SOME EXAMPLES HOW TO USE MEASURE CLASSIFICATION IN NUMBER THEORY

ELON LINDENSTRAUSS

1. Introduction

1.1. Ergodic theory has proven itself to be a powerful method to tackle difficult number theoretical problems, particularly problems which involve equidistribution.

A typical application involves three parts: (a) translating the number theoretical problems into a problem about specific algebraically defined actions; (b) classifying invariant measures for the action; (c) deducing the desired equidistribution statement from this measure classification.

1.2. All the actions we will consider are of the following form: the space on which the action takes place is a quotient space $X = \Gamma\backslash G$ where G is a linear algebraic group, and $\Gamma < G$ a lattice[1]. Any subgroup H of the group of affine transformations[2] on G mapping Γ-cosets to Γ-cosets acts on X. In particular, any subgroup $H < G$ acts on X by right translations $h.x = xh^{-1}$.

This is a fairly broad class of actions. Typically, for specific number theoretic applications one needs to consider on the a specific action. For example, in §3 we give a proof of Furstenberg to the equidistribution of $n^2\alpha \bmod 1$ by studying the $\mathbb{Z}$ action generated by the affine map $(x, y) \mapsto (x + \alpha, y + 2x + \alpha)$ on the space $X = \mathbb{R}^2/\mathbb{Z}^2$. Margulis proved the long-standing Oppenheim conjecture by studying the action of $\mathrm{SO}(2, 1)$, i.e. the group of linear transformations preserving a fixed indefinite quadratic form (say $x_1^2 + x_2^2 - x_3^2$) in three variables, on $X = \mathrm{SL}(3, \mathbb{Z})\backslash \mathrm{SL}(3, \mathbb{R})$, the space of covolume one lattices in $\mathbb{R}^3$. In §6 we present results from [EKL04] towards Littlewood's conjecture regarding simultaneous Diophantine approximations by studying the action of the group of 3×3 diagonal matrices of determinant one on the same space $X = \mathrm{SL}(3, \mathbb{Z})\backslash \mathrm{SL}(3, \mathbb{R})$.

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[1] I.e. a discrete subgroup of finite covolume.

[2] I.e. the group of maps $G \to G$ generated by right translations and automorphisms.

1.3. Weyl's original proof the equidistribution of $n^2\alpha$ mod 1 is not very complicated and unlike most ergodic theoretic methods gives quantitative results regarding equidistribution rates, but the elegance of Furstenberg's proof is quite striking. Furthermore, it is a good illustration of the general scheme discussed in §1.1 and serves as a simple model for the other, more complicated, results we discuss and quote later.

1.4. A very general measure classification theorem which lies at the heart of numerous deep number theoretical applications is Ratner's measure classification theorem (§4.6). We discuss this theorem, and particularly how it can be applied to prove equidistribution in §4. Returning to the general scheme presented in §1.1, the first step of translating a number theoretic question to one related to dynamics seems to be more of an art than a science. Ratner's measure classification theorem is a deep and complicated theorem. The reader could certainly profit from learning some of the ideas involved but this seems beyond the scope of this paper; besides, the recent book [Mor05] seems to cover this quite well. Therefore we focus on the third part of the general scheme, namely how to use this measure classification effectively; techniques introduced by Dani and Margulis seem to be particularly useful in this respect. Even with regard to this part, we only attempt to illustrate clearly the issues that need to be patent; their reader who really wants to master this important technique should study in detail one of the papers quoted where such an application is carried out.

1.5. Ratner theorem does not cover all algebraic actions which arise naturally from number theoretic problems. A good example is the action of the full diagonal group on $\mathrm{SL}(n,\mathbb{Z})\backslash special \in yourgroup(n,\mathbb{R})$.

A good understanding of entropy theory is absolutely essential to applying what results we have regarding invariant measures for these actions to number theory. Therefore we devote considerable space in §5 to present some of the fundamentals regarding entropy. After that we present in detail in §6 one application due to Einsiedler, Katok and the author [EKL04] of a partial measure classification result to estimating the set of possible exceptions to Littlewood's conjecture. Our treatment is quite close to that of [EKL04] though some of the results are presented in a slightly more explicit form.

1.6. Finally, in §7 we explain how measure classification is related to the behavior of Laplacian eigenfunctions on arithmetic quotient spaces — specifically the arithmetic quantum unique ergodicity question. At first sight the measure classification problem one is led to does not seem to be a promising one as there are too many invariant measures, but hidden symmetries and restrictions save the day.

1.7. This paper was written with a fairly narrow aim: to aid graduate students who are interested in understanding the interplay between ergodic theory and number theory and are willing to spend some effort doing so. Of course, other people may find this paper helpful or at least entertaining.

This paper is certainly not a survey, in the traditional sense of the word. Some topics are given detailed even technical treatment, while some are discussed only superficially. It is certainly not meant to be comprehensive and the choice of topics is fairly subjective and arbitrary. While I have made some effort to give correct attributions, doubtless some inaccuracies remain — the reader interested in a detailed historical account should look elsewhere.

It is the author's intention to continue updating this tutorial, and eventually to publish an expanded and more detailed version elsewhere. As it is, it already contains quite a bit of material and (supplemented with pertinent references) can probably be used as a basis for a one semester graduate course on homogeneous dynamics and applications.

1.8. A word about notations: the paragraphs in this paper are numbered, and this numbering is logically identified with the numbering of theorems, definitions, etc.; e.g. "Ratner's measure classification theorem (§4.6)" and "Theorem 4.6" are synonyms. Hopefully this will survive the typesetting.

2. Dynamical systems: some background

2.1. Definition. *Let X be a locally compact space, equipped with an action of a noncompact (but locally compact) group*[1] *H. An H-invariant probability measure μ on X is said to be* ergodic *if one of the following equivalent conditions holds:*

(i) *Suppose $A \subset X$ is an H-invariant set, i.e. $h.A = A$ for every $h \in H$. Then $\mu(A) = 0$ or $\mu(A^{\complement}) = 0$.*
(ii) *Suppose f is a measurable function on X with the property that for every $h \in H$, for μ-a.e. x, $f(h.x) = f(x)$. Then f is constant a.e.*
(iii) *μ is an extreme point of the convex set of all H-invariant Borel probability measures on X.*

[1] All groups will be assumed to be second countable locally compact, all measures Borel probability measures unless otherwise specified.

2.2. A stronger condition which implies ergodicity is mixing:

Definition. *Let X, H and μ be as in Definition 2.1. The action of H is said to be mixing if for any sequence $h_i \to \infty$ in H*[1] *and any measurable subsets $A, B \subset X$,*

$$\mu(A \cap h_i.B) \to \mu(A)\mu(B) \qquad \text{as } i \to \infty.$$

2.3. A basic fact about H-invariant measures is that any H-invariant measure is an average of ergodic measures, i.e. there is some auxiliary probability space (Ξ, ν) and a (measurable) map attaching to each $\xi \in \Xi$ an H-invariant and ergodic probability measure μ_ξ on X so that

$$\mu = \int_\Xi \mu_\xi d\nu(\xi).$$

2.4. Definition. *An action of a group H on a locally compact topological space X is said to be uniquely ergodic if there is only one H-invariant probability measure on X.*

2.5. The simplest example of a uniquely ergodic transformation is the map $T_\alpha : x \mapsto x + \alpha$ on the one dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ where α is irrational. Clearly Lebesgue measure m on $\mathbb{T}$ is T_α-invariant; we need to show it is the only such probability measure.

To prove this, let μ be an arbitrary T_α-invariant probability measure. Since μ is T_α-invariant,

$$\hat{\mu}(n) = \int_\mathbb{T} e(nx) d\mu(x) = \int_\mathbb{T} e(n(x + \alpha)) d\mu(x) = e(n\alpha)\hat{\mu}(n),$$

where as usual $e(x) = \exp(2\pi i x)$. Since α is irrational, $e(n\alpha) \neq 1$ for all $n \neq 0$, hence $\hat{\mu}(n) = 0$ for all $n \neq 0$ and $\mu = m$.

2.6. Definition. *Let X be a locally compact space, and suppose that $H = \{h_t\} \cong \mathbb{R}$ acts continuously on X. Let μ be a H-invariant measure on X. We say that $x \in X$ is* generic for μ *if for every $f \in C_0(X)$ we have*[2]*:*

$$\frac{1}{T} \int_0^T f(h_t.x)\, dt \to \int_X f(y)\, d\mu(y) \qquad \text{as } T \to \infty.$$

Equidistribution is another closely related notion:

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[1] I.e. a sequence so that for any compact $C \subset H$ only finitely many of the h_i are in C.

[2] Where $C_0(X)$ denotes the space of continuous functions on X which decay at infinity, i.e. so that for any $\epsilon > 0$ the set $\{x : |f(x)| \geq \epsilon\}$ is compact.

2.7. Definition. *A sequence of probability measures μ_n on a locally compact space X is said to be* equidistributed *with respect to a (usually implicit) measure m if they converge to m in the weak* topology, i.e. if $\int f\, d\mu_n \to \int f\, dm$ for every $f \in C_c(X)$.*

A sequence of points $\{x_n\}$ in X is said to be equidistributed if the sequence of probability measures $\mu_N = N^{-1} \sum_{n=1}^{N} \delta_{x_n}$ is equidistributed, i.e. if for every $f \in C_0(X)$

$$\frac{1}{N} \sum_{n=1}^{N} f(x_n) \to \int_X f(y)\, dm(y) \qquad \text{as } N \to \infty.$$

Clearly there is a lot of overlap between the two definitions, and in many situations" equidistributed" and "generic" can be used interchangeably.

2.8. For an arbitrary $H \cong \mathbb{R}$-invariant measure μ on X, the Birkhoff pointwise ergodic theorem shows that μ-almost every point $x \in X$ is generic with respect to some H-invariant and ergodic probability measure on X. If μ is ergodic, mu-a.e. $x \in X$ is generic for μ.

If X is compact, and if the action of $H \cong \mathbb{R}$ on X is uniquely ergodic with μ being the unique H-invariant measure, then something much stronger is true: *every* $x \in X$ is generic for μ!

Indeed, let μ_T be the probability measures

$$\mu_T = \frac{1}{T} \int_0^T \delta_{h_t . x}\, dt$$

then any weak* limit of the μ_T will be H-invariant. But there is only one H-invariant probability measure on X, namely μ, so $\mu_T \to \mu$, i.e. x is generic for μ.

3. Equidistribution of $n^2\alpha$ mod 1

3.1. A famous theorem of Weyl states that for any irrational α, the sequence $n^2\alpha$ mod 1 is equidistributed. In this section we give an alternative proof, due to Furstenberg, which proves this theorem by classifying invariant measures on a suitable dynamical system. We follow Furstenberg's treatment in [Fur81, §3.3].

3.2. The dynamical system we will study is the following: the space will simply be the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and the action will be the one generated by the map

$$T : (x, y) \mapsto (x + \alpha, y + 2x + \alpha). \tag{3.2.a}$$

One easily proved using induction that

$$T^n(x, y) = (x + n\alpha, y + 2nx + n^2\alpha). \tag{3.2.b}$$

We will prove below that $(\mathbb{T}^2, T)$ is uniquely ergodic. By §2.8, it follows that for *every* (x, y) the orbit $\{T^n(x, y)\}_{n=1}^{\infty}$ is equidistributed. In particular, the orbit of the point $(0, 0)$ is equidistributed, i.e.

$$\left\{(n\alpha \bmod 1, n^2\alpha \bmod 1) : n \in \mathbb{N}\right\}$$

is equidistributed. We see that unique egodicity of T implies not only equidistribution of $n^2\alpha \bmod 1$ but the stronger fact that $(n\alpha \bmod 1, n^2\alpha \bmod 1)$ is equidistributed in $\mathbb{T}^2$.

The same proof, with minor modifications, can be used to show equidistribution of $p(n) \bmod 1$ for any polynomial with an irrational leading coefficient (see Exercise 3.8 below).

3.3. The proof that (X, T) is uniquely ergodic is harder than for irrational rotations on $\mathbb{T}$ (cf. §2.5). The basic scheme, which is not unusual in such proofs, is that we first prove that Lebesgue measure m on $\mathbb{T}^2$, which is obviously invariant under T, is also ergodic. A separate argument is then used to bootstrap the ergodicity of Lebesgue measure to unique ergodicity.

3.4. Proposition. *Lebesgue measure m on $\mathbb{T}^2$ is ergodic under T.*

Proof. Let $f \in L^2(m)$ be T-invariant. Expand f to a Fourier series

$$f(x, y) = \sum_{n,m} \hat{f}_{n,m} e(nx + my).$$

By T-invariance,

$$\hat{f}_{n,m} = \hat{f}_{n+2m,m} e((n + m)\alpha). \tag{3.4.a}$$

In particular, $\left|\hat{f}_{n,m}\right| = \left|\hat{f}_{n+2m,m}\right|$. By the Riemann-Lebesgue Lemma, $\hat{f}_{n,m} \to 0$ as $(n, m) \to \infty$, hence $\hat{f}_{n,m} = 0$ if $m \neq 0$.

For $m = 0$, however, (3.4.a) becomes $\hat{f}_{n,0} = e(n\alpha)\hat{f}_{n,0}$, so $\hat{f}_{n,m} = 0$ for all $(n, m) \neq 0$.

It follows that f is constant a.e., and m is ergodic. □

This argument cannot be applied directly for T-invariant probability measures, as the Fourier transform of a measure does not satisfy the Riemann-Lebesgue Lemma.

3.5. The bootstrapping argument which we use to upgrade ergodicity to unique ergodicity is a simple positivity argument.

Proposition. *Let g be a measurable function $\mathbb{T} \to \mathbb{T}$, and $T_g : \mathbb{T}^2 \to \mathbb{T}^2$ be the map*

$$T_g(x, y) = (x + \alpha, y + g(x))$$

with α irrational. Then if the Lebesgue measure m is T_g-ergodic, then in fact it is the only T_g-invariant probability measure, i.e. $(\mathbb{T}^2, T_g)$ is uniquely ergodic.

Proof. Suppose $\mu \neq m$ is another T_g-invariant probability measure on $\mathbb{T}^2$. Let R_a denote the map $(x, y) \mapsto (x, y + a)$. Then since T_g and R_a commute, for any $a \in \mathbb{T}$, $(R_a)_*\mu$ is also T_g-invariant. Consider the measure

$$m' = \int_{\mathbb{T}^2} (R_a)_*\mu. \tag{3.5.a}$$

Clearly m' is invariant under R_a for every a, and its projection to the first coordinate has to be a probability measure invariant under the rotation $x \mapsto x + \alpha$, hence Lebesgue. It follows that $m' = m$. But by assumption, m is ergodic, and hence is an extreme points in the convex set of T_g invariant probability measures on $\mathbb{T}^2$. Therefore m cannot be presented as a nontrivial linear combination of other T_g invariant probability measures, in contradiction to (3.5.a). $\square$

3.6. Proposition 3.4 and Proposition 3.5 together clearly imply

Corollary. *The map $T : (x, y) \mapsto (x + \alpha, y + 2x + \alpha)$ on $\mathbb{T}^2$ is uniquely ergodic for every irrational α.*

As discussed in §3.2, equidistribution of $\{n^2\alpha \bmod 1\}$ is now an easy consequence of this corollary.

3.7. The proof we have given for the equidistribution of $\{n^2\alpha \bmod 1\}$ is very elegant, but it has one serious drawback compared to Weyl's original method: it does not give rates. The ambitious reader is encouraged to try and figure out how to modify Furstenberg's proof to obtain a more quantitative result regarding the rate of equidistribution. Such a quantification of a qualitative ergodic theoretic argument is often referred to as effectivization, and often can be quite entertaining and worthwhile.

3.8. Exercise. Generalize this argument to give an ergodic theoretic proof for the equidistribution of $p(n)$ mod 1 for any polynomial $p(n)$ with an irrational leading coefficient.

4. Unipotent flows and Ratner's theorems

4.1. A very general and important measure classification theorem has been proved by Ratner, in response to conjectures by Dani and Raghunathan. For simplicity, we restrict our treatments to the case of Lie groups, even though the extension of Ratner's theorems to products of real and p-adic groups [Rat95, MT94] is just as important for number theoretical applications; for a recent and striking example, see [EV06]).

4.2. Definition. *An element* $g \in \mathrm{GL}(n, \mathbb{R})$ *is said to be* unipotent *if all its (real or complex) eigenvalues are equal to one. An element* g *in a Lie group* G *is said to be* Ad-unipotent *if* $\mathrm{Ad}(g)$ *is a unipotent element of* $\mathrm{GL}(\mathfrak{g})$*, with* $\mathfrak{g}$ *the Lie algebra of* G*.*

4.3. Definition. *Let* X *be a topological space, and* H *a locally compact group acting continuously on* X*. An orbit* $H.x$ *is said to be* periodic *if it has a finite* H*-invariant measure.*[1]

Example. Suppose $H = \mathbb{Z}$, and that the action of H on X is generated by the map $T : X \to X$. Then x has a periodic H-orbit iff $T^n x = x$ for some $n \in \mathbb{N}$.

4.4. We remark that in the locally homogeneous context, i.e. $X = \Gamma\backslash G$ and H a subgroup of G acting on X by right translations, every periodic H-orbit is also a closed subset of X [Rag72, Theorem 1.13].

4.5. Let G be a Lie group, $\Gamma < G$ a discrete subgroup, and $H < G$. One obvious class of H invariant probability measures on $\Gamma\backslash G$ are L-invariant probability measures on single periodic L-orbits for (closed) subgroups $L < G$ containing H. We shall call such measures *homogeneous*; equally common in this context is the adjective *algebraic.*

4.6. Theorem (Ratner's measure classification theorem [Rat91a])**.** *Let* G *be a Lie group,* $\Gamma < G$ *a discrete subgroup, and* $H < G$ *a closed connected subgroup generated by Ad-unipotent one parameter groups. Then any* H*-invariant and ergodic probability measure on* $\Gamma\backslash G$ *is homogeneous (in the sense of §4.5).*

While the statement of Theorem 4.6 the group Γ is not assumed to be a lattice[2], for most applications this assumption is necessary, as otherwise the assumption that the H-invariant measure under consideration is a *probability* measure is not a natural one.

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[1] More formally, there is a nontrivial finite H-invariant measure ν on X so that $\nu(X - H.x) = 0$.

[2] I.e. a discrete subgroup of finite covolume

For the remainder of this section, unless otherwise specified, Γ will be a *lattice* in G.

4.7. The proof of Theorem 4.6 is beyond the scope of this paper. The ambitious reader is encouraged to study the proof; helpful references are the recent book [Mor05] (particularly Chapter 1), Ratner's treatment of a "baby case" in [Rat92], and a simplified self contained proof of the special case $H \cong \mathrm{SL}(2,\mathbb{R})$ (but general G and Γ) in [Ein06]. A more advanced reference (in addition to Ratner's original papers) is Margulis and Tomanov's proof of this result [MT94] which in particular uses entropy theory as a substitute to some of Ratner's arguments. A useful survey paper which covers much of what we discuss in this section is [KSS02], particularly [KSS02, §3].

4.8. Consider for simplicity first the case of H itself a unipotent one parameter flow (in particular, as an abstract group, $H \cong \mathbb{R}$). If there are no H-invariant probability measures on $\Gamma\backslash G$ other than the G-invariant measure, then as we have seen in §2.8 it is fairly straightforward to deduce from the measure classification theorem information regarding how each individual orbit is distributed, and in particular classify the possible orbit closures (which in the uniquely ergodic case can be only $\Gamma\backslash G$ itself, i.e. the H-flow is *minimal*).

4.9. Exercise.
- (i) Let $G = \mathrm{SL}(2,\mathbb{R})$ and $\Gamma < G$ a *cocompact* lattice. Let H be the group $\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$. Deduce from Ratner's theorem that the action H on $\Gamma\backslash G$ is uniquely ergodic.
- (ii) Let $H < G$ be as in (i), but take $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. What are the H-invariant measures in this case?

The proof that H acting on $\Gamma\backslash G$ in (i) above is uniquely ergodic predated Ratner's theorem by about 20 years and is due to Furstenberg [Fur73]. The classification in (ii) is due to Dani [Dan78]. There are much simpler proofs now, the simplest (and quite quantitative) proof is via the mixing of the geodesics flow.

4.10. In most cases of interest, however, there is more than one invariant measure in Theorem 4.6, and in this case deducing information about individual orbits from a measure classification theorem is far less obvious (cf. Exercise 4.15 below). Nonetheless measure classification is the main ingredient in the proof of the following two important results of Ratner:

4.11. Theorem (Ratner's genericity theorem [Rat91b])**.** *Let G be a Lie group, Γ a lattice, and H a unipotent one parameter subgroup of G.*

Then every $x \in \Gamma\backslash G$ *is generic for a homogeneous measure supported on a periodic orbit* $L.y$ *containing* x.

4.12. Theorem (Ratner's orbit closure classification theorem [Rat91b])**.** *Let* G *be a Lie group,* Γ *be a lattice in* G*, and* $H < G$ *closed connected subgroup generated by Ad-unipotent one parameter groups. Then for any* $x \in \Gamma\backslash G$*, the orbit closure* $\overline{H.x}$ *is a single periodic orbit for some group* $H \leq L \leq G$*.*

4.13. Ratner's theorems give us very good understanding of the dynamics of groups generated by unipotents on finite volume quotients $\Gamma\backslash G$. Much less is known about the case when Γ is a discrete subgroup with infinite covolume. For example, we do not know how to classify Radon measures[1] invariant under unipotent groups in the infinite covolume case, and we do not understand orbit closures in this case except in very special cases (for example, see [Bur90, LS05a, Rob03]).

The action of groups H which are not generated by unipotents on $\Gamma\backslash G$ (even in the finite covolume case) is also not well understood at present. This topic will be discussed in detail in §6.

4.14. Ratner's proof of Theorem 4.12 via measure classification is not the only approach to classifying orbit closures. In particular, for the important special case of $H = \mathrm{SO}(2,1)$[2] Dani and Margulis [DM89] (following earlier work of Margulis) classified all possible orbit closures $\overline{H.x}$ in $\mathrm{SL}(3,\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{R})$ before Ratner's work.

The action of this group $H = SO(2,1)$ on $\mathrm{SL}(3,\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{R})$ is closely connected to the Oppenheim conjecture regarding values of indefinite quadratic forms which was posed in the 1920's, and was only solved in the 1980's by Margulis (see e.g. [Mar89]) using a partial classification of orbit closures of this action. An accessible self-contained treatment of this result is [DM90].

It is not clear (at least to me) exactly what is the limit of these topological methods, and whether they can be pushed to give a full proof of Theorem 4.12.

4.15. Exercise. Let $X = \{0,1\}^{\mathbb{Z}}$ and $\sigma : X \to X$ be the shift map $(\sigma(x))_i = x_{i+1}$. Let $n_i \uparrow \infty$ be an increasing sequence of integers with $n_i/i \to 0$. Define Y to be the set of those sequences $x \in X$ with the property that for every $i \in \mathbb{N}$, the sequence "01" does not

[1] I.e. locally finite but possibly infinite measures.

[2] I.e. the group of determinant one matrices preserving a fixed quadratic form of signature 2,1 — e.g. $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$.

appear more than i times in any stretch of $n_i + 1$-digits, i.e. if $B = \{x \in X : x_0 = 0, x_1 = 1\}$,

$$Y = \left\{ x \in X : \forall i \in \mathbb{N}, \max_j \sum_{k=j}^{j+n_i-1} 1_B(\sigma^k x) \le i \right\}.$$

(i) Show that there are precisely two σ-invariant and ergodic probability measures on Y. What are they?
(ii) Prove that there are $y \in Y$ which are not generic for any σ-invariant probability measure on Y.
(iii) Show that there are uncountably many possible orbit closures for σ (i.e. sets of the form $\overline{\{\sigma^n y : n \in \mathbb{Z}\}}$ with $y \in Y$).

4.16. Exercise 4.15 shows that one cannot deduce Ratner's strong rigidity statements about individual orbit from her measure classification theorem by purely formal means. We follow the approach of Dani and Margulis [DM93] to these issues which give more uniform and flexible versions of Theorem 4.11 that are often highly useful in number theoretic applications. See also [Sha94] for other closely related results.

4.17. The main difficulty in passing from a measure classification theorem to a theorem about behavior of individual orbits is that orbits may for some stretch of time behave according to some invariant measure, and then after a relatively short transition period start behaving according to a different invariant measure.

There is an extra difficulty in the locally homogeneous context in that more often than not the space we consider is not compact, bringing in another complication: to pass from measure classification to statements regarding individual orbits one needs to be able to control how much time an orbit spends far away (i.e. outside big compact sets). Both of these difficulties (which are closely related) can be addressed by the following basic estimates.

4.18. Definition. *Let G be a Lie group and $\Gamma < G$ a discrete subgroup. For any subgroup $H < G$ define the* singular set relative to H, *denoted by $\mathcal{S}(H)$, as the union of all periodic orbits in $X = \Gamma\backslash G$ of all closed subgroups $L < G$ containing H.*

If H is a one parameter Ad-unipotent group then by Theorem 4.11, $\mathcal{S}(H)$ is precisely the set of all $x \in X$ which fail to be generic for the G-invariant measure on X with respect to the action of H.

It is worthwhile to delve a bit into the structure of this singular set $\mathcal{S}(H)$. Suppose that $L_1.x$ is a periodic orbit with $L_1 > H$ and

$x = \pi_\Gamma(g)$. Let $L = {}^gL_1$ where we use the notations ${}^gL = gLg^{-1}$ and $L^g = g^{-1}Lg$. Then since $L_1.x$ has finite volume, $\Gamma_L = L \cap \Gamma$ is a lattice in L. Let

$$X(L, H) = \left\{h \in G : hHh^{-1} \subset L\right\}.$$

For any $h \in G$, since $L_1.x$ is periodic, the orbit of $y = \pi_\Gamma(h)$ under $L_2 = {}^{h^{-1}}L$ is periodic. If $h \in X({}^gL, H)$, we have that the natural probability measure on this periodic orbit $L_2.y$ is H-invariant. In this way we get a "tube" of periodic orbits $\pi_{\Gamma_L}(X(L, H))$ on $\Gamma_L\backslash G$ which descends to a family of periodic orbits $\pi_\Gamma(X(L, H))$ on X. Of course, for some L and H this family may be empty or consist of a single periodic orbit.

By [DM93, Proposition 2.3],

$$\mathcal{S}(H) = \bigcup_{L \in \mathcal{H}} \pi_\Gamma(X(L, H))$$

where $\mathcal{H}$ is a countable collection of closed connected subgroups of G[1].

Exercise. Work this decomposition out explicitly for $G = \mathrm{SL}(2, \mathbb{R})$, $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, and $H = \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$.

4.19. A careful understanding of this singular set (cf. [DM90, §3]) is important to control the amount of time a unipotent trajectory can spend near a lower dimensional invariant subspace. For instance, it can be used to show the following:

Theorem (Dani and Margulis [DM93, Theorem 1])**.** *Let H be a closed connected subgroup of G generated by Ad-unipotent elements. Let $F \subset X - \mathcal{S}(H)$ be compact. Then for any $\epsilon > 0$, there is a neighborhood Ω of $\mathcal{S}(H)$ such that for any Ad-unipotent one parameter subgroup $\{u_t\}$ of G, any $x \in F$, and any $T \geq 0$,*

$$\mathit{Leb}\left(\{t \in [0, T] : u_t.x \in \Omega\}\right) \leq \epsilon T.$$

Note that typically $\mathcal{S}(H)$ is dense so the condition on F above is rather harsh. A more precise result is [DM93, Theorem 7.3] which essentially shows that the only way the trajectory $\{u_t.\pi_\Gamma(g) : t \in [0, T]\}$ spends a substantial amount of time near the singular set is if g is so close to some $X(L_0, H)$ so that *for all* $t \in [0, T]$ the point gu_{-t} is close to $X(L_0, H)$.

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[1]Namely, the collection of all closed connected subgroups $L < G$ satisfying that (a) $\dim L < \dim G$, (b) $L \cap \Gamma$ is a lattice in L, and (c) the image of $\Gamma \cap L$ under the adjoint representation is Zariski dense in the image of L.

4.20. In order to control the related question of how much time an arbitrary orbit of the one parameter unipotent subgroup spends in a neighborhood of infinity we have the following, which follows from several papers of Dani and Margulis starting with [Mar71].

Theorem (Dani and Margulis [DM93, Theorem 6.1])**.** *Let G be a Lie group and $\Gamma < G$ a lattice. Then for any compact $F \subset \Gamma\backslash G$ and any $\epsilon > 0$ there is a compact $C \subset X$ so that for any Ad-unipotent one parameter subgroup $\{u_t\}$ of G, any $x \in F$ and any $T > 0$*

$$\mathit{Leb}\left(\{t \in [0,T] : u_t.x \notin C\}\right) \leq \epsilon T.$$

There are many extensions and variations on this result, some of them quite important. A nice place to read about some of these developments (and indeed also about the basic method) is [KM98].

4.21. As a basic example of how Theorems 4.19 and 4.20 can be used, we show how with the aid of these theorems, Ratner's theorem about generic points (Theorem 4.11) can be deduced from her measure classification theorem (Theorem 4.6)[1].

Proof. Let $x \in X$, and for any $T > 0$ set μ_T to be the probability measure

$$\mu_T = \frac{1}{T}\int_0^T \delta_{u_t.x}\, dt.$$

Without loss of generality, we may assume that $x \notin \mathcal{S}(\{u_t\})$ for otherwise we may replace X (and G and Γ accordingly) with a (lower dimensional) periodic orbit containing $\{u_t.x\}$.

What we want to prove is that for any $f \in C_0(X)$

$$\int_X f\, d\mu_T \overset{?}{\to} \int_X f\, dm \qquad \text{as } T \to \infty \tag{4.21.a}$$

where m is the G invariant probability measure on X, i.e. that μ_T converge weak* to m.

By Theorem 4.20 applied to $F = \{x\}$, for any $\epsilon > 0$ there is a compact set $C \subset X$ so that for all T we have $\mu_T(C) > 1 - \epsilon$. It follows that there is a sequence of $T_i \uparrow \infty$ for which μ_{T_i} converge in the weak* topology to a *probability* measure μ_∞.

This limiting measure μ_∞ is invariant under u_t. By Theorem 4.6 and the ergodic decomposition, μ_∞ is a linear combination of m and the natural probability measures on periodic orbits of groups L containing H. In particular, $\mu_\infty = \alpha m + (1-\alpha)\mu'$ with μ' a probability measure on $\mathcal{S}(\{u_t\})$.

[1]This is not how Ratner proved Theorem 4.11!

Applying Theorem 4.19 to $F = \{x\}$, we get for any $\epsilon > 0$ an open set $\Omega \supset \mathcal{S}(\{u_t\})$ with $\mu_T(\Omega) < \epsilon$ for all T. It follows that

$$\mu_\infty(\Omega) \leq \varliminf_{i\to\infty} \mu_{T_i}(\Omega) \leq \epsilon$$

and so $\alpha \geq 1 - \epsilon$. Since ϵ was arbitrary, we see that $\mu_\infty = m$ and Theorem 4.11 follows. □

4.22. Exercise. Use Theorem 4.20 to show that (4.21.a) holds for any continuous bounded f (not necessarily decaying at infinity). This slightly stronger form of Theorem 4.11 is the one given in [Rat91b].

4.23. Exercise. Use a similar arguments to prove the following ([DM90, Theorem 2]):

Let $u_t, u_t^{(1)}, u_t^{(2)}, \dots$ be one parameter Ad-unipotent subgroups of G with $u_t^{(i)} \to u_t$, x_i be a sequence of points in X converging to $x \in X - \mathcal{S}(\{u_t\})$, and $T_i \uparrow \infty$. Then for any continuous bounded f

$$\frac{1}{T_i}\int_0^{T_i} f(u_t^{(i)}.x_i)\,dt \to \int_X f\,dm.$$

Hint: show first that without loss of generality we can assume $x_i \notin \mathcal{S}(\{u_t\})$.

4.24. We end this section with an interesting application, presented in the form of an exercise, of Ratner's theorems and the related results of Dani-Margulis to equidistribution of the points of Hecke correspondences. This application was first suggested by Burger and Sarnak [BS91] and a detailed proof was given by Dani and Margulis in [DM93]. Recently Eskin and Oh [EO06] gave a further generalization of this approach.

All these results are quite general, but we consider only the simplest case of $X = \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$. This case is also discussed in Venkatesh' contribution to these proceedings [Ven06].

4.25. We begin our discussion by defining the Hecke correspondences for the case of $G = \mathrm{SL}(2,\mathbb{R})$, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$; as usual let $X = \Gamma\backslash G$. We say that an integer matrix $\gamma \in M_2(\mathbb{Z})$ is irreducible if there is no nontrivial integer dividing all its coefficients.

Definition. *Let n be an integer ≥ 2. The n-*Hecke correspondence *is a map which assigns to a point $x = \pi_\Gamma(g) \in X$ a finite subset $T_n(x)$ of X (with the number of points in $T_n(x)$ depending only on n) by*

$$T_n(x) = \left\{\pi_\Gamma(n^{-\frac{1}{2}}\gamma g) : \gamma \in M_2(\mathbb{Z}) \text{ irreducible with } \det\gamma = n\right\}.$$

While it is not completely obvious from the formula, $T_n(x)$ is a finite collection of points of X, and its cardinality can be given explicitly and depends only on n.

Using the Hecke correspondence we can define operators (also denoted T_n) on $L^2(X)$ by

$$T_n(f)[x] = c_n \sum_{y \in T_n(x)} f(y);$$

where we take[1] $c_n = |T_n(x)|^{-1}$.

4.26. For example[2], if $n = p$ is prime, and $x = \pi_\Gamma(g)$ as above, $T_p(x)$ consists of the $p+1$ points

$$T_p(x) = \left\{ \pi_\Gamma \left(p^{-1/2} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} g \right), \pi_\Gamma \left(p^{-1/2} \begin{pmatrix} p & 0 \\ 1 & 1 \end{pmatrix} g \right), \dots, \right.$$
$$\left. \pi_\Gamma \left(p^{-1/2} \begin{pmatrix} p & 0 \\ p-1 & 1 \end{pmatrix} g \right), \pi_\Gamma \left(p^{-1/2} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} g \right) \right\};$$

and for any n, the set $T_{n^2}(x)$, which contains roughly n^2 points, contains in particular the $\phi(n)$ points

$$\pi_\Gamma \left(\begin{pmatrix} 1 & k/n \\ 0 & 1 \end{pmatrix} g \right) \qquad (k, n) = 1.$$

The operators T_n respects the action of $G = \mathrm{SL}(2, \mathbb{R})$ by translations on X, i.e. $T_n(g.x) = g.T_n(x)$. For future reference, we note that this shows that the Hecke operators T_n descend to correspondences on $\Gamma \backslash \mathbb{H} \cong \Gamma \backslash G / K$ with $K = \mathrm{SO}(2, \mathbb{R})$.

4.27. The following theorem was discussed in Venkatesh' contribution to these proceedings:

Theorem. *For any $x \in X$, the points of the Hecke correspondences $T_n(x)$ become equidistributed* [3] *as $n \to \infty$.*

There a spectral approach to the theorem is discussed, using the known bounds towards the Ramanujan Conjecture, which gives much sharper results than what one can presently get using ergodic theory. However, it is quite instructive to deduce this equidistribution statement from Ratner's theorems.

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[1] This is not the standard normalization; the standard normalization is $c_n = n^{-1/2}$.

[2] Which the reader should verify!

[3] To be more precise, the sequence of probability measures $|T_n(x)|^{-1} \sum_{y \in T_n(x)} \delta_y$ become equidistributed in a sense of Definition 2.7.

4.28. Exercise. Let $G^2 = G \times G$, $\Gamma^2 = \Gamma \times \Gamma$, $X^2 = \Gamma^2 \backslash G^2$ and $G_\Delta < G^2$ the subgroup $G_\Delta = \{(g,g) : g \in G\}$. Also let $u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $u_\Delta(t) = (u(t), u(t))$. Let m_X and denote the G invariant measure on X and similarly for X^2.

The purpose of this exercise[1] is to prove the following, which is essentially equivalent[2] to the equidistribution of $T_{n^2}(x_0)$ for every $x_0 \in X$ as $n \to \infty$ along the lines of [BS91, DM93]

For any $f, g \in L^2(X)$, we have that

$$\int T_{n^2}(f) g \, dm_X \to \int f \, dm_X \int g \, dm_X \qquad \text{as } n \to \infty. \tag{4.28.a}$$

Let $\alpha \in (0,1) - \mathbb{Q}$ be arbitrary, and let $k(n)$ be a sequence of integers satisfying (a) $(k(n), n) = 1$ and (b) $k(n)/n \to \alpha$.

(i) Let $y_n^2 = \pi_{\Gamma^2}\left(\begin{pmatrix} 1 & k(n)/n \\ 0 & 1 \end{pmatrix}, e\right)$, with e denoting the identity. Show that
$$G_\Delta . y_n^2 = \{(x,y) : x \in X, y \in T_{n^2}(x)\}.$$
Let $y_\infty^2 = \pi_{\Gamma^2}\left(\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, e\right)$.

(ii) Let μ_n denote the natural measure on the periodic orbit $G_\Delta . y_n^2$. Show, using the well known ergodicity of the action of $u(t)$ on $\Lambda \backslash G$ (for any lattice Λ), that $u_\Delta(t)$ acts ergodically on μ_n.

(iii) Deduce from Theorem 4.20, the fact that $y_n^2 \to y_\infty^2$, the ergodicity of $u_\Delta(t)$ acting on μ_n and the pointwise ergodic theorem that μ_n converge weak* to a probability measure μ.

(iv) Show that $y_\infty^2 \notin \mathcal{S}(G_\Delta)$, and deduce similarly from Theorem 4.19 that $\mu(\mathcal{S}(G_\Delta)) = 0$.

(v) Use Ratner's measure classification theorem (§4.6) to deduce $\mu = m_{X^2}$.

(vi) Consider $F(x,y) = f(x)g(y)$. Show that
$$\int F(x,y) \, d\mu_n = \int T_{n^2}(f) g dm_X.$$
Deduce (4.28.a).

[1]This exercise is somewhat advanced and may use more than we assume in the rest of this paper.

[2]Only in this particular instance, because of the equivariance of T_p under G.

4.29. The reader is encouraged to look at other applications of Ratner's theorem to equidistribution and counting problems, for example [EMS96, EMM98, EM04, EO].

5. Entropy of dynamical systems: some more background

5.1. A very basic and important invariant in ergodic theory is entropy. It can be defined for any action of a (not too pathological) unimodular amenable group H preserving a probability measure [OW87], but for our purposes we will only need (and only consider) the case $H \cong \mathbb{R}$ or $H \cong \mathbb{Z}$.

Entropy was lurking behind the scenes already in the study of the action of unipotent groups considered in §4[1], but plays a much more prominent role in the study of diagonalizable actions which we will consider in the next section.

5.2. Let (X, μ) be a probability space. The entropy $H_\mu(\mathcal{P})$ of a finite or countable partition of X is defined to be

$$H_\mu(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

One basic property of entropy is sub-additivity; the entropy of the refinement $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$ satisfies

$$H_\mu(\mathcal{P} \vee \mathcal{Q}) \leq H_\mu(\mathcal{P}) + H_\mu(\mathcal{Q}). \tag{5.2.a}$$

However, this is just a starting point for many more natural identities and properties of entropy, e.g. equality holds in (5.2.a) if and only if $\mathcal{P}$ and $\mathcal{Q}$ are independent.

5.3. The ergodic theoretic entropy $h_\mu(T)$ associated to a measure preserving map $T : X \to X$ can be defined using the entropy function H_μ as follows:

Definition. *Let μ be a probability measure on X and $T : X \to X$ a measurable map preserving μ. Let $\mathcal{P}$ be either a finite partition of X or a countable partition with $H_\mu(\mathcal{P}) < \infty$. The entropy of the four-tuple $(X, \mu, T, \mathcal{P})$ is defined to be*[2]

$$h_\mu(T, \mathcal{P}) = \lim_{N \to \infty} \frac{1}{N} H_\mu\left(\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P}\right). \tag{5.3.a}$$

[1] In particular, in [MT94] Margulis and Tomanov give a substantially shorter proof of Ratner's measure classification theorem using entropy theory.

[2] Note that by the subadditivity of the entropy function H_μ the limit in (5.3.a) exists and is equal to $\inf_N \frac{1}{N} H_\mu(\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P})$.

The ergodic theoretic entropy of (X, μ, T) *is defined to be*

$$h_\mu(T) = \sup_{\mathcal{P}: H_\mu(\mathcal{P}) < \infty} h_\mu(T, \mathcal{P}).$$

The ergodic theoretic entropy was introduced by A. Kolmogorov and Ya. Sinai and is often called the Kolmogorov-Sinai entropy; it is also somewhat confusingly called the metric entropy (even though it has nothing to do with any metric that might be defined on X!).

If μ is a T-invariant but not necessarily ergodic measure, it can be shown that the entropy of μ is the average of the entropy of its ergodic components: i.e. if μ has the ergodic decomposition $\mu = \int \mu_\xi d\nu(\xi)$, then

$$h_\mu(T) = \int h_{\mu_\xi}(T) d\nu(\xi). \tag{5.3.b}$$

5.4. A partition $\mathcal{P}$ is said to be a *generating partition* for T and μ if the σ-algebra $\bigvee_{n=-\infty}^{\infty} T^{-n}\mathcal{P}$ (i.e. the σ-algebra generated by the sets $\{T^n P : n \in \mathbb{Z}, P \in \mathcal{P}\}$) separates points; that is, for μ-almost every x, the atom of x with respect to this σ-algebra is $\{x\}$.[1] The Kolmogorov-Sinai theorem asserts the non-obvious fact that $h_\mu(T) = h_\mu(T, \mathcal{P})$ whenever $\mathcal{P}$ is a generating partition.

5.5. We also want to define the ergodic theoretic entropy also for flows (i.e. for actions of groups $H \cong \mathbb{R}$). Suppose $H = \{a_t\}$ is a one parameter group acting on X. Then it can be (fairly easily) shown that for $s \neq 0$, $\frac{1}{|s|} h_\mu(x \mapsto a_s.x)$ is independent of s. We define the entropy of μ with respect to $\{a_t\}$, denoted $h_\mu(a_\bullet)$, to be this common value of $\frac{1}{|s|} h_\mu(x \mapsto a_s.x)$.[2]

5.6. Suppose now that (X, d) is a compact metric space, and that $T : X \to X$ is a homeomorphism (the pair (X, T) is often implicitly identified with the generated $\mathbb{Z}$-action and is called a dynamical system).

Definition. *The* $\mathbb{Z}$ *action on* X *generated by* T *is said to be* expansive *if there is some* $\delta > 0$ *so that for every* $x \neq y \in X$ *there is some* $n \in \mathbb{Z}$ *so that* $d(T^n x, T^n y) > \delta$.

If X is expansive then any measurable partition $\mathcal{P}$ of X for which the diameter of every element of the partition is $< \delta$ is generating (with respect to any measure μ) in the sense of §5.4.

---

[1] Recall that the atom of x with respect to a countably generated σ-algebra $\mathcal{A}$ is the intersection of all $B \in \mathcal{A}$ containing x and is denoted by $[x]_{\mathcal{A}}$.

[2] Note that $h_\mu(a_\bullet)$ depends not only on H as a group but on the particular parametrization a_t.

5.7. For the applications presented in the next section, an important fact is that for many dynamical systems (X,T) the map $\mu \mapsto h_\mu(T)$ defined on the space of T-invariant probability measures on X is semi-continuous. This phenomenon is easiest to see when (X,T) is expansive.

Proposition. *Suppose (X,T) is expansive, and that μ_i, μ are T-invariant probability measures on X with $\mu_i \to \mu$ in the weak* topology. Then*

$$h_\mu(T) \geq \varlimsup_{i\to\infty} h_{\mu_i}(T).$$

In less technical terms, for expansive dynamical systems, a "complicated" invariant measure might be approximated by a sequence of "simple" ones, but not vice versa.

Proof. Let $\mathcal{P}$ be a partition of X such that for each $P \in \mathcal{P}$

(i) $\mu(\partial P) = 0$

(ii) P has diameter $< \delta$ (δ as in the definition of expansiveness).

Since $\mu(\partial P) = 0$ and $\mu_i \to \mu$ weak*, for every $P \in \mathcal{P}$ we have that $\mu_i(P) \to \mu(P)$. Then

$$\begin{aligned}\frac{1}{N} H_\mu\left(\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P}\right) &= \lim_{i\to\infty} \frac{1}{N} H_{\mu_i}\left(\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P}\right)\\ &\geq \varlimsup_{i\to\infty} h_{\mu_i}(T,\mathcal{P}) \overset{\text{(by (ii))}}{=} \varlimsup_{i\to\infty} h_{\mu_i}(T).\end{aligned}$$

Taking the limit as $N \to \infty$ we get

$$h_\mu(T) = h_\mu(T,\mathcal{P}) = \lim_{N\to\infty} \frac{1}{N} H_\mu\left(\bigvee_{n=0}^{N-1} T^{-n}\mathcal{P}\right) \geq \varlimsup_{i\to\infty} h_{\mu_i}(T).$$

□

Note that we have used both (ii) and expansiveness only to establish

(ii′) $h_\nu(T) = h_\nu(T,\mathcal{P})$ for $\nu = \mu, \mu_1, \dots$.

We could have used the following weaker condition: for every ϵ, there is a partition $\mathcal{P}$ satisfying (i) and

(ii″) $h_\nu(T) \leq h_\nu(T,\mathcal{P}) + \epsilon$ for $\nu = \mu, \mu_1, \dots$.

5.8. We are interested in dynamical systems of the form $X = \Gamma\backslash G$ (G a connected Lie group and $\Gamma < G$ a lattice) and $T : x \mapsto g.x$. If G has rank ≥ 2,[1] this system will not be expansive, and furthermore in the most interesting case of $X = \mathrm{SL}(n,\mathbb{Z})\backslash \mathrm{SL}(n,\mathbb{R})$ the space X is not compact.

[1] For example, $G = \mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$.

Even worse, e.g. on $X = \mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ one may have a sequence of probability measures μ_i ergodic and invariant under the one parameter group $\left\{a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}\right\}$ with $\underline{\lim}_{i\to\infty} h_{\mu_i}(a_\bullet) > 0$ converging weak* to a measure μ which is not a probability measure and furthermore has zero entropy[1].

However, one has the following "folklore theorem"[2] :

Proposition. *Let G be a connected Lie group, $\Gamma < G$ a lattice, and $H = \{a_t\}$ a one parameter subgroup of G. Suppose that μ_i, μ are H-invariant* probability *measures on X with $\mu_i \to \mu$ in the weak* topology. Then*

$$h_\mu(a_\bullet) \geq \overline{\lim_{i\to\infty}}\, h_{\mu_i}(a_\bullet).$$

For X compact (and possibly by some clever compactification also for general X), this follows from deep (and complicated) work of Yomdin, Newhouse and Buzzi (see e.g. [Buz97] for more details); however Proposition 5.8 can be established quite elementarily. In order to prove this proposition, one shows that any sufficiently fine finite partition of X satisfies §5.7.(ii″).

5.9. The following example shows that this semicontinuity does not hold for a general dynamical system:

Example. Let $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, 0\right\}$, and $X = S^{\mathbb{Z}}$ (equipped with the usual Tychonoff topology). Let $\sigma : X \to X$ be the shift map[3].

Let μ_n be the probability measure on X obtained by taking the product of the probability measures on S giving equal probability to 0 and $\frac{1}{n}$, and $\delta_{\mathbf{0}}$ the probability measure supported on the fixed point $\mathbf{0} = (\dots, 0, 0, \dots)$ of σ. Then $\mu_n \to \delta_{\mathbf{0}}$ weak*, $h_{\mu_n}(\sigma) = \log 2$ but $h_{\delta_{\mathbf{0}}}(\sigma) = 0$.

5.10. Let (X, d) be a compact metric space, $T : X \to X$ continuous[4]. Two points $x, x' \in X$ are said to be *k, ϵ-separated* if for some $0 \leq \ell < k$ we have that $d(T^\ell.x, T^\ell.x') \geq \epsilon$. Let $N(X, T, k, \epsilon)$ denote the maximal cardinality of a k, ϵ-separated subset of X.

---

[1] Strictly speaking, we define entropy only for probability measures, so one needs to rescale μ first.

[2] Which means in particular that there seems to be no good reference for it. A special case of this proposition is proved in [EKL04, Section 9]. The proof of this proposition is left as an exercise to the energetic reader.

[3] See Exercise 4.15 for a definition of the shift map.

[4] For X which is only locally compact, one can extend T to a map $\tilde{T}$ on its one-point compactification $\tilde{X} = X \cup \{\infty\}$ fixing ∞ and define $h_{\mathrm{top}}(X, T) = h_{\mathrm{top}}(\tilde{X}, \tilde{T})$

Definition. *The* topological entropy *of* (X,T) *is defined by*

$$H(X,T,\epsilon) = \varliminf_{k\to\infty} \frac{\log N(X,T,k,\epsilon)}{k}$$
$$h_{\rm top}(X,T) = \lim_{\epsilon\to 0} H(X,T,\epsilon).$$

The topological entropy of a flow $\{a_t\}$ is defined as in §5.5 and denoted by $h_{\rm top}(X,a_\bullet)$.

5.11. Topological entropy and the ergodic theoretic entropy are related by the *variational principle* (see e.g. [Gla03, Theorem 17.6] or [KH95, Theorem 4.5.3])

Proposition. *Let X be a compact metric space and $T : X \to X$ a homeomorphism.*[1] *Then*

$$h_{\rm top}(X,T) = \sup_\mu h_\mu(T)$$

where the sup *runs over all T-invariant probability measures supported on X.*

Note that when $\mu \mapsto h_\mu(T)$ is upper semicontinuous (see §5.7) the supremum is actually attained by some T-invariant measure on X.

6. Diagonalizable actions and the set of exceptions to Littlewood's conjecture

6.1. As we have seen in §4, the action of a group H on a locally homogeneous space $X = \Gamma\backslash G$ for H generated by unipotent subgroups is quite well understood. The action of one parameter Ad-diagonalizable groups is also reasonably well understood; at least sufficiently well understood to see that there is no useful measure classification theorem in this case, since there are simply too many invariant measures (but cf. [LS05b, Question 1] and in a different direction §7).

Our understanding of the action of multidimensional groups H which are not generated by (Ad-)unipotents is much less satisfactory. If H contains some unipotents one can typically get quite a bit of mileage by investigating first the action of the subgroup generated by these unipotent elements (see e.g. [MT96]). A typical case which is at present not well understood is the action of abelian groups H which are Ad-diagonalizable[2] over $\mathbb{R}$ with $\dim H \geq 2$, in which case one expects a "Ratner like" measure classification theorem should be true.

[1] This proposition also easily implies the analogous statement for flows $\{a_t\}$.

[2] I.e. groups H whose image under the adjoint representation is diagonalizable over $\mathbb{R}$

6.2. Following is an explicit conjecture (essentially this is [Mar00, Conjecture 2]; similar conjectures were given by Katok and Spatzier in [KS96] and Furstenberg (unpublished)):

Conjecture. *let G be a connected Lie group, $\Gamma < G$ a lattice, and $H < G$ a closed connected group generated by elements which are Ad-diagonalizable over $\mathbb{R}$. Let μ be a H-invariant and ergodic probability measure. Then at least one of the following holds:*

(i) *μ is homogeneous (cf. §4.5)*
(ii) *μ is supported on a single periodic orbit $L.x$ which has an algebraic rank one factor.*[1]

6.3. The existence of the second, not quite algebraic, alternative in Conjecture 6.2 (§6.2.(ii)) is a complication (one of many…) we have not encountered in the theory of unipotent flows. Fortunately in some cases, in particular in the case we will focus on in this section of the full diagonal group acting on $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$, this complication can be shown not to occur, e.g. by explicitly classifying the possible H-invariant periodic orbits (not necessarily of the group H) and verifying none of them have rank one factors.[2]

6.4. The study of such multiparameter diagonalizable actions has a long history and there are contributions by many authors. Instead of surveying this history we refer the reader to [Lin05b, EL06]. Rather we focus here on a specific case: $G = \mathrm{SL}(n,\mathbb{R})$, $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, and $H < G$ the group of all diagonal matrices, mostly for $n = 3$, and present results from one paper [EKL04]. For the remainder of this section, we set $X_n = \mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$.

6.5. In this case, Conjecture 6.2 specializes to the following:

Conjecture. *Let H be the group of diagonal matrices in $\mathrm{SL}(n,\mathbb{R})$, $n \geq 3$. Then any H-invariant and ergodic probability measure μ on X_n is homogeneous.*

It is not hard to classify the possible homogeneous measures (see e.g. [LW00]). For n prime, the situation is particularly simple: any H-invariant homogeneous measure on X_n is either the natural measure on a H-periodic orbit, or the G invariant measure m on X_n.

[1] Formally: there exists a continuous epimorphism ϕ of L onto a Lie group F such that $\phi(\mathrm{stab}_L(x))$ is closed in F and $\phi(H)$ is a one parameter subgroup of F containing no nontrivial Ad-unipotent elements.

[2] This complication does occur [Ree82] when classifying invariant probability measures for certain other lattices Γ in $\mathrm{SL}(n,\mathbb{R})$ (and H the full group of diagonal matrices), and even in $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})$ if one considers also infinite Radon measures.

6.6. In [EKL04] we give the following partial result towards Conjecture 6.5:

Theorem (Einsiedler, Katok and L.[EKL04, Theorem 1.3])**.** *Let H be the group of diagonal matrices as above and $n \geq 3$. Let μ be an H-invariant and ergodic probability measure on X_n. Then one of the following holds:*

(i) *μ is an H-invariant homogeneous measure which is not supported on a periodic H-orbit.*

(ii) *for every one-parameter subgroup $\{a_t\} < H$, $h_\mu(a_\bullet) = 0$.*

By the classification of H-invariant homogeneous measures alluded to in §6.5, if (i) holds μ is not compactly supported.

6.7. Theorem 6.6 is proved by combining two techniques: a "low entropy" method developed in [Lin06] and a "high entropy" method developed in [EK03]. Techniques introduced by Ratner in her study of horocycle flows in [Rat82] and subsequent papers are used in the former method. Ratner's measure classification theorem (§4.6) is also used in the proof.

As in §4, the proof of Theorem 6.6 is beyond the scope of this paper; some hints on these methods can be found in [Lin05b], but the reader who wants study the proof should consult [EK03, Lin06, EKL04].

6.8. In §4 the fact that there were many invariant measures, even though they were explicitly given and came from countably many nice families had caused considerable difficulties when we tried to actually use this measure classification. One would think that the partial measure classification given in Theorem 6.6 would be even more difficult to use. Fortunately, this is not the case, and the key is the semicontinuity of entropy (§5.8). Using the semicontinuity one can sometimes, such as in the case of arithmetic quantum unique ergodicity considered in §7, verify positive entropy of a limiting measure by other means (see also [ELMV06a]), and sometimes, such as in the case of Littlewood's conjecture considered in this section or [ELMV06b] obtain partial but meaningful results which at present cannot be obtained using alternative techniques.

6.9. The following is a well known conjecture of Littlewood:

Conjecture (Littlewood (c. 1930))**.** *For every $u, v \in \mathbb{R}$,*

$$\varliminf_{n\to\infty} n\|nu\|\|nv\| = 0, \tag{6.9.a}$$

where $\|w\| = \min_{n\in\mathbb{Z}} |w - n|$ is the distance of $w \in \mathbb{R}$ to the nearest integer.

It turns out that this conjecture would follow from Conjecture 6.5. The reduction is nontrivial and is essentially due to Cassels and Swinnerton-Dyer [CSD55], though there is no discussion of invariant measures in that paper[1].

We need the following criterion for when α, β satisfy (6.9.a):

6.10. Proposition. (α, β) *satisfy* (6.9.a) *if and only if the orbit of*

$$x_{\alpha,\beta} = \pi_\Gamma\left(\begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

under the semigroup

$$H^+ = \{a(s,t) : s, t \geq 0\} \qquad a(s,t) = \begin{pmatrix} e^{s+t} & 0 & 0 \\ 0 & e^{-s} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

is unbounded[2]*. Moreover, for any $\delta > 0$ there is a compact $C_\delta \subset X_3$, so that if $\underline{\lim}_{n\to\infty} n \|n\alpha\| \|n\beta\| \geq \delta$ then $H^+.x_{\alpha,\beta} \subset C_\delta$.*

6.11. Before we prove Proposition 6.10 we need to understand better what it means for a set $E \subset X_3$ to be bounded. For this one has the following important criterion (see e.g. [Rag72, Chapter 10]):

Proposition (Mahler's compactness criterion)**.** *Let $n \geq 2$. A set $E \subset X_n$ is bounded if and only if there is some $\epsilon > 0$ so that for any $x = \pi_\Gamma(g) \in X_n$ there is no vector v in the lattice spanned by the rows of g with $\|v\|_\infty < \epsilon$.*

6.12. We now prove Proposition 6.10.

Proof. We prove only that $H^+.x_{\alpha,\beta}$ unbounded $\implies$ (6.9.a); the remaining assertions of this proposition follow similarly and are left as an exercise to the reader.

Let $\epsilon \in (0, 1/2)$ be arbitrary. By Mahler's compactness criterion (§6.11), if $H^+.x_{\alpha,\beta}$ is unbounded, there is a $h \in H^+$ such that in the lattice generated by the rows of $x_{\alpha,\beta}h^{-1}$ there is a nonzero vector v with $\|v\|_\infty < \epsilon$. This vector v is of the form

$$v = (ne^{-s-t}, (n\alpha - m)e^s, (n\beta - k)e^t)$$

[1]It is worthwhile to note that this remarkable paper appeared in 1955, many years before Conjectures 6.2 and 6.5 were made, and even before 1967 when Furstenberg made his related discoveries about scarcity of invariant sets and measures for the maps $x \mapsto 2x \bmod 1$ and $x \mapsto 3x \bmod 1$ on $\mathbb{R}/\mathbb{Z}$! The same paper also implicitly discusses the connection between Oppenheim's conjecture and the action of $\mathrm{SO}(2,1)$ on X_3.

[2]I.e. $\overline{H^+.x_{\alpha,\beta}}$ is not compact.

where n, m, k are integers at least one of which is nonzero, and $s, t \geq 0$. Since $\|v\|_\infty < 1/2$, $n \neq 0$ and $\|n\alpha\| = (n\alpha - m)$, $\|n\beta\| = (n\beta - k)$. Without loss of generality $n > 0$ and

$$n \|n\alpha\| \|n\beta\| \leq \|v\|_\infty^3 < \epsilon^3.$$

□

6.13. We now turn to answering the following question: With the partial information given in Theorem 6.6, what information, if any, do we get regarding Littlewood's conjecture?

Theorem (Einsiedler, Katok and L. [EKL04, Theorem 1.5])**.** *For any $\delta > 0$, the set*

$$\Xi_\delta = \left\{ (\alpha, \beta) \in [0,1]^2 : \underline{\lim}_{n \to \infty} n \|n\alpha\| \|n\beta\| \geq \delta \right\}$$

has zero upper box dimension[1,2]*.*

6.14. We present a variant of the proof of this theorem given in [EKL04]. The first step of the proof, which is where Theorem 6.6 is used, is an explicit sufficient criterion for a single point α, β to satisfy Littlewood's conjecture (§6.9).

Let $a_{\sigma,\tau}(t) = a(\sigma t, \tau t)$, with $a(s,t)$ as in §6.10.

Proposition. *Let α, β be such that for some $\sigma, \tau \geq 0$, the topological entropy of $a_{\sigma,\tau}$ acting on*

$$\overline{\{a_{\sigma,\tau}(t).x_{\alpha,\beta} : t \in \mathbb{R}^+\}}$$

is positive. Then α, β satisfies (6.9.a)*.*

6.15. It will be useful for us to prove a slightly stronger result:

Proposition. *Let $\sigma, \tau \geq 0$, and suppose that for $x_0 \in X_3$, the topological entropy of the action of $a_{\sigma,\tau}$ on $\overline{\{a_{\sigma,\tau}(t).x_0 : t \in \mathbb{R}^+\}}$ is positive. Then $H^+.x_0$ is unbounded.*

Note that by Proposition 6.10 the proposition above does indeed imply Proposition 6.14.

Proof. Let x_0 be as in the proposition. By the variational principal, there is a $a_{\sigma,\tau}$-invariant measure μ supported on $\overline{\{a_{\sigma,\tau}(t).x_0 : t \in \mathbb{R}^+\}}$ with $h_\mu(a_{\sigma,\tau}) > 0$.

---

[1] I.e., for every $\epsilon > 0$, for every $0 < r < 1$, one can cover Ξ_δ by $O_{\delta,\epsilon}(r^{-\epsilon})$ boxes of size $r \times r$.

[2] Since (6.9.a) depends only on α, β mod 1 it is sufficient to consider only $(\alpha, \beta) \in [0,1]^2$.

Assume in contradiction to the proposition that $H^+.x_0$ is bounded. Define for any $S > 0$

$$\mu_S = \frac{1}{S^2} \iint_0^S a(s,t).\mu \, ds \; dt,$$

with $a(s,t).\mu$ denoting the push forward of μ under the map $x \mapsto a(s,t).x$. Since $a(s,t)$ commutes with the one parameter subgroup $a_{\sigma,\tau}$, for any $a_{\sigma,\tau}$-invariant measure μ' the entropy

$$h_{\mu'}(a_{\sigma,\tau}) = h_{a(s,t).\mu'}(a_{\sigma,\tau}).$$

If μ has the ergodic decomposition $\int \mu_\xi \, d\nu(\xi)$, the measure μ_S has ergodic decomposition $S^{-2} \iint_0^S \int a(s,t).\mu_\xi \, d\nu(\xi) \, ds \, dt$ and so by §5.3, for every S

$$h_{\mu_S}(a_{\sigma,\tau}) = h_\mu(a_{\sigma,\tau}).$$

All μ_S are supported on the compact set $\overline{H^+.x_0}$, and therefore there is a subsequence converging weak* to some compactly supported probability measure μ_∞, which will be invariant under the full group H. By semicontinuity of entropy (§5.8),

$$h_{\mu_\infty}(a_{\sigma,\tau}) \geq h_\mu(a_{\sigma,\tau}) > 0,$$

hence by Theorem 6.6 the measure μ_∞ is not compactly supported[1] — a contradiction. □

6.16. Proposition 6.14 naturally leads us to the question of the size of the set of $(\alpha, \beta) \in [0,1]^2$ for which $h_{\mathrm{top}}(X_{\alpha,\beta}, a_{\sigma,\tau}) = 0$. This can be answered using the following general observation:

Proposition. *Let X' be a metric space equipped with a continuous $\mathbb{R}$-action $(t,x) \mapsto a_t.x$. Let X'_0 be a compact $a_\bullet$-invariant[2] subset of X' such that for any $x \in X'_0$,*

$$h_{\mathrm{top}}(Y_x, a_\bullet) = 0 \qquad Y_x = \overline{\{a_t.x : t \in \mathbb{R}^+\}}.$$

Then $h_{\mathrm{top}}(X'_0, a_\bullet) = 0$.

Proof. Assume in contradiction that $h_{\mathrm{top}}(X'_0, a_\bullet) > 0$. By the variational principle (§5.11), there is some $a_\bullet$-invariant measure μ on X'_0 with $h_\mu(a_\bullet) > 0$.

[1] Notice that a priori there is no reason to believe μ_∞ will be H-ergodic, while Theorem 6.6 deals with H-ergodic measures. So an implicit exercise to the reader is to understand why we can still deduce from $h_{\mu_\infty}(a_{\sigma,\tau}) > 0$ that μ_∞ is not compactly supported.

[2] Technical point: we only use that $a_t.X' \subset X'$ for $t \geq 0$. The variational principle (§5.11) is still applicable in this case.

By the pointwise ergodic theorem, for μ-almost every $x \in X_0'$ the measure μ is supported on Y_x. Applying the variational principle again (this time in the opposite direction) we get that

$$0 = h_{\mathrm{top}}(Y_x, a_\bullet) \geq h_\mu(a_\bullet) > 0$$

a contradiction. □

6.17. Corollary. *Consider, for any compact $C \subset X_3$ the set*

$$X_C = \left\{x \in X_3 : H^+.x \subset C\right\}.$$

Then for any $\sigma, \tau \geq 0$, it holds that $h_{\mathrm{top}}(X_C, a_{\sigma,\tau}) = 0$.

Proof. By Proposition 6.15, for any $x \in X_C$ the topological entropy of $a_{\sigma,\tau}$ acting on $\overline{\{a_{\sigma,\tau}(t).x : t \in \mathbb{R}^+\}}$ is zero. The corollary now follows from Proposition 6.16. □

6.18. We are now in position to prove Theorem 6.13, or more precisely to deduce the theorem from Theorem 6.6:

Proof. To show that Ξ_δ has upper box dimension zero, we need to show, for any $\epsilon > 0$, that for any $r \in (0,1)$ the set Ξ_δ can be covered by $O_\epsilon(r^{-\epsilon})$ boxes of side r, or equivalently that any r-seperated set (i.e. any set S such that for any $x, y \in S$ we have $\|x - y\|_\infty > r$) is of size $O_{\delta,\epsilon}(r^{-\epsilon})$.

Let C_δ be as in Proposition 6.10. Let d denote a left invariant Riemannian metric on $G = \mathrm{SL}(3,\mathbb{R})$. Then d induces a metric, also denote by d on X_3. For $a, b \in \mathbb{R}$ let

$$g_{a,b} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since C_δ is compact, and d induced from a left invariant Riemannian metric, there will be r_0, c_0 such that for any $x \in C_\delta$ and $|a|, |b| < r_0$

$$d(x, g_{a,b}.x) \geq c_0 \max(|a|, |b|).$$

For any $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$ we have that

$$x_{\alpha,\beta} = g_{\alpha'-\alpha,\beta'-\beta}.x_{\alpha',\beta'}$$

and more generally for any t

$$a_{1,1}(t).x_{\alpha,\beta} = g_{e^{3t}(\alpha'-\alpha),e^{3t}(\beta'-\beta)}.a_{1,1}.x_{\alpha',\beta'}.$$

It follows that if $S \subset \Xi_\delta$ is r separated for $r = e^{-3t}r_0 \in (0, r_0)$ then

$$S' = \{x_{\alpha,\beta} : (\alpha,\beta) \in S\}$$

is $(t, c_0 r_0)$-separated for $a_{1,1}$ in the sense of (§5.10). By definition of C_δ and Ξ_δ, we have that (in the notations of §6.17) the set $S' \subset X_{C_\delta}$, a set which has zero topological entropy with respect to the group $a_{1,1}$. It follows that the cardinality of a maximal $(t, c_0 r_0)$-separated set in S' is at most $O_{\delta,\epsilon}(\exp(\epsilon t))$; hence for $r < r_0$ the cardinality of a maximal r-separated subset of Ξ_δ is $O_{\delta,\epsilon}(r^{-\epsilon})$. □

6.19. The alert and optimistic reader[1] may hope that there is some choice of (σ, τ), e.g $(1/3, 2/3)$, for which the condition of Proposition 6.14 holds for all $(\alpha, \beta) \in [0,1]^2$. To avoid trivial counterexamples of e.g. $\alpha, \beta \in \mathbb{Q}$, we can require that for every $(\alpha, \beta) \in [0,1]^2$ either the condition of Proposition 6.14 (and hence Littlewood's conjecture) holds or $\{a_{\sigma,\gamma}(t).x_{\alpha,\beta} : t \in \mathbb{R}^+\}$ is unbounded (in which case Littlewood's conjecture follows readily from Proposition 6.10). Though I could not immediately come up with a counterexample for $(1/3, 2/3)$, this is extremely unlikely to be true.

We recall the following well-known conjecture of Furstenberg (which dates back to the time of [Fur67] but is not stated there; one place where it is explicitly stated is [Mar00, Conjecture 5]):

Conjecture (Furstenberg)**.** *for any irrational* $x \in \mathbb{R}/\mathbb{Z}$

$$\frac{h_{\text{top}}\left(\overline{\{2^n x : n \in \mathbb{N}\}}, \times 2\right)}{h_{\text{top}}(\mathbb{R}/\mathbb{Z}, \times 2)} + \frac{h_{\text{top}}\left(\overline{\{3^n x : n \in \mathbb{N}\}}, \times 3\right)}{h_{\text{top}}(\mathbb{R}/\mathbb{Z}, \times 3)} \geq 1; \tag{6.19.a}$$

In particular, the conjecture implies that for any irrational x one of the entropies in the numerators in (6.19.a) is positive. Even this is not known.

6.20. In analogy with this conjecture, we give the following, which in view of Proposition 6.14 would imply Littlewood's conjecture (§6.9):

Conjecture. *Suppose* $\alpha, \beta \in [0,1]^2$ *is such that* $\{a_{\sigma,\tau}(t).x_{\alpha,\beta} : t \in \mathbb{R}^+\}$ *is bounded for both* $(\sigma, \tau) = (1/3, 2/3)$ *and* $(\sigma, \tau) = (2/3, 1/3)$.

Then α, β *satisfy the conditions of Proposition 6.14*[2] *either for* $(\sigma, \tau) = (1/3, 2/3)$ *or for* $(\sigma, \tau) = (2/3, 1/3)$ *(or both).*

One may wonder whether there are any α, β satisfying the boundedness assumptions of the conjecture. It is strongly expected that there should be many such pairs (α, β), but currently this is an open problem; indeed even the existence of one such pair is open:

[1] May he prove many theorems.

[2] I.e. the topological entropy of $a_{\sigma,\tau}$ acting on $\overline{\{a_{\sigma,\gamma}(t).x_{\alpha,\beta} : t \in \mathbb{R}^+\}}$ is positive.

6.21. Conjecture (W. Schmidt [Sch83, p. 274]). *There is some* α, β *for which* $\{a_{\sigma,\tau}(t).x_{\alpha,\beta} : t \in \mathbb{R}^+\}$ *is bounded for both* $(\sigma, \tau) = (1/3, 2/3)$ *and* $(\sigma, \tau) = (2/3, 1/3)$.

The relation between this conjecture and Littlewood's conjecture is that if it is false, Littlewood's conjecture is true[1]...

7. Applications to quantum unique ergodicity

7.1. In this section we present an application of measure classification to equidistribution; but it is slightly unusual as we deal not with equidistribution of points or orbits but of *eigenfunctions*.

This equidistribution problem, a.k.a. the quantum unique ergodicity problem, is part of a much larger topic, namely the study of quantum mechanical behavior of classically chaotic systems. A basic overview of the subject was given by de Bièvre in this volume [dB06], and a discussion of quantized versions of toral automorphisms by Rudnick [Rud06].

Our discussion will be quite brief, in part because of these two contributions, and in part because the topic seems to be well covered by other sources, e.g. [Lin05a].

7.2. Let M be a compact surface of constant negative curvature. Such a surface can be presented as $M = \Gamma\backslash\mathbb{H}$ with $\Gamma < \mathrm{SL}(2, \mathbb{R})$ a torsion free lattice acting on $\mathbb{H}$ by Mobius transformations. The hyperbolic plane $\mathbb{H}$ possesses a differential operator, the hyperbolic Laplacian $\Delta = y^2(\partial_x^2 + \partial_y^2)$, which is invariant under all Mobius transformations. Because of this invariance property Δ can be viewed also as a differential operator also on M.

Since M is compact, $L^2(M)$ is spanned by the Laplacian eigenfunctions, i.e. by functions ϕ_i satisfying $\Delta\phi_i = -\lambda_i\phi_i$ with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$; we also normalize the ϕ_i so that $\|\phi_i\|_2 = 1$.

It is natural to consider the measures $\tilde{\mu}_i$ defined by $d\tilde{\mu}_i = |\phi_i|^2\, dm$, m being the Riemannian area. For example, in quantum mechanics the eigenfunctions ϕ_i correspond to the steady states of a simple[2] particle restricted to the surface M, and given a nice measurable $A \subset M$, $\tilde{\mu}_i(A)$ is the probability of finding our particle in A (assuming we have some physical contraption which is able to measure if our particle is in A or not).

[1] Of course the converse implication does not hold.

[2] Nonrelativistic, spinless, etc.

7.3. Exploiting the connections between quantum and classical mechanics, Šnirel′man, Colin de Verdière and Zelditch [Šni74, CdV85, Zel87] have shown that $N^{-1}\sum_{i=0}^{N-1}\tilde{\mu}_i$ converge weak* to $m(M)^{-1}m$. This phenomenon is called *quantum ergodicity*. Rudnick and Sarnak conjectured that there is no need to take the average, i.e.:

Conjecture (Quantum Unique Ergodicity [RS94])**.** *For $M = \Gamma\backslash\mathbb{H}$ we have that $\tilde{\mu}_i$ converges weak* to $m(M)^{-1}m$.*

Both the quantum ergodicity theorem and the quantum unique ergodicity conjecture extend to more general M: the quantum ergodicity theorem extends to a general compact manifold M for which the geodesic flow is ergodic[1], the conjecture to manifolds with negative sectional curvature (not necessarily constant) in any dimension. The quantum unique ergodicity conjecture is not expected to hold in the full generality of the quantum ergodicity theorem (cf. [Don03]).

7.4. So far no dynamics seem to enter; however, a key point in the proof of the quantum ergodicity theorem is that the measures $\tilde{\mu}_i$ "lift" to probability measures on the unit cotangent bundle which become increasingly invariant under the geodesic flow. Specializing to the case $M = \Gamma\backslash\mathbb{H}$ we consider, S^*M is essentially equal to[2] $X = \Gamma\backslash\operatorname{SL}(2,\mathbb{R})$, with the geodesic flow corresponding to the action of the diagonalizable one parameter group $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. More formally, the lifting procedure allows us to define for any i measures measures μ_i on X so that

(i) for any $f \in C^\infty(M)$ (viewed as a subspace of $C^\infty(X)$)
$$\left| \int f\, d\tilde{\mu}_i - \int f\, d\mu_i \right| \to 0 \qquad \text{as } i \to \infty$$

(ii) for any $h \in C^\infty(X)$,
$$\int \left.\frac{dh(a_t.x)}{dt}\right|_{t=0} d\mu_i(x) \to 0 \qquad \text{as } i \to \infty.$$

We call any weak* limit of the lifted measures μ_i a *quantum limit*. It follows from (i) and (ii) above that any quantum limit is a_t-invariant and that any weak* limit of the measures $\tilde{\mu}_i$ is the image of a quantum limit under the projection $\pi_{;K} : X \to M = X/K$ where K is the compact group $\operatorname{SO}(2,\mathbb{R})$.

[1] For the Liouville measure on the unit cotangent bundle of M.

[2] More precisely, X is a double cover of the unit contangent bundle.

7.5. Without inputting any additional information on the measures $\tilde{\mu}_i$ and μ_i we cannot go further because there is no useful measure classification theorem for the action of the one parameter group a_t. Very recently, Nalini Anantharaman [Ana04] has been able to use the WKB approximation to obtain additional such information, proving in particular that any quantum limit has positive entropy[1] However, even with this entropy bound there is no useful measure classification.

7.6. We vary the setting by taking Γ to be a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ or of certain lattices that arise from quaternionic division algebras over $\mathbb{Q}$ that are unramified over $\mathbb{R}$. The latter lattices are slightly harder to define[2] but have the advantage that X is compact.

7.7. In both cases, for all but finitely many primes p, we have a map — the *Hecke correspondence* — T_p from X to $(p+1)$-tuples of points of X, and a corresponding operator, also denoted by T_p on $L^2(X)$ preserving $L^2(M)$ — the *Hecke operator*. For $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ these are defined in §4.25. These Hecke operators play a very important role in the spectral theory of M. In particular, the subspace of $L^2(M)$ spanned by L^2-eigenfunctions of Δ is spanned by joint eigenfunctions of Δ and all Hecke operators[3]. Let ϕ_i be such a sequence of L^2-normalized eigenfunctions of all these operators, and define using the ϕ_i measures $\tilde{\mu}_i$ and μ_i as above. Any weak* limit of the μ_i will be called an *arithmetic quantum limit.*

7.8. In addition to being invariant under the flow a_t, arithmetic quantum limits have the following subtle additional property (see [Lin06, §8]: T_p-recurrence.

Definition. *A measure μ on X is said to be* T_p-recurrent *for a fixed prime p if for any set $E \subset X$ with $\mu(E) > 0$, for μ-almost every $x \in E$ and any n, there exists an m so that*

$$(T_p)^m(x) \cap \left(E - \bigcup_{k=1}^{n} (T_p)^k(x) \right) \neq \emptyset.$$

An arithmetic quantum limit is T_p-recurrent for every prime p for which T_p is defined.

[1]Since quantum limits a priori need not be ergodic, this does not show that quantum limits give zero measure to an a_t-periodic orbits. It does, however, show that e.g. no quantum limits gives full measure to a countable union of a_t-periodic orbits.

[2]We do not do this here; the interested readers is referred to [Lin06] for further details.

[3]If M is not compact it is no longer true that L^2-eigenfunctions of Δ.

7.9. This extra recurrence assumption seems to be almost as good as having invariants under an additional one parameter group. Indeed, we conjecture the following:

Conjecture. *Let* Γ *be a congruence lattice as in §7.6. Let* μ *be an* a_t*-invariant probability measure on* $X = \Gamma\backslash G^1$ *which is also* T_p*-recurrent for a (single) prime* p*. Then* μ *is a linear combination of* a_t*-invariant algebraic measures.*

As in §6, this is currently known under an entropy assumption:

Theorem ([Lin06, Theorem 1.1])**.** *Let* X *be as in the conjecture. Let* μ *be an* a_t*-invariant probability measure on* $X = \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ *which is also*

(i) T_p*-recurrent for a (single) prime* p

(ii) $h_{\mu_\xi}(a_\bullet) > 0$ *for every* a_t*-invariant ergodic component* μ_ξ *of* μ*.*

then μ *is the* G*-invariant probability measure on* X*.*

This theorem is proved using the "low entropy" method mentioned in §6.7.

7.10. Condition (ii) in Theorem 7.9 was verified for quantum limits by Bourgain and the author in [BL03] using combinatorial properties of Hecke points, and using all Hecke operators. No microlocal analysis was involved. Thus combining the results of [BL03] and [Lin06] one gets

Theorem. *Let* Γ *be a congruence lattice as in §7.6. Then any arithmetic quantum limit is of the form* cm*,* m *being the* G*-invariant measure on* X*.*

One would have liked to prove that c has to be $1/m(X)$, but this is currently unknown in the noncompact case (though of course this is true in the compact case).

7.11. Theorem 7.10 is equivalent to the statement that for any $f \in C_0(X)$ with $\int f\,dm = 0$

$$\int_X f(x)\,d\mu_i(x) \to 0 \qquad \text{as } i \to \infty. \tag{7.11.a}$$

In particular, if ϕ_i is a sequence of joint (say real) eigenfunctions as above

$$\int_M \phi_j(x)\phi_i(x)^2\,dx \to 0 \qquad \text{as } i \to \infty.$$

[1] Where as above $G = \mathrm{SL}(2,\mathbb{R})$.

An identity of Watson [Wat01] expresses this triple product in terms of L-functions; specifically for $M = \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$

$$\left| \int_M \phi_j(x)\phi_i(x)^2\,dx \right|^2 = \frac{\pi^6 \Lambda(1/2, \phi_i \times \phi_i \times \phi_j)}{6^6 \Lambda(1, \mathrm{Sym}^2\,\phi_i)^2 \Lambda(1, \mathrm{Sym}^2\,\phi_j)}$$

with Λ denoting the completed L-function. All the terms in the denominator are well understood, the numerator is much more mysterious. Currently our understanding of the numerator is not sufficient to deduce (7.11.a); but if one knew e.g. the Riemann hypothesis for $\Lambda(s, \phi_i \times \phi_i \times \phi_j)$ one would get (7.11.a) with an optimal[1] rate of convergence. Conversely, a more quantative form of (7.11.a) would yield improved ("subconvex") estimates on $\Lambda(1/2, \phi_i \times \phi_i \times \phi_j)$.

7.12. Anantharaman's recent results discussed in §7.5 give only the information on the entropy of μ, i.e. on the average of the entropy of the ergodic components of μ but use no Hecke operators. This can be used to give weaker variants of the above theorem even if one assumes that the ϕ_i are eigenfunctions of Δ and one additional Hecke operator.

7.13. Using the same general strategy, Silberman and Venkatesh have been able to prove a version of arithmetic quantum unique ergodicity for other $\Gamma\backslash G/K$, specifically for locally symmetric spaces arising from division algebras of prime degree. While the strategy remains the same, several new ideas are needed for this extension, in particular a new micro-local lift for higher rank groups [SV04, Sil05].

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