Borel measure μ with full support, and it is clear that the natural way to prove this is to consider first the connection between isometric measurable extensions and isometric topological extensions.

Our result can be seen as a generalization of Abramov's quasi-discrete spectrum results [1], which gives a simple structure, that can be viewed as a topological structure, for special measurably distal transformations (transformations with quasi-discrete spectrum).

In order to do this, we prove a partial generalization of a result due to D. Rudolph [7] for cocycles from a measurable **Z**-action on a compact metric space to a connected compact group. This theorem may be of interest by itself.

While we prove that every measurably distal system has a topologically distal model, it is not in general true that this model can be chosen to be uniquely ergodic: indeed, a finite to one measurable extension of $\mathbf{R}/\mathbf{Z}$ (equipped with Haar measure and the m.p.t. corresponding to addition of an irrational number α) that is not a Kronecker system does not have a uniquely ergodic model. We discuss the issue of unique ergodicity in section 4.

The results described in this paper are part of the author's PHD thesis, conducted under the guidance of Professor Benjamin Weiss from the Hebrew University. The author is grateful to Professor Eli Glasner of Tel Aviv University and to B. Weiss for many discussions about these topics. Some of the ideas presented in this paper originated from these discussions. The problem of the relation between measurable distal systems and topological distal system was raise by M. Foreman in connection to certain descriptive set theoretical questions about distal systems [2].

§2. Preliminaries

In order to state this result we must first recall some definitions. Let M be a homogeneous compact metric space, i.e. a compact metric space such that for every two points x, $y \in M$ there is an isometry of M mapping x to y.

Definition (2.1) *Let X and Y be compact metric spaces, and let pi be a continuous map from X onto Y. Then X is an M* **bundle over** *Y if there is a function $\rho(x_1, x_2)$ defined for all pairs x_1 and x_2 from the same fiber (i.e. $\pi(x_1) = \pi(x_2)$), such that*

a. ρ is a continuous function on the subset of $X \times X$ defined by the condition $\pi(x_1) = \pi(x_2)$.

b. For each $y \in Y$, ρ is a metric on the fiber $\pi^{-1}(y)$ under which $\pi^{-1}(y)$ is isometric to M.

Definition (2.2) *Let (Y, S) and (X, T) be dynamical systems, with X an M bundle over Y. Let π, $\rho(\cdot, \cdot)$ as in definition (2.1). Then the dynamical system (X, T) is a* **(topological) isometric extension** *of (Y, S) if the following two conditions hold:*

a. $\pi \circ T = S \circ \pi$.

b. For every x_1 and x_2 from the same fiber of X over Y, $\rho(x_1, x_2) = \rho(Tx_1, Tx_2)$.

One possible way to construct an isometric extension of (Y, S) is the following:

Example (2.3) *If (X, T) is a dynamical system, M a homogeneous compact metric space with $K = \mathrm{Isom}(M)$ its group of isometries, and $f: X \to K$ a continuous function then $X \times M$ equipped with the transformation*

$$T: (x, m) \mapsto (Tx, f(x)m)$$

is an isometric topological extension of (X, T)*. We shall call a dynamical system derived by this process the* **(topological) skew product of** X **and** M **corresponding to** f*.*

Note that the above example makes sense even if f is not continuous but merely measurable. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic system, and let M be a symmetric space for the compact metric group K. To any measurable function $f: X \to K$ there corresponds a measurable transformation $T_f: X \times M \to X \times M$ defined as follows:

$$T_f: (x, m) \mapsto (Tx, f(x) \cdot m).$$

If ν is a measure on $X \times M$ preserved by T_f that projects into the measure μ on X then the resulting measure preserving system will be called a **skew product of** X **and** M **corresponding to** f **(and** ν**)**, or, alternatively, **an extension of** X **by** M.

Definition (2.4) *A measurable extension* $(Y, \mathcal{D}, \nu, S)$ *of a system* $(X, \mathcal{B}, \mu, T)$ *is a* **measurable isometric extension** *if* $(Y, \mathcal{D}, \nu, S)$ *is a skew product of* X *and some homogeneous space* M *of a compact metric group.*

Different functions can lead to isomorphic skew products. Indeed for any measurable function $h: X \to K$ the mapping $Id_h: (x, k) \mapsto (x, h(x)k)$ is an isomorphism between the skew product corresponding to $f(x)$ and ν, and the skew product corresponding to $h(Tx)f(x)h(x)^{-1}$ and $Id_h(\nu)$ as can be seen in the following diagram:

$$\begin{array}{ccc}
\big(x, k\big) & \overset{Id_h}{\longmapsto} & \big(x, h(x)k\big) \\
\Big\downarrow{\scriptstyle T_f} & & \Big\downarrow{\scriptstyle T_{h\circ T\cdot f\cdot h^{-1}}} \\
\big(Tx, f(x)k\big) & \overset{Id_h}{\longmapsto} & \big(Tx, h(Tx)f(x)k\big)
\end{array}$$

The discussion above motivates the following definition of an equivalence relation between measurable functions $X \to K$:

Definition (2.5) *Two measurable functions* $X \to K$*,* f *and* g*, are called* **cohomologous** *if there exists a measurable* $h: X \to K$ *such that*

$$g(x) \equiv h(Tx)f(x)h(x)^{-1}.$$

Suppose now that X is a compact metric space, and T continuous. Thus (X, T) can be viewed both as a m.p.s. and a topological dynamical system. What we want to show is that any measurable isometric extension of (X, T) can be realized as a topological isometric extension of (X, T) (in fact, even as a topological skew product of X and a symmetric space) equipped with some Borel measure. If we prove this then a simple inductive argument will prove our main theorem.

A measurable isometric extension (over a given symmetric space) is given by a function $f: X \to K$, and a measure ν. If we show that f is cohomologous to a continuous function (say by some function h) then indeed we get an isomorphism of measurable system from our original measurable extension to a topological isometric extension with some Borel measure $(I_h(\nu))$.

Unfortunately, it is not always true that one can find a continuous representative in every cohomology class. For instance if X is connected, and K is totaly disconnected, then any continuous function from X to K is constant, but in general there are non trivial measurable extensions of X by K.

The main technical result we need in this paper is that if K is a connected compact metric group then every cohomology class contains a continuous representative. This will prove that any measurable isometric extension of a system (X,T) by a symmetric space of a connected group is isomorphic to a topological isometric extention of (X,T) equipped with some Borel measure on the extended system.

We shall assume in this paper that all measure spaces are Lebesgue, and all groups metric.

§3. Cohomology classes for extensions by a compact and connected group

In this section we will prove our main technical result, namely:

Theorem (3.1) *Let X be a compact metric space, μ a borel measure on X and $T\colon X \to X$ a measure preserving transformation such that the set of periodic points has measure zero. Let K be a compact connected metric group. Then any measurable function from X to K is cohomologous to a continuous function.*

We first prove an easy Lemma (essentialy a rephrasing of a well known result):

Lemma (3.2) *Let K be a compact connected group, U an open neighbourhood of the identity $e \in K$. Then for every N large enough there exist N measurable functions $r_1, \ldots, r_N\colon K^N \to U$ so that for every $g_1, g_2, \ldots, g_N$ in K,*

$$r_1(g_1, g_2, \ldots, g_N)\, g_1\, r_2(g_1, g_2, \ldots, g_N) \cdots g_{N-1}\, r_N(g_1, g_2, \ldots, g_N) = g_N^{-1}. \tag{3.2.1}$$

Proof of Lemma.

We first prove that for any open $U \ni e$ it is possible to define such functions $r_1, \ldots, r_N$, then that we can choose them so as to be measurable.

To prove the existence of (not necessarily measurable) functions with the property (3.2.1), it is sufficient to prove that for N large enough, for any $g_1, \ldots, g_{N-1}$,

$$Ug_1Ug_2 \cdots g_{N-1}U = K,$$

or equivalently that for any $g_1, \ldots, g_{N-1}$

$$A^{g_1}(U)A^{g_2}(U) \cdots A^{g_{N-1}}(U) = K, \tag{3.2.2}$$

where $A^g(U) := g^{-1}Ug$. In a compact connected group any open set U satisfies $U^N = K$ for N large enough. And so we see that to prove (3.2.2) it suffice to prove that the set

$$\bigcap_{g \in K} A^g(U)$$

contains an open neighbourhood of e.

Let $V \ni e$ be an open symetric (i.e. $V^{-1} = V$) set such that $V^3 \subset U$. Then for any $k \in K$ and any $g \in Vk$,

$$A^k(V) \subset A^g(U).$$

Since K is compact there are a finite number of elements $\gamma_1, \ldots, \gamma_r$ of K so that

$$\bigcup_{i=1}^{r} V\gamma_i = K,$$

and so

$$\bigcap_{i=1}^{r} A^{\gamma_i}(V) \subset \bigcap_{g\in K} A^g(U).$$

Since the left hand side of the above inequality is an open set containing the unit e we have proved the first part of this lemma.

It remains to show we can chose the functions $r_1, \ldots, r_N$ in a measurable way. Let $U' \ni e$ be an open symetric set with ${U'}^2 \subset U$. We apply the first part of the proof to the set U', and suppose N is large and $k_1, \ldots, k_r \in K$ are such that

$$\bigcup_{i=1}^{r} U'k_i = K.$$

We know that for every sequence $i_1, \ldots, i_N$, there are $r_1^{i_1,\ldots,i_N}, \ldots, r_N^{i_1,\ldots,i_N}$ satisfying

$$r_1^{i_1,\ldots,i_N} k_{i_1} \cdots k_{i_{N-1}} r_N^{i_1,\ldots,i_N} = k_{i_N}^{-1}.$$

Now, Let

$$\mathcal{P} := \left\{P^{i_1,\ldots,i_N}\right\}_{i_1,\ldots,i_N\in\{1,\ldots,r\}}$$

be a partition of K^N into measurable sets, with each element $P^{i_1,\ldots,i_N}$ satisfying

$$P^{i_1,\ldots,i_N} \subset U'k_{i_1} \times U'k_{i_2} \times \cdots \times U'k_{i_N}.$$

We can now define the functions $r_j \colon K^N \to U$ for $j = 1, \ldots, N$, by

$$r_j(g_1, \ldots, g_N) := r_j^{i_1,\ldots,i_N} k_{i_j} g_j^{-1},$$

where $P^{i_1,\ldots,i_N}$ is the unique element of $\mathcal{P}$ that contains $(g_1, \ldots, g_N)$.

Since $g_j \in U'k_{i_j}$, that is $k_{i_j} g_j^{-1} \in {U'}^{-1}$, we have

$$r_j(g_1, \ldots, g_N) \subset U' \cdot {U'}^{-1} \subset U,$$

and

$$r_1(g_1,\dots,g_N)g_1\cdots g_{N-1}r_N(g_1,\dots,g_N) =$$
$$r_1^{i_1,\dots,i_N}k_{i_1}g_1^{-1}\cdot g_1\cdots k_{i_{N-1}}g_{N-1}^{-1}\cdot g_{N-1}\cdot r_N^{i_1,\dots,i_N}k_{i_N}g_N^{-1} =$$
$$r_1^{i_1,\dots,i_N}k_{i_1}\cdots k_{i_{N-1}}r_N^{i_1,\dots,i_N}k_{i_N}g_N^{-1} = g_N^{-1}.$$

□

Let K be a compact metrizable group. Then K has some bounded bi-invariant metric d, and we can suppose without loss of generality that the diameter of K, according to the metric d is 1. For any open set $U \in K$, let

$$M(X;U) = \text{measurable functions } X \to U,$$

and

$$C(X;U) = \text{continuous functions } X \to U.$$

We will use two metrics d^1 and d^∞ on $M(X) := M(X;K)$:

$$d^1(f,g) := \int_X d(f(x),g(x))d\mu(x),$$

and

$$d^\infty(f,g) := \operatorname*{ess\,sup}_{x\in X} d(f(x),g(x)).$$

For our proof, we will need to show that functions in $M(X;U)$ can be approximated (in the d^1 metric) by functions of $C(X;U)$ for many U's.

Lemma (3.3) *Let K, X be as above. If K is connected then there is a neighbourhood basis $\{U(n)\}$ of $e \in K$ such that $C(X;U(n))$ is dense in $M(X;U(n))$ in the d_1 metric (with $U(0) = k$).*

Proof.

Since K is a compact connected metric group, it is the inverse limit of the Lie groups L_i (say), with $L_i \subset \mathrm{SU}(n_i)$. Let $\pi_i^j : L_j \to L_i$ and $\pi_i : K \to L_i$ be the projections that arise from the definition of an inverse limit, $\mathcal{L}_i \subset su(n_i)$ be the Lie algebra associated with L_i, and $\Pi_i^j : \mathcal{L}_j \to \mathcal{L}_i$ the lie algebra homomorphism corresponding to π_i^j.

Define

$$U(n) := \pi_n^{-1}\exp\left(u(n)\right),$$

with $u(n) \subset \mathcal{L}_n$ a small convex open set containing 0. Since by definition $\operatorname{diam}\pi_n^{-1}(e) \to 0$ as $n \to \infty$, it is clear that if the $u(n)$ are small enough, $U(n)$ will be a neighbourhood basis of e. We choose $u(0) = \mathcal{L}_0$, and so $U(0) = K$.

We now wish to show that $C(X;U(n))$ is dense in $M(X;U(n))$. Let f be any function in $M(X;U(n))$, and $\epsilon > 0$. We will find $\bar{f} \in C(X;U(n))$ such that $d^1(f,\bar{f}) < 3\epsilon$.

Since X is compact we can find N disjoint closed sets $A_i \subset X$ such that

$$\mu(X \setminus \bigcup_1^N A_i) < \epsilon,$$

and such that

$$\operatorname{diam}(f(A_i)) < \epsilon.$$

For every i let f_i be some point in $f(A_i)$, let $m \geq n$ be a number large enough so that $\operatorname{diam}(\pi_m^{-1}(e)) < \epsilon$ and

$$\eta < \frac{1}{2} \min_{i,j} d(A_i, A_j).$$

Since the following diagram commutes

$$\begin{array}{ccc} L_m & \xleftarrow{\exp} & \mathcal{L}_m \\ \downarrow{\scriptstyle \pi_n^m} & & \downarrow{\scriptstyle \Pi_n^m} \\ L_n & \xleftarrow{\exp} & \mathcal{L}_n \end{array},$$

we see that

$$U(n) = \pi_n^{-1} \exp(u(n)) = \pi_m^{-1} \exp\left(\Pi_n^{m\,-1} u(n)\right).$$

Notice that $\Pi_n^{m\,-1} u(n)$ is a convex set (since Π_n^m is linear). As f_i is in $U(n)$, there is for every i an $\hat{f}_i \in \left(\Pi_n^{m\,-1} u(n)\right)$ such that $\pi_m f_i = \exp \hat{f}_i$.

At last we are ready to begin to construct $\bar{f}(x)$ itself. As a first step let us define $\bar{f}_m(x) := \pi_m \bar{f}(x)$, and then show how to lift this function to a function $X \to K$. For $x \in A_i$, we define $\bar{f}_m(x) = \pi_m(f_i)$. If $\min_i d(x, A_i) \geq \eta$, we define $\bar{f}_m(x) = e$. If $x \in X \setminus \bigcup_1^N A_i$, then x can be at a distance of η or smaller from only one set A_i, say $A_{i'}$. In this case we define

$$\bar{f}_m(x) = \exp\left(\left(1 - \eta^{-1} d(x, A_{i'})\right) \hat{f}_{i'}\right).$$

$\bar{f}_m(x)$ is a continuous function $X \to L_m$, and if $\bar{f}(x)$ is any function from $X \to K$ such that $\bar{f}_m = \pi_m \bar{f}$ then since $\operatorname{diam} \pi_m^{-1}(e) < \epsilon$ we have for any $x \in A_i$,

$$d(f(x), \bar{f}(x)) < d(f(x), f_i) + d(f_i, \bar{f}(x)) < 2\epsilon.$$

Thus if we can lift $\bar{f}_m(x)$ to a function $\bar{f}(x)$, we will have

$$\begin{aligned} d^1(f, \bar{f}) &= \int_{\bigcup A_i} d(f(x), \bar{f}(x)) d\mu + \int_{X \setminus \bigcup A_i} d(f(x), \bar{(}f(x))) d\mu \\ &\leq \int_X 2\epsilon d\mu + \mu(X \setminus \bigcup A_i) \leq 3\epsilon \end{aligned}.$$

$\bar{f}_n$ was defined using the one parameter subgroups $\left\{\exp(t\hat{f}_i)\right\}_{t \in \mathbf{R}}$ of L_m, for $i = 1, \ldots,$ N. One parameter subgroups have the nice property that if $L \to L'$ is a homomorphism of Lie groups then one parameter subgroups of L' can be lifted to be one parameter subgroups of L. Thus inductively we see that the one parameter subgroups that interest us can be lifted (in a consistent way) to every $L_{m'}$, and so eventually to K. If for every i we lift $\left\{\exp(t\hat{f}_i)\right\}_{t \in \mathbf{R}}$ to the one parameter subgroup $\{k_i(t)\}$ of K, we can define $\bar{f}(x)$ as follows:

$$\bar{f}(x) := \begin{cases} k_i\left(1 - \eta^{-1} d(x, A_i)\right), & \text{if } d(x, A_i) < \eta; \\ e, & \text{otherwise.} \end{cases}$$

$\square$

Remark Since the $U(n)$ constructed in the above lemma are a neighbourhood basis for e, we can assume $\mathrm{diam}(U(n)) \leq 2^{-n}$.

We now return to prove theorem (3.1)

Theorem (3.1) *Let X be a compact metric space, μ a Borel measure on X, T a measurable transformation such that the set of T periodic points has measure 0. Let K be a connected compact metric group. Then any function in $M(X;K)$ is cohomologous to a function in $C(X;K)$.*

Proof.

Let $m \in M(X;K)$. We will construct sequences of functions $c_n \in C(X;K)$ and $h_n \in M(X;K)$ so that for every $n \geq 0$,

$$c_n(x) \in h_n(Tx)^{-1} m(x) h_n(x) U(n) \tag{3.4.1}$$

(and so in particular

$$d^\infty\left(h_n(Tx)c_n(x)h_n(x)^{-1}, m(x)\right) < 2^{-n}\Big),$$

and for every $n \geq 1$ we also demand that

$$\mu(x : h_n(x) \neq h_{n-1}(x)) < 2^{-n}), \tag{3.4.2}$$

and

$$d^\infty(c_n(x), c_{n-1}(x)) < 2^{-n}. \tag{3.4.3}$$

Given such a sequence of functions, for a.e. x, we have by the Borel-Canteli Lemma that $h_n(x)$ is a constant a.e. (which we shall call $h(x)$) for n large enough. Furthermore, $c_n(x)$ is a uniformly Cauchy sequence and so converges uniformly to a continuous function $c(x)$. Using (3.4.1) it is immediate that for any $\epsilon > 0$, if n is large enough then for any x not in some set of measure smaller then ϵ (the set $h(x) \neq h_n(x)$),

$$d\left(h(Tx)c_n(x)h(x)^{-1}, m(x)\right) < \frac{\epsilon}{2},$$

and so, since if n is large enough $d(c_n(x), c(x)) < \frac{\epsilon}{2}$ for all x (and the metric d is bi-invariant)

$$d\left(h(Tx)c(x)h(x)^{-1}, m(x)\right) < \epsilon,$$

for all x not in some set of measure smaller then ϵ. Thus, since ϵ is arbitrary, for a.e. x,

$$h(Tx)c(x)h(x)^{-1} = m(x).$$

It remains to construct such sequences of functions. Chose $c_0(x) \equiv e$ and $h_0(x) \equiv e$ — this satisfies condition (3.4.1) for $n = 0$.

Suppose now that $c_{n-1}(x)$ and $h_{n-1}(x)$ are already defined. Use (3.2) to find an N large enough so that (3.2.1) is satisfied for U_n and N. Since $(X, \mathcal{B}, \mu, T)$ is ergodic, it follows from the Rochlin Tower Lemma that there exists some set B so that

$$B \cap T^{-k}B = \emptyset \qquad \text{for } k < N-1,$$

but

$$X \subset \bigcup_{k=0}^{N} T^{-k}B. \tag{3.4.4}$$

We can rephrase (3.4.4) by saying that for every $x \in X$ there is a measurable 'projection' $b(x)\colon X \to B$ such that for all x there is a $0 \geq k \geq N$ with

$$T^k b(x) = x,$$

and $x \in B$ iff $x = b(x)$. For every $x \in B$ let

$$n(x) := \min(k : T^k x \in B).$$

We know from (3.4.1) that

$$\phi(x) := h_{n-1}(x)^{-1} m(x)^{-1} h_{n-1}(Tx) c_{n-1}(x) \in U(n-1),$$

and so $\phi \in M(X; U(n-1))$. Since $C(X; U(n-1))$ is dense in $M(X; U(n-1))$ we can find a continuous $\xi \in M(X; U(n-1))$ such that

$$\phi(x)\xi(x)^{-1} \in U(n)$$

for all x not in a set R_n of measure smaller then $\frac{2^{-n}}{(N+1)}$. Enlarge the set of bad points to $\bar{R}_n$ as follows:

$$\bar{R}_n = \{x : \exists y \in R_n \text{ s.t. } b(x) = b(y)\}.$$

Clearly $\mu(\bar{R}_n) \leq (N+1)\mu(R_n) \leq 2^{-n}$.

Set

$$c_n(x) = c_{n-1}(x)\xi(x)^{-1} \qquad \text{for all } x,$$

and

$$h_n(x) = h_{n-1}(x) \qquad \text{for all } x \notin R_n.$$

Already we see that (3.4.2) and (3.4.3) hold, and also (3.4.1)for all $x \notin \bar{R}_n$. We now wish to define $h_n(x)$ on the set of bad points $\bar{R}_n$, and we already commit ourselves that $h_n(x) = h_{n-1}(x)$ on $x \in b(\bar{R}_n)$.

Recall the conditions h_n needs to satisfy. One is that for every $x \in b(\bar{R}_n)$ and $k < n(x)$, the element

$$u_n^k(x) := h_n(T^k x)^{-1} m(T^k x)^{-1} h_n(T^{k+1}x) c_n(T^k x),$$

is in $U(n)$, the other that $h_n(T^{n(x)}x) = h_{n-1}(T^{n(x)}x)$. The first condition translates to

$$h_n(T^{k+1}x) = m(T^k x)h_n(T^k x)u_n^k(x)c_n(T^k x)^{-1}, \tag{3.4.5}$$

and so we need to show that we can chose measurably $u_n^k(x)$ so that

$$h_{n-1}(T^{n(x)}x) = h_n(T^{n(x)}x) = \\ m(T^{n(x)-1}x)\cdots m(x)h_{n-1}(x)u_n^0(x)c_n(x)^{-1}\cdots u_n^{n(x)-1}(x)c_n(T^{n(x)-1}x)^{-1}.$$

Define for every $x \in b(\bar{R}_n)$,

$$M'(x) := \Big(m(T^{n(x)-1}x)\cdots m(x)h_{n-1}(x)\Big)^{-1} h_{n-1}(T^{n(x)}x).$$

$M'(x)$ is a measurable function, and we are left with trying to find in a measurable way $u_n^0(x), \ldots, u_n^{n(x)-1}(x) \in U(n)$ satisfying

$$u_n^0(x)c_n(x)^{-1}\cdots u_n^{n(x)-1}(x)c_n(T^{n(x)-1}x)^{-1} = M'(x),$$

and this is an immediate corollary of lemma (3.2). Given the u_n^k's we can define measurably $h_n(T^k x)$ for all $x \in b(\bar{R}_n)$ and $0 \le k < n(x)$ using equation (3.4.5).

Thus we can define inductively h_n and c_n if we are given h_{n-1} and c_{n-1}, and we have seen that from the existence of sequences h_n and c_n with properties (3.4.1)– (3.4.3) the theorem follows. □

As we have seen in §2, the following is an immediate corollary from this theorem:

Corollary (3.5) *Any measurable isometric extension of a dynamical system (X, T), together with an ergodic measure μ by a homogeneous space M of a connected compact metric group G can be represented as the measurable system derived from a topologically isometric extension of X using some Borel measure ν on the extended system that projects to the measure μ on X. In fact this topologically isometric extension can be chosen to be a skew product of the system X with the homogeneous space M.*

§4. Topologically distal and measurably distal systems

In this section we use the results of the previous section to prove via a straight-forward induction argument our main result. We first give an exact definition of measurable distal systems, and quote H. Furstenberg's structure theorem for minimal topologically distal systems [3]. Note that both of these depend on the definitions given in §2.

Definition (4.1) *A system $(X, \mathcal{B}, \mu, T)$ is called* **measurably distal** *if it has a sequence of (measurable) factors indexed by ordinals X_η, $\eta \le \eta_0$ such that $X = X_{\eta_0}$, X_0 the trivial system, $X_{\eta+1}$ a measurable isometric extension of X_η and $X_{\lim \xi} = \text{inverse}\lim_{\xi' < \xi} X_{\xi'}$.*

Theorem (4.2) *Every minimal topologically distal system has a sequence of (topological) factors indexed by ordinals X_η, $\eta \le \eta_0$ such that $X = X_{\eta_0}$, X_0 the trivial system, $X_{\eta+1}$ a topologically isometric extension of X_η and $X_{\lim \xi} = \text{inverse}\lim_{\xi' < \xi} X_{\xi'}$. Every dynamical system that has such a sequence is topologically distal.*

Definition (4.3) *For both the topological and measurable case, the sequence X_η togather with the collection of (compatible) factor transformations $X_\eta \to X_{\eta'}$ $(\eta > \eta')$ is called a* **distal tower** *and η_0 is called the* **height** *of the distal tower.*

Since we assume that the measure space of our measurably distal system is Lebesgue, and that the topologically distal system is metricable it is easy to see that in both cases one can assume that $\eta_0 < \Omega$, the smallest non-denumerable ordinal.

We now return to our main result:

Theorem (4.4) *Every ergodic measurable distal system $(X, \mathcal{B}, \mu, T)$ can be represented as a minimal topologically distal system equipped with a borel measure of full support.*

Proof.

We first prove the theorem by induction on the height of the distal tower for the measurably distal system, initially without the additional condition on the full support of the measure. If $\{(X_\eta, \mathcal{B}_\eta, \mu_\eta, T_\eta\}$ is the measurable distal tower, we will construct a topological distal tower $\{(\hat{X}_\eta, \hat{T}_\eta)\}$ and measures $\{\mu_\eta\}$ such that

$$(X_\eta, \mathcal{B}_\eta, \mu_\eta, T_\eta) \cong (\hat{X}_\eta, \mathcal{B}(\hat{X}_\eta), \hat{\mu}_\eta \hat{T}_\eta) \tag{4.4.1}$$

and such that these isomorphisms are consistent with the factor transformations of both distal towers.

If the height of the tower is 0, then X consists of a single point and is also topologically distal (of same height).

If we have a tower of height $\eta + 1$, we first find a topologically distal system $\hat{X}_\eta$ with tower of height η and a Borel measure $\hat{\mu}_\eta$ so that (4.4.1) is satisfied for η. Now using this isomorphism, we can view $X_{\eta+1}$ as a measurable isometric extension of $\hat{X}_\eta$, say by a homogeneous space M of the metric compact group K. Since unitary representations separate points of any compact group, and if, in addition, the group is metric a countable number suffice, without loss of generality we see that K imbeds in

$$G := \mathrm{SU}(1) \times \mathrm{SU}(2) \times \cdots \times \mathrm{SU}(n) \times \cdots,$$

and so a symmetric space $M \cong K/L$ (where L is the closed subgroup of G that fixes one arbitrary element of M) is imbedded in the symmetric space G/L. Now the original measure on $X \times M$ can be seen as a measure on $X \times (G/L)$. Since G is connected we can use corollary (3.5) to construct a topologically isometric extension of $\hat{X}_\eta$, say $\hat{X}_{\eta+1}$ and measure $\hat{\mu}_\eta$ so that (4.4.1) is satisfied for $\eta + 1$ and such that the factor transformation $\hat{X}_{\eta+1} \to \hat{X}_\eta$ projects the measure $\mu_{\eta+1}$ to μ_η. This guarantees that the isomorphisms $X_{\eta+1} \cong \hat{X}_{\eta+1}$ and $X_\eta \cong \hat{X}_\eta$ are compatible with the factor transformations $X_{\eta+1} \to X_\eta$ and $\hat{X}_{\eta+1} \to \hat{X}_\eta$.

For limiting ordinals ξ define

$$\hat{X}_\xi := \text{inverse} \lim_{\xi' < \xi} \hat{X}_{\xi'}$$

and the appropriate limiting measure $\hat{\mu}_\xi$ can be defined by the condition

$$\hat{\mu}_\xi(A) = \hat{\mu}_{\xi'}(A) \qquad \text{for every } A \in \mathcal{B}(\hat{X}_\xi).$$

(4.4.1) and the consistency criterion are still satisfied — this is an easy consequence of them being satisfied for $\xi' < \xi$ (since we can assume $\xi \leq \mu_0 < \Omega$ the inverse limit is taken for a denumerable sequence of factors we need not worry about pathological difficulties arising from non denumerable limits).

Thus we have proven by induction that X is isomorphic to a topologically distal system $\hat{X}$ with measure $\hat{\mu}$. If we restrict ourselves to $\bar{X} := \mathrm{supp}_{\hat{X}}(\hat{\mu})$ we get a subsystem of $\hat{X}$ isomorphic to X as a measure preserving system and such that $\hat{\mu}$ has full support on $\bar{X}$. An easy consequence of the naive topological definition of distallity as systems for which every two points never get arbitrarily close is that topological distallity is inherited to subsystems, and so $\bar{X}$ is also topologically distal.

Since $\bar{X}$ is topologically distal, and has an ergodic measure $\hat{\mu}$ with full support it must also be minimal. This can be seen as follows: Any distal system is the disjoint union of minimal systems, and for any ergodic measure on a topological dynamical system there is a point whose orbit is dense in the support of the measure (this is an easy consequence of the Pointwise Ergodic Theorem). This point (and thus its whole orbit) must be in one of the minimal components of $\bar{X}$, and so the support of $\hat{\mu}$, which is $\bar{X}$, must be a minimal component. ◻

§5. Unique ergodicity

In this section we begin to investigate the question whether the topologically distal model of a measurable distal system can be chosen so as to be uniquely ergodic.

Proposition (5.1) *If $(X, \mathcal{B}, \mu, T)$ is a measurably distal system, with a distal tower such that all the measurable isometric extensions are by compact connected groups, then X has a uniquely ergodic topological distal model (in fact, the topologically distal system constructed in the previous section will be uniquely ergodic in this case).*

This proposition is a corollary of the following observation, due to H. Furstenberg. Recall a point $x \in X$ is called generic with respect to μ if for every continuous function $X \to \mathbf{R}$,

$$\frac{1}{n}\sum_{i=1}^{n} f(T^i x) \to \int f d\mu,$$

as $n \to \infty$.

Lemma (5.2) *Let (X, T) be a uniquely ergodic dynamical system, and let (Y, S) be a skew product of (X, T) and a compact group G corresponding to a continuous function $h: X \to G$ (see Example (2.3)). Let μ be the unique invariant measure on X, and let $\tilde{\mu}$ be the measure $\mu \times \mathcal{H}$ on Y, where $\mathcal{H}$ is the Haar measure on G. If $\tilde{\mu}$ is ergodic, then it is the only S invariant measure on Y (i.e. Y is uniquely ergodic).*

Proof. Suppose that y_0 is any generic point for $(Y, S, \tilde{\mu}, \mathcal{D})$. Then any $g \in G$ acts as an $\tilde{\mu}$-preserving transformation of Y (taking $(x, g') \mapsto (x, g'g)$). Recall that S takes $(x, g') \mapsto (Tx, h(x)g')$, so S and g commute. Then for any continuous $f: Y \to \mathbf{R}$,

$$\frac{1}{N}\sum_{i=0}^{N-1} f(S^i(y_0 g)) = \frac{1}{N}\sum_{i=0}^{N-1} f(S^i(y_0)g) \to$$
$$\int f(yg)\, d\tilde{\mu}(y) = \int f(y)\, d\tilde{\mu}(y) \qquad \text{as } N \to \infty,$$

and so $y_0 g$ is also a generic point. Thus if y_0 is generic, the whole fiber of y_0 over X is generic. By ergodicity, $\tilde{\mu}$-a.e. $y \in Y$ is $\tilde{\mu}$-generic, i.e. for almost every x the fiber of Y over x contains only $\tilde{\mu}$-generic points (and thus does not contain ν-generic points for any other S invariant measure).

Suppose now that ν is another S invariant measure on Y. As ν must project to the measure μ on X, and since ν-a.e point is ν-generic, for a.e. $x \in X$ the fiber of Y over x must contain a ν-generic point— a contradiction. $\square$

Proposition (5.1) follows, since the system $\hat{X}$ constructed in the proof of Theorem (4.4) can easily be seen to be uniquely ergodic using the above lemma (as all the homogeneous spaces involved are connected compact groups by the assumptions of the Proposition). And if $\hat{X}$ is uniquely ergodic, then clearly $\bar{X}$, the support of the (unique) invariant measure on $\hat{X}$, must be uniquely ergodic.

We now show that there are measurable distal systems without any uniquely ergodic topologically distal models.

Lemma (5.3) *Let $T_\alpha: \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ be the mapping $x \mapsto x + \alpha \bmod 1$ for an irrational α. Then any (topological) isometric extension (Y, S) of $(\mathbf{R}/\mathbf{Z}, T_\alpha)$ with finite fibers is a Kronecker system.*

Proof. Let $\pi: Y \to \mathbf{R}/\mathbf{Z}$ be the finite to one (say p to 1) projection of Y onto the factor $\mathbf{R}/\mathbf{Z}$. By compactness of $\mathbf{R}/\mathbf{Z}$ there is an $\epsilon > 0$ such that for any $x \in \mathbf{R}/\mathbf{Z}$, $\pi^{-1}((x-\epsilon, x+\epsilon))$ is composed of p disjoint intervals homeomorphic to $(x-\epsilon, x+\epsilon)$. The metric $d(\cdot,\cdot)$ defined by $d(y_1, y_2) = |\pi(y_1) - \pi(y_2)|$, if $|\pi(y_1) - \pi(y_2)| < \epsilon$ and y_1 and y_2 on the same connected component of $\pi^{-1}(\pi((y_1 - \epsilon, y_1 + \epsilon)))$, and $d(y_1, y_2) = \epsilon$ otherwise, is an invariant metric of Y. $\square$

More general results of this type can be found in [5] and [8]. Let λ denote the Lebesgue measure on $\mathbf{R}/\mathbf{Z}$.

Lemma (5.4) *Any (measurable) factor of $(\mathbf{R}/\mathbf{Z}, T_\alpha, \mathcal{B}, \lambda)$ with factor transformation ϕ is (measurably) isomorphic to $(\mathbf{R}/\mathbf{Z}, T_{n\alpha}, \mathcal{B}, \lambda)$ with the factor transformation $x \mapsto nx \bmod 1$ for some $n \in \mathbf{Z}$.*

This result is rather standard, and can be proved e.g. by considering the L^2 space of the factor system as a closed subspace of the L^2 space of the original system.

Claim (5.5) *Suppose $(Y, T, \mathcal{D}, \nu)$ is an isometric extension of $(\mathbf{R}/\mathbf{Z}, T_\alpha, \mathcal{B}, \lambda)$ by a finite group, such that Y is not a Kronecker system. Then $(Y, T, \mathcal{D}, \nu)$ does not have a uniquely ergodic distal model.*

Proof. Suppose (Y', T') is a uniquely ergodic model of $(Y, T, \mathcal{D}, \nu)$. Clearly Y' cannot be a monothetic group. According to Furstenbergs structural theorem for topologically distal transformations, (Y', T') must have a distal factor (Y'', T'') which is not Kronecker but is an isometric extension of the maximal Kronecker factor of (Y', T'). Notice that the maximal topological Kronecker factor of Y' is in particular a (measurable) Kronecker factor of $(Y, T, \mathcal{D}, \nu)$. Thus is must be a factor of the maximal measurable Kronecker factor of Y, $(\mathbf{R}/\mathbf{Z}, T_\alpha, \mathcal{B}, \lambda)$. Using Lemma (5.4) we conclude that the maximal topological Kronecker factor of (Y', T') (or equivalently (Y'', T'')) is (measurably) isomorphic to $(\mathbf{R}/\mathbf{Z}, T_{n\alpha}, \mathcal{B}, \lambda)$ for some $n \in \mathbf{Z}$.

We can now apply Lemma (5.3), since (Y', T') (and so (Y'', T'')) is a finite to one extension of its maximal measurable Kronecker factor, which in turn is a finite extension of the maximal topological Kronecker factor (by Lemma (5.4)). Thus we see that (Y'', T'') is Kronecker - a contradiction. $\square$

§6. Final remarks

The relatively more difficult part of this paper is theorem (3.1). It would be interesting to strengthen this theorem. Two major ways in which this result can be generalized is by looking at cohomology classes of functions from X to homogeneous spaces of locally compact groups and to consider more general group actions than $\mathbf{Z}$.

If one considers cohomology classes of function whose range is (say) a non-compact, non-abelian locally compact group then it seems the approximization technique used in this paper is hopelessly insufficient. As a simple but interesting example one can take the subgroup $\left\{ \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix} \right) \right\}$ of $SL_2(R)$. Either a counter-example or a proof would be nice.

For more general group actions, say a $\mathbf{Z}^2$ action, the statement that two cocycles are of the same cohomology class is much stronger than for $\mathbf{Z}$ actions, and even for simple target groups such as $\mathbf{R}/\mathbf{Z}$ it would be surprising if an analogue of (3.1) holds.

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