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Equidistribution in homogeneous spaces and number theory

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Abstract. We survey some aspects of homogeneous dynamics — the study of algebraic group actions on quotient spaces of locally compact groups by discrete subgroups. We give special emphasis to results pertaining to the distribution of orbits of explicitly describable points, especially results valid for the orbits of *all* points, in contrast to results that characterize the behavior of orbits of *typical* points. Such results have many number theoretic applications, a few of which are presented in this note. Quantitative equidistribution results are also discussed.

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1. Introduction

1.1. In this note we discuss a certain very special class of dynamical systems of algebraic origin, in which the space is the quotient of a locally compact group G by a discrete subgroup Γ and the dynamics is given by the action of some closed subgroup $H < G$ on G/Γ by left translations, or more generally by the action of a subgroup of the group of affine transformations on G that descends to an action on G/Γ. There are several natural classes of locally compact groups one may consider — connected Lie groups, linear algebraic groups (over $\mathbb{R}$, or $\mathbb{Q}_p$, or perhaps general local field of arbitrary characteristic), finite products of linear algebraic groups over different fields, or the closely related case of linear algebraic groups over adeles of a global field such as $\mathbb{Q}$.

1.2. Such actions turn out to be of interest for many reasons, but in particular are intimately related to deep number theoretic questions. They are also closely connected to another rich area: the spectral theory of such quotient spaces, also known as the theory of automorphic forms, which has so many connections to both analytic and algebraic number theory that they are hard to separate.

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From the point of view of these connections between dynamics and number theory, perhaps the most interesting quotient space is the space X_d of lattices in $\mathbb{R}^d$ up to homothety, which is naturally identified with $\mathrm{PGL}(d,\mathbb{R})/\mathrm{PGL}(d,\mathbb{Z})$. There are several historical sources for the use of this space in number theory. One prominent historical source is H. Minkowski's work on Geometry of Numbers c. 1895; and while (like most mathematical research areas) it is hard to draw the precise boundaries of the Geometry of Numbers, certainly at its heart is a systematic use of lattices, and implicitly the space of lattices, to the study of number theoretic problems of independent interest.

The use of tools and techniques of ergodic theory and dynamical systems, and perhaps no less importantly the use of the dynamical point of view, to study these actions has proven to be a remarkably powerful method with applications in several rather diverse areas in number theory and beyond, but in particular for many of the problems considered in the Geometry of Numbers. This is a very active direction of current research sometimes referred to as Flows on Homogeneous Spaces, though the shorter term Homogeneous Dynamics seems to be gaining popularity.

1.3. We present below a Smörgåsbord of topics from the theory. The selection is somewhat arbitrary, and is biased towards aspects that I have personally worked on. A brief overview of the topics discussed in each section is given below:

§2. Actions of unipotent and diagonalizable groups are discussed. Thanks to the deep work of several mathematicians the actions of unipotent groups are quite well understood (at least on a qualitative level). The actions of diagonalizable groups are much less understood. These diagonalizable actions behave quite differently depending on whether the acting group is one dimensional or of higher dimensions; in the latter case there are several long-standing conjectures and a few partial results toward these conjectures that are powerful enough to have applications of independent interest.

§3. We consider why the rigidity properties of an action of a multiparameter diagonalizable group is harder to understand than actions of unipotent groups (or groups generated by unipotents), and highlight one difference between these two classes of groups: growth rates of the Haar measure of norm-balls in these groups.

§4. Three applications of the measure classification results for multiparameter diagonalizable groups are presented: results regarding Diophantine approximations and Littlewood's Conjecture, Arithmetic Quantum Unique Ergodicity, and an equidistribution result for periodic orbits of the diagonal group in X_3 (a problem considered by Linnik with strong connections to L-functions and automorphic forms).

§5. We present recent progress in the study of actions of another natural class of groups that share with unipotent groups the property of large norm-balls: Zariski dense subgroups of semisimple groups or more generally groups generated by unipotents.

§6. We conclude with a discussion of the quantitative aspects of the density and equidistribution results presented in the previous sections regarding orbits of group actions on homogeneous spaces.

2. Actions of unipotent and diagonalizable groups

2.1. Part of the beauty of the subject is that for a given number theoretic application one is led to consider a very concrete dynamical system. Perhaps the best way to illustrate this point is by example. An important and influential milestone in the theory of flows on homogeneous spaces has been Margulis' proof of the longstanding Oppenheim Conjecture in the mid 1980's [Mar87]. The Oppenheim Conjecture states that if $Q(x_1, \dots, x_d)$ is an indefinite quadratic form in $d \geq 3$ variables, not proportional to a form with integral coefficients, then

$$\inf \left\{ |Q(v)| : v \in \mathbb{Z}^d \setminus \{0\} \right\} = 0. \tag{2.1}$$

By restricting Q to a suitably chosen rational subspace, it is easy to reduce the conjecture to the case of $d = 3$, and instead of considering the values of an arbitrary indefinite ternary quadratic form on the lattice $\mathbb{Z}^d$ one can equivalently consider the values an arbitrary lattice ξ in $\mathbb{R}^d$ attains on the fixed indefinite ternary quadratic form, say $Q_0(x, y, z) = 2xz - y^2$. The symmetry group

$$\mathrm{SO}(1,2) = \left\{ h \in \mathrm{SL}(3,\mathbb{R}) : Q_0(v) = Q_0(hv) \text{ for all } v \in \mathbb{R}^3 \right\}$$

is a noncompact semisimple group. By the definition of H, for every $h \in H = \mathrm{SO}(1,2)$ and $\xi \in X_3$ the set of values Q_0 attains at nonzero vectors of the lattice ξ coincides with the set of values this quadratic form attains at nonzero vectors of the lattice $h.\xi$, i.e. the lattice obtained from ξ by applying the linear map h on each vector. It is now an elementary observation, using Mahler's Compactness Criterion, that for $\xi \in X_3$,

$$\inf \{ |Q_0(v)| : v \in \xi \setminus \{0\} \} = 0 \quad \iff \quad \text{the orbit } H.\xi \text{ is unbounded.}$$

G. A. Margulis established the conjecture by showing that any orbit of H on X_3 is either periodic or unbounded (see [DM90a] for a highly accessible account); the lattices corresponding to periodic orbits are easily accounted for, and correspond precisely to indefinite quadratic forms proportional to integral forms. Here and throughout, an orbit of a group H acting on a topological space X is said to be *periodic* if it is closed and supports a finite H-invariant measure.

We note that the homogeneous space approach for studying values of quadratic forms was noted by M.S. Raghunathan who also gave a much more general conjecture in this direction regarding orbit closures of connected unipotent groups in the quotient space G/Γ. In retrospect one can identify a similar approach in the remarkable paper [CSD55] by Cassels and Swinnerton-Dyer.

2.2. This example illustrates an important point: in most cases it is quite easy to understand how a typical orbit behaves, e.g. to deduce from the ergodicity of H acting on X_3 that for almost every ξ the orbit $H.\xi$ is dense in X_3; but for many number theoretical applications one needs to know how orbits of individual points behave — in this case, one needs to understand the orbit $H.\xi$ for **all** $\xi \in X_3$.

2.3. Raghunathan's Conjecture regarding the orbit closures of groups generated by one parameter unipotent subgroups, as well as an analogous conjecture by S.G. Dani regarding measures invariant under such groups [Dan81] have been established in their entirety[1] in a fundamental series of papers by M. Ratner [Ra91a, Ra90a, Ra90b, Ra91b].

Theorem 1 (Ratner). *Let G be a real Lie group, $H < G$ a subgroup generated by one parameter* Ad*-unipotent groups, and Γ a lattice in G. Then:*

(i) Any H-invariant and ergodic probability measure μ on G/Γ is an L-invariant measure supported on a single periodic L-orbit of some subgroup $L \leq G$ containing H

(ii) For any $x \in G/\Gamma$, the orbit closure $\overline{H.x}$ is a periodic orbit of some subgroup $L \leq G$ containing H.

A measure μ as in (i) above will be said to be *homogeneous* .

This fundamental theorem of Ratner, which in applications is often used in conjunction with the work of Dani and Margulis on nondivergence of unipotent flows [Mar71, Dan86] and related estimates on how long a unipotent trajectory can spend near a periodic trajectory of some other group (e.g. as developed in [DM90b, DM93] or [Ra91b]) give us very good (though non-quantitative) understanding of the behavior of individual orbits of groups H generated by one parameter unipotent subgroups, such as the group $\mathrm{SO}(1,2)$ considered above. It has been extended to algebraic groups over $\mathbb{Q}_p$ and to S-algebraic groups (products $G = \prod_{p \in S} \mathbb{G}_i(\mathbb{Q}_p)$ with the convention that $\mathbb{Q}_\infty = \mathbb{R}$) by Ratner [Ra95] and Margulis-Tomanov [MT94].

2.4. These theorems on unipotent flows have numerous number theoretical applications, much too numerous to list here. A random sample of such applications, to give a flavor of their diverse nature, is the substantial body of work regarding counting of integer and rational points on varieties, e.g. Eskin, Mozes and Shah [EMS96] who give the asymptotic behavior as $T \to \infty$ of the number of elements $\gamma \in \mathrm{SL}(d,\mathbb{Z})$ with a given characteristic polynomial satisfying $\|\gamma\| < T$ (see also H. Oh's survey [Oh10] for some more recent counting results of interest); Vatsal's proof of a conjecture of Mazur regarding non-vanishing of certain L-functions associated to elliptic curves at the critical point [Vat02]; Elkies and McMullen's study of gaps in the sequence $\sqrt{n}$ mod 1 [EM04]; and Ellenberg and Venkatesh theorems on representing positive definite integral quadratic forms by other forms [EV08].

[1] Special cases of Raghunathan's Conjecture were established by Dani and Margulis [DM90b] using a rather different approach.

2.5. The action of one parameter diagonalizable groups on homogeneous spaces, such as the action of $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ on X_2 is fairly well understood (at least in some aspects), but these $\mathbb{R}$-actions behave in a drastically different way than e.g. one parameter unipotent groups. The case of a_t acting on X_2 is particularly well studied. There is a close collection between this action and the continued fraction expansion of real numbers that has been used already by E. Artin [Art24], and was further elucidated by C. Series [Ser85] and others, that essentially allows one to view this system as a flow over a simple symbolic system. Any ergodic measure preserving flow of sufficiently small entropy can be realized as an invariant measure for the action of a_t on X_2, and there is a wealth of irregular orbit closures. There is certainly also a lot of mystery remaining regarding this action and in particular due to the lack of rigidity it is extremely hard to understand the behavior of specific orbits of the action, e.g.:

Question 1. *Is the orbit of the lattice*

$$\begin{pmatrix} 1 & \sqrt[3]{2} \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2$$

under the semigroup $\{a_t : t \geq 0\}$ *dense in* X_2*?*

Even showing that this orbit is unbounded is already equivalent to the continued fraction expansion of $\sqrt[3]{2}$ being unbounded, a well known and presumably difficult problem. While Artin constructs in [Art24] a point in X_2 which has a dense a_t-orbit in a way that can be said to be explicit, I do not know of any construction of a lattice in X_2 generated by vectors with algebraic entries that is known to have a dense a_t-orbit.

2.6. Actions of higher rank diagonal groups are much more rigid than one parameter diagonal group, though not quite as rigid as the action of groups generated by unipotents. Many of the properties such actions are expected to satisfy are still conjectural, though there are several quite usable partial results that can be used to obtain nontrivial number theoretic consequences. A basic example of such actions is the action of the $(d-1)$-dimensional diagonal group $A < \mathrm{PGL}(d, \mathbb{R})$ on the space of lattices X_d for $d \geq 3$. A similar phenomenon is exhibited in a somewhat more elementary setting by the action of a multiplicative semigroup Σ of integers containing at least two multiplicative independent elements on the 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. This surprising additional rigidity of multidimensional diagonalizable groups has been discovered by Furstenberg [Fur67] in the context of multiplicative semigroups acting on $\mathbb{T}$, and is in a certain sense implicit in the work of Cassels and Swinnerton-Dyer [CSD55].

2.7. Actions of diagonalizable groups also appear naturally in many contexts. In the aforementioned paper of Cassels and Swinnerton-Dyer [CSD55] the following conjecture is given:

Conjecture 2. *Let $F(x_1,\dots,x_d)=\prod_{i=1}^d\left(\sum_{j=1}^d g_{ij}x_j\right)$ be a product of d-linearly independent linear forms in d variables, not proportional to an integral form (as a homogeneous polynomial in d variables), with $d\geq 3$. Then*

$$\inf\left\{|F(v)| : v\in\mathbb{Z}^d\smallsetminus\{0\}\right\}=0. \tag{2.2}$$

This conjecture in shown in [CSD55] to imply Littlewood's Conjecture (see §4.1), and seems to me to be the more fundamental of the two. As pointed out by Margulis, e.g. in [Mar97], Conjecture 2 is equivalent to the following:

Conjecture 2'. *Any A-orbit $A.\xi$ in X_d for $d\geq 3$ is either periodic or unbounded.*

2.8. A somewhat more elementary action with similar features was studied by Furstenberg [Fur67]. Let Σ be the multiplicative semigroup of $\mathbb{N}$ generated by two multiplicative independent integers a,b (i.e. $\log a/\log b\notin\mathbb{Q}$). In stark contrast to cyclic multiplicative semigroups, Furstenberg has shown that any Σ-invariant closed subset $X\subset\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is either finite or $\mathbb{T}$ and gave the following influential conjecture:

Conjecture 3. *Let $\Sigma=\{a^nb^k : n,k\geq 0\}$ be as above. The only Σ-invariant probability measure on $\mathbb{R}/\mathbb{Z}$ with no atoms is the Lebesgue measure.*

This conjecture can be phrased equivalently in terms of measures on G/Γ invariant under left translation by a rank two diagonalizable group H for an appropriate *solvable* group G and lattice $\Gamma<G$; e.g. if a,b are distinct primes, we can take

$$\begin{aligned}
H&=\left\{(s,t,r) : s\in\mathbb{R}^\times, t\in\mathbb{Q}_a^\times, r\in\mathbb{Q}_r^\times, |s|\cdot|t|_a\cdot|r|_b=1\right\}\\
G&=H^\times\ltimes(\mathbb{R}\times\mathbb{Q}_a\times\mathbb{Q}_b)\\
\Gamma&=\{(s,s,s) : s=a^nb^m,\ n,m\in\mathbb{Z}\}\ltimes\{(t,t,t) : t\in\mathbb{Z}[\tfrac{1}{ab}]\}.
\end{aligned}$$

2.9. Ergodic theoretic entropy is a key invariant in ergodic theory whose introduction in the late 1950s by Kolmogorov and Sinai completely transformed the subject. At first sight it seems quite unrelated to the type of questions considered above. However, it has been brought to the fore in the study of multiparameter diagonalizable actions by D. Rudolph (based on earlier work of R. Lyons [Lyo88]), who established an important partial result towards Furstenberg's Conjecture (Conjecture 3): Rudolph classified such measures under a positive entropy condition [Rud90]. A. Katok and R. Spatzier were the first to extend this type of results to flows on homogeneous spaces [KS96], but due to a subtle question regarding ergodicity of subactions their results do not seem to be applicable in the number theoretic context.

2.10. Some care needs to be taken when stating the expected measure classification result for actions of multiparameter diagonalizable groups on a quotient space G/Γ, even for $G=\mathrm{PGL}(3,\mathbb{R})$ and A the full diagonal group, since as pointed out by M. Rees [Ree82] (see also [EK03, §9]), any such conjecture should take into account possible scenarios where the action essentially degenerates into a one parameter action where no such rigidity occurs. An explicit conjecture regarding measures invariant under multiparameter diagonal flows was given by Margulis in [Mar00, Conjecture 2]; a similar but less explicit conjecture by Katok and Spatzier was given in [KS96], and by Furstenberg (unpublished). For the particular case of the action of the diagonal group A on the space of lattice in X_d such degeneration cannot occur[2] and one has the following conjecture:

Conjecture 4. *Let μ be an A-invariant and ergodic probability measure on X_d for $d \geq 3$ (and $A < \mathrm{PGL}(3,\mathbb{R})$ the group of diagonal matrices). Then μ is homogeneous (cf. §2.3).*

More generally, we quote the following from [EL06]:

Conjecture 5. *Let S be a finite set of places for $\mathbb{Q}$ and for every $v \in S$ let G_v be a linear algebraic group over $\mathbb{Q}_v$. Let $G_S = \prod_{v\in S} G_v$, $G \leq G_S$ closed, and $\Gamma < G$ discrete. For each $v \in S$ let $A_v < G_v$ be a maximal $\mathbb{Q}_v$-split torus, and let $A_S = \prod_{v\in S} A_v$. Let A be a closed subgroup of $A_S \cap G$ with at least two independent elements. Let μ be an A-invariant and ergodic probability measure on G/Γ. Then at least one of the following two possibilities holds:*

(i) *μ is homogeneous, i.e. is the L-invariant measure on a single, finite volume, L-orbit for some closed subgroup $A \leq L \leq G$.*

(ii) *There is some S-algebraic subgroup L_S with $A \leq L_S \leq G_S$, an element $x \in G/\Gamma$, an algebraic homeomorphism $\phi : L_S \to \tilde{L}_S$ onto some S-algebraic group $\tilde{L}_S$, and a closed subgroup $H < \tilde{L}_S$ with $H \geq \phi(\Gamma)$ so that (i) $\mu((L_S \cap G).x_\Gamma) = 1$, (ii) $\phi(A)$ does not contain two independent elements and (iii) the image of μ to $\tilde{L}_S/H$ is not supported on a single point.*

2.11. To obtain a measure classification result in the homogeneous spaces setting with only an entropy assumption and no assumptions regarding ergodicity of subactions (which are nearly impossible to verify in most applications of the type considered here) requires a rather different strategy of proof than [KS96], using two different and complementary methods. The first, known as the *high entropy method*, was developed by M. Einsiedler and Katok [EK03] and utilizes non-commutativity of the unipotent subgroups normalized by the acting group, and e.g. in the case of A acting on X_d for $d \geq 3$ allows one to conclude that any measure of sufficiently high entropy (or positive entropy in "sufficiently many directions") is the uniform measure. The other method, the *low entropy method*, was developed by the author [Lin06] where in particular an analogue to Rudolph's theorem for the action of the maximal $\mathbb{R}$-split torus[3] on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\Gamma$ is given. Even though the measure under study is invariant under a diagonalizable group and a priori has no invariance under any unipotent element, ideas from the theory

[2] For *probability* measures; there are non-homogeneous A-invariant and ergodic Radon measures on X_d.

[3] Which in this case is simply the product of the diagonal group from each factor

of unipotent flows, particularly from a series of papers of Ratner on the horocycle flow [Ra82a, Ra82b, Ra83], are used in an essential way. These two methods can be combined successfully as was done in a joint paper with Einsiedler and Katok [EKL06] where the following partial result toward Conjecture 4 is established:

Theorem 2 ([EKL06]). *Let A be the group of diagonal matrices as above and $d \geq 3$. Let μ be an A-invariant and ergodic probability measure on X_d. If for some $a \in A$ the entropy $h_\mu(a) > 0$ then μ is homogeneous.*

2.12. The high entropy method was developed further by Einsiedler and Katok in [EK05] and the low entropy method was developed further by Einsiedler and myself in [EL08]; these can be combined to give in particular the following theorem, which we state for simplicity for real algebraic groups but holds in the general S-algebraic setting of Conjecture 5 (see [EL06, §2.1.4] for more details[4]):

Theorem 3. *Let G be a semisimple real algebraic group, $A < G$ the connected component of a maximal $\mathbb{R}$-split torus, and $\Gamma < G$ an irreducible lattice. Let μ be an A-invariant and ergodic probability measure on G/Γ. Assume that:*

(i) the $\mathbb{R}$-rank of G is ≥ 2

(ii) there is no reductive proper subgroup $L < G$ so that μ is supported on a single periodic L-orbit

(iii) there is some $a \in A$ for which $h_\mu(a) > 0$.

Then μ is the uniform measure on G/Γ.

If (ii) does not hold, one can reduce the classification of A-invariant measures μ on this periodic L-orbit to the classification of $A \cap [L, L]$-invariant and ergodic measures μ' on $[L, L]/\Lambda$, with Λ a lattice in $[L, L]$. If Λ is reducible, up to finite index, $[L, L]/\Lambda = \prod_{i=1}^s L_i/\Lambda_i$ and $\mu = \prod_{i=1}^s \mu'_i$, with μ'_i an $A \cap L_i$-invariant measure on L_i/Λ_i. As long as there is some L_i with $\mathbb{R}$-rank ≥ 2 and some element $a' \in A \cap L_i$ with $h_{\mu'_i}(a') > 0$, one can apply Theorem 3 recursively to obtain a more explicit, but less concise measure classification result.

New ideas seem to be necessary to extend Theorem 3 to non-maximally split tori; in part this seems to be related to the fact that for non-maximal A much more general groups L, even solvable ones, need to be considered in case (ii).

3. A remark on invariant measures, individual orbits, and size of groups

[4] There is a slight inaccuracy in the statement of [EL06, Thm. 2.4]: either one needs to assume to begin with that $h_\mu(a) > 0$ for some $a \in A$ or one needs to allow the trivial group $H = \{e\}$ in the first case listed there.

3.1. One important difference between a group H generated by unipotent one parameter subgroups (considered as a subgroup of some ambient algebraic group G, which for simplicity we assume in this paragraph to be simple) and diagonalizable groups such as the group A of diagonal matrices in $G = \mathrm{PGL}(d, \mathbb{R})$ is the size of norm-balls in the groups H or A respectively under any nontrivial finite dimensional representation ρ of G (in particular, the adjoint representation): if λ_H and λ_A denote Haar measure on H and A respectively,

$$\lambda_H\left(\{h \in H : \|\rho(h)\| < T\}\right) \geq CT^{\alpha} \qquad \text{for some } \alpha = \alpha(\rho) > 0 \tag{3.1}$$

while

$$\lambda_A\left(\{a \in A : \|\rho(a)\| < T\}\right) \asymp (\log T)^{d-1}. \tag{3.2}$$

We shall loosely refer to groups as in (3.1) for which the volume of norm-balls is polynomial as *thick* in G, and groups where this volume is polylogarithmic as in (3.2) as *thin*.

3.2. Such norm balls appear naturally when one studies how orbits of nearby points x and y diverge — an important element of Ratner's proof of Theorem 1. Suppose e.g. G is a linear algebraic group over $\mathbb{R}$, $\Gamma < G$ a lattice and $H < G$ some closed subgroup . If $x = exp(w).y$ for $w \in \mathrm{Lie}(G)$ small, $h.x = exp(\mathrm{Ad}\, h(w)).h.y$ and these will still be reasonably close for all $h \in H$ with $\|Ad(h)\| < \|w\|^{-1}$. One can gain in the range of usable elements of H by allowing $h.x$ to be compared with a more carefully chosen point $h'.y \in H.y$, but in any case the range of usable $h \in H$ includes elements of norm bounded at most by a polynomial in $\|w\|^{-1}$. The entropy condition of Theorems 2 and 3 can be thought of as a partial compensation for the fact that the acting group is thin.

3.3. The size of norm-balls also plays an important role in another important aspect of the dynamics, namely the extent to which the behavior of individual orbits relates to any possible classification of invariant measures. We recall the following definition due to Furstenberg:

Definition 1. Let X be a locally compact space, and H an amenable group acting continuously on X. A point $x \in X$ will be said to be *generic* for an H-invariant measure μ along a Følner sequence[5] $\{F_n\}$ in H (that is usually kept implicit) if for any $f \in C_c(X)$

$$\lim_{n\to\infty} \frac{\int_{F_n} f(h.x)\, d\lambda_H(h)}{\lambda_H(F_n)} \to \int_X f(y)\, d\mu(y)$$

where λ_H is the left invariant Haar measure on H.

[5] A sequence of sets $F_n \subset H$ is said to be a Følner sequence if for any compact $K \subset G$ we have that $\lambda_H(F_n \triangle K F_n)/\lambda_H(F_n) \to 0$ as $n \to \infty$; a group H is said to be amenable if it has a Følner sequence.

By the pointwise ergodic theorem (which in this generality can be found in [Lin01]) and separability of $C_c(X)$, if $\{F_n\}$ is a sufficiently nice Følner sequence (e.g. for $H = \mathbb{R}^k$, F_n can be taken to be any increasing sequence of boxes whose shortest dimension $\to \infty$ as $n \to \infty$), and if μ is an H-invariant and ergodic probability measure, then μ almost every $x \in X$ is generic for μ along $\{F_n\}$.

3.4. As is well-known, if X is uniquely ergodic, i.e. there is a *unique* H-invariant probability measure μ on X (which will necessarily be also H-ergodic, as the ergodic measures are the extreme points of the convex set of all H-invariant probability measures) then something much stronger is true: *every* $x \in X$ is generic for μ along *any* Følner sequence (we will also say in this case that the H-orbit of x is μ-equidistributed in X along any Følner sequence).

Even if there are only two H-invariant and ergodic probability measures on X, or even if there is a unique H-invariant and ergodic probability measure on X but X is not compact, individually orbits may behave in somewhat complicated ways, failing to be generic for any measure on X. The most one can say is that if $\{F_n\}$ is Følner sequence, for large n the push forward of $(\lambda_H(F_n))^{-1}\lambda_H|_{F_n}$ restricted to a large Følner set F_n under the map $h \mapsto h.x$ is close to a linear combination (depending on n) of the two H-invariant and ergodic measures in the former case, or to c times the unique H-invariant probability measure in the latter case for some $c \in [0,1]$ (which again may depend on n).

3.5. For unipotent flows, the connection between distribution properties of individual orbits and the ensemble of invariant probability measures is exceptionally sharp. In [Ra91b] Ratner has shown that if u_t is a one parameter unipotent group, G a real Lie group, and $\Gamma < G$ a lattice then any $x \in G/\Gamma$ is generic for some homogeneous measure μ whose support contains x. A uniform version where one is allowed to vary the unipotent group as well as the starting point was given by Dani and Margulis [DM93, Thm. 2]. Another useful result in the same spirit by Mozes and Shah [MS95] classifies limits of sequences of homogeneous probability measures $(m_i)_i$ in G/Γ that are invariant and ergodic under some one parameter unipotent subgroup of G (possibly different for different i); such a limiting measure is also a homogeneous probability measure. Often if the volume of the corresponding sequence of periodic orbits goes to ∞ one can show that these homogeneous probability measures converge to the uniform measure on G/Γ. In the proof of all these results, the thickness of unipotent groups (and groups generated by unipotents), under the guise of the polynomial nature of unipotent flows, plays a crucially important role.

Even for $G = \mathrm{SL}(2,\mathbb{R})$, the connection between invariant measures and distribution properties of individual orbits for the action of unipotent groups on *infinite* volume quotients is not well understood outside the geometrically finite case, though there is some interesting work in this direction, e.g. [SS08].

3.6. For diagonalizable flows, the connection between invariant measures and behavior of individual orbits is much more tenuous. Certainly if $X = G/\Gamma$ is compact then for any $\xi \in X$ the A-orbit closure $\overline{A.\xi}$ supports an A-invariant measure:

but this measure may not be unique, nor does the support of μ have to coincide with $\overline{A.\xi}$. Counterexamples given by Maucourant [Mau10] to the topological counterpart of Conjecture 5 in [Mar00] are of precisely this type: they give an A orbit whose limit set is the support of two (or more) different homogeneous measures. An example in a similar spirit has been given by U. Shapira [Sha10, LS10] for the action of the full diagonal group A on X_3: Here ξ is the lattice

$$\xi = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} \mathbb{Z}^3$$

which for a typical $a \in \mathbb{R}$ will spiral between two *infinite* homogeneous measures supported on the closed orbits through the standard lattice $\mathbb{Z}^d$ of the groups

$$H_1 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

3.7. In special cases *isolation results* give a weak substitute for diagonal actions to the "linearization" techniques used in [DM93, MS95, Ra91b] for unipotent flows. An isolation result of this type for the action of A on X_d for $d \geq 3$ by Cassels and Swinnerton-Dyer [CSD55][6] gives in particular that if $\xi, \xi_0 \in X_d$, with

$$A.\xi_0 \subset \overline{A.\xi} \smallsetminus (A.\xi) \quad \text{and} \quad A.\xi_0 \text{ periodic} \tag{3.3}$$

then $A.\xi$ is unbounded; this has been strengthened by Barak Weiss and myself [LW01] to show that under the same assumptions $\overline{A.\xi}$ is a periodic orbit of some closed connected group H with $A \leq H \leq \mathrm{PGL}(d, \mathbb{R})$ (such periodic orbits are easily classified and in particular unless $H = A$ are unbounded). Results of this nature under somewhat less restrictive conditions than (3.3), along with some Diophantine applications, were recently given by U. Shapira and myself [LS10].

Using the Cassels Swinnerton-Dyer isolation result it is easy to show that Conjecture 4 implies Conjecture 2: indeed, if $A.\xi$ is a bounded orbit in X_d then $\overline{A.\xi}$ supports an A-invariant probability measure, and hence by the ergodic decomposition $\overline{A.\xi}$ supports an A-invariant and ergodic probability measure. Assuming Conjecture 4 this measure will be homogeneous, and by the classification alluded to in the previous paragraph the only compactly supported A-invariant homogeneous probability measures are the probability measures on periodic A-obits. Thus $\overline{A.\xi}$ contains an A-periodic measure, and unless $A.\xi$ is itself periodic we get a contradiction to the Cassels-Swinnerton-Dyer Isolation Theorem.

3.8. The field of arithmetic combinatorics has witnessed dramatic progress over the last few years with remarkable applications. One of the basic results is the following exponential sum estimate by Bourgain, Glibichuk and Konyagin [BGK06]:

[6] In the paper, Cassels and Swinnerton-Dyer treat only the case of $d = 3$, but the general case is similar.

for any δ there are $c, \epsilon > 0$ so that if p is prime, $\tilde{H}$ a subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$ with $|\tilde{H}| > p^\delta$,

$$\max_{b \in (\mathbb{Z}/p\mathbb{Z})^\times} \frac{\left|\sum_{h \in \tilde{H}} e(bh/p)\right|}{|\tilde{H}|} < cp^{-\epsilon}$$

with $e(x) = \exp(2\pi i x)$. Bourgain has proved a similar estimate with p replaced by an arbitrary integer N; this involves considerable technical difficulties since one is interested in a result in which the error term does not depend on the decomposition of N into primes. If $\tilde{H}$ is the reduction modulo N of some multiplicative semigroup $H \subset \mathbb{Z}^\times$, we can interpret this estimate as saying that for any $0 \leq b < N$, the periodic H-orbit $\left\{\frac{hb}{N} \bmod 1 : h \in \tilde{H}\right\}$ is close to being equidistributed in $\mathbb{T}$ in a quantitative way provided $|\tilde{H}| > N^\delta$.

3.9. Of particular interest to us is the semigroup $H = \left\{a^n b^k : n, k \in \mathbb{N}\right\}$ where a, b are multiplicatively independent integers. For a certain sequence of N_i (relatively prime to ab) it may well happens that $|H \bmod N| > N^\delta$ for a fixed δ, even though H is a thin sequence in the sense of §3.1. For such a sequence N_i and any choice of $b_i \in (\mathbb{Z}/N_i\mathbb{Z})^\times$, the sequence of periodic H-orbits $H.\frac{b_i}{N_i} \bmod 1$ would become equidistributed in a quantitative way as $i \to \infty$ by the theorem of Bourgain quoted above (§3.8). However there are sequences of N for which $|H \bmod N|$ is rather small — $(\log N)^{c \log\log\log N}$ [APR83]. A trivial lower bound on $|H \bmod N|$ is

$$|H \bmod N| \geq (\log_a N)(\log_b N)/2,$$

and if there were infinitely many N_i with $|H \bmod N_i| \ll (\log N_i)^2$ then the orbits $H.\frac{1}{N_i} \bmod 1$ would spend a positive proportion of their mass very close to 0, and hence fail to equidistribute.

Using the Schmidt Subspace Theorem (more precisely, its S-algebraic extension by Schlickewei) in an elegant and surprising way Bugeaud, Corvaja and Zannier [BCZ03] show that

$$\lim_{N \to \infty} \frac{|H \bmod N|}{(\log N)^2} \to \infty$$

giving credence to the following conjecture, presented as a question by Bourgain in [Bou09]:

Conjecture 6. *Let $H = \left\{a^n b^k : n, k \in \mathbb{N}\right\}$, with a, b multiplicatively independent. Then for any sequence $\left\{(b_i, N_i)\right\}$ with $N_i \to \infty$ and $b_i \in (\mathbb{Z}/N_i\mathbb{Z})^\times$ the sequence of H-periodic orbits $H.\frac{b_i}{N_i} \bmod 1$ becomes equidistributed as $i \to \infty$, i.e. for any $f \in C(\mathbb{T})$,*

$$|H|^{-1} \sum_{h \in H} f\left(h.\frac{b_i}{N_i}\right) \to \int_{\mathbb{T}} f\,dx.$$

Even if one assumes (or proves) Conjecture 3 regarding H-invariant measures, this conjecture seems challenging due to the absence of a strong connection between individual orbits and invariant measures for diagonalizable group actions (cf. §3.6).

4. Some applications of the rigidity properties of diagonalizable group actions

4.1. The partial measure classification results for actions of diagonalizable groups mentioned above, e.g. Theorems 2 and 3, have several applications. We give below a sample of three theorems, in the proof of which one of the major ingredients is the classification of positive entropy invariant measures. Several other applications are discussed in Einsiedler's notes for his lecture at this ICM [Ein10][7].

Multiparameter diagonal groups and Diophantine approximations.

4.2. Using the variational principle relating topological entropy and ergodic theoretic entropy, together with an averaging argument and use of semicontinuity properties of entropy for measures supported on compact subsets of X_d in [EKL06] the following partial result towards Conjecture 2 was deduced from Theorem 2 (see either [EKL06] or [EL10, §12] for more details):

Theorem 4 (Einsiedler, Katok and L. [EKL06]). *The set of degree d homogeneous polynomials $F(x_1, \dots, x_d)$ that can be factored as a product of d linearly independent forms in d variables that fail to satisfy* (2.2) *have Hausdorff dimension zero.*

By Conjecture 2 above, the set of such F is expected to be countable; the trivial upper bound on the dimension of the set of such F is $d(d-1)$.

4.3. Recall the following well known conjecture of Littlewood regarding simultaneous Diophantine approximations:

Conjecture 7 (Littlewood). *For any $x, y \in \mathbb{R}^2$,*

$$\inf \left\{ n \, |nx - m| \, |ny - k| : (n, m, k) \in \mathbb{Z}^3, \ n \neq 0 \right\} = 0. \tag{4.1}$$

Similar ideas as in the proof of Theorem 4 allows one to prove that the Hausdorff dimension of the set of exceptional pairs $(x, y) \in \mathbb{R}^2$ that do not satisfy (4.1) is zero. Indeed, one can be a bit more precise: for a sequence of integers $(a_k)_{k\in\mathbb{N}}$ define its combinatorial entropy as

$$h_{comb}((a_k)) = \lim_{n\to\infty} \frac{\log W_n((a_k))}{n}$$

where $W_n((a_k))$ counts the number of possible n-tuples $(a_k, a_{k+1}, \dots, a_{k+n-1})$ (if (a_k) is unbounded, $W_n((a_k)) = \infty$). Then the techniques of [EKL06] gives the following explicit sufficient criterion for a real number x to satisfy Littlewood's conjecture for all $y \in \mathbb{R}$:

[7] Einsiedler and I have worked together for some years on many aspects of the action of diagonalizable groups, and there is some overlap between this paper and Einsiedler's [Ein10], as well as our joint contribution to the proceedings of the previous ICM in Madrid [EL06]. However the selection of topics and style is quite different in these three papers.

Theorem 5. *Let* $x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}$ *be the continued fraction expansion of* $x \in \mathbb{R}$*. If* $h_{comb}((a_k)) > 0$ *then for every* $y \in \mathbb{R}$ *equation* (4.1) *holds.*

Periodic orbits of diagonal groups.

4.4. Unlike the case for groups generated by unipotents, it is not hard to give a sequence of A-periodic orbits $A.x_i$ in X_d (for any $d \geq 2$) so that the associated probability measures $m_{A.x_i}$ fail to converge to the uniform measure (cf. [ELMV09, §7]). Indeed, as pointed out to me by U. Shapira, such an example is implicit already in an old paper by Cassels [Cas52].

4.5. However, when the periodic orbits are appropriately grouped their behavior improves markedly: define for any A-periodic $\xi \in X_d$ an order in the ring D of $d \times d$ (possibly singular) diagonal matrices by

$$\mathcal{O}(\xi) = \left\{ h \in D : h.\bar{\xi} \subseteq \bar{\xi} \right\}$$

where $\bar{\xi}$ is a lattice representing the homothety equivalence class ξ. This is a discrete subring of D containing 1; $\operatorname{stab}_A(\xi) = \{a \in A : a.\xi = \xi\}$ is precisely the set of invertible elements of $\mathcal{O}(\xi)$ and moreover $\mathbb{Z}[\operatorname{stab}_A(\xi)] \subseteq \mathcal{O}(\xi)$. Since ξ is A-periodic, $\operatorname{stab}_A(\xi)$ contains $d-1$-independent units and $\mathcal{O}(\xi)$ is a lattice in D (considered as an additive group), isomorphic as a ring to an order in a totally real number field K of degree d over $\mathbb{Q}$. For a given order $\mathcal{O} < D$ set

$$\mathcal{C}(\mathcal{O}) = \{A.y : \mathcal{O}(y) = \mathcal{O}\} ;$$

for any A-periodic $\xi \in X_d$ the collection $\mathcal{C}(\mathcal{O}(\xi))$ can be shown to be finite.

Theorem 6 (Einsiedler, Michel, Venkatesh and L. [ELMV10]). *Let* $A.x_i$ *can be a sequence of distinct* A*-periodic orbits in* X_3*, and set* $\mathcal{C}_i = \mathcal{C}(\mathcal{O}(x_i))$*. Then for any* $f \in C_c(X_3)$ *we have that*

$$\frac{1}{|\mathcal{C}_i| \cdot |A/\operatorname{stab}_A(x_i)|} \sum_{A.y \in \mathcal{C}_i} \int_{A/\operatorname{stab}_A(x_i)} f(a.y)\, da \to \int_{X_3} f.$$

For $d = 2$ the corresponding statement is a theorem of Duke [Duk88] proved using the theory of automorphic forms, with some previous substantial partial results by Linnik and Skubenko (see [Li68]). Weaker results about the distribution of periodic A-orbits for $d \geq 3$ in substantially greater generality were obtained in [ELMV09].

4.6. In the case of periodic A-orbits $A.\xi$ whose corresponding order $\mathcal{O}(\xi)$ is maximal (equivalently, is isomorphic to the full integer ring $\mathcal{O}_K$ of a totally real number field K), $\mathcal{C}(\xi)$ can be identified with the ideal class group of $\mathcal{O}_K$, and in particular has a natural structure of a group. It is quite challenging to make use of the group structure of $\mathcal{C}(\xi)$ in the dynamical context. In particular, it would be of interest to prove equidistribution of the collection of A-orbits corresponding to (possibly quite small) subgroups of the ideal class group.

4.7. We refer the reader to the comprehensive survey [MV06] by Michel and Venkatesh for more details on this and related equidistribution questions.

Diagonal flows and Arithmetic Quantum Unique Ergodicity.

4.8. In [RS94], Z. Rudnick and P. Sarnak conjectured the following:

Conjecture 8. *Let M be a compact Riemannian manifold of negative sectional curvature. Let ϕ_i be an orthonormal sequence of eigenfunctions of the Laplacian on M. Then*

$$\int_M f(x)\,|\phi_i(x)|^2\,d\,\mathrm{vol}(x) \to \frac{1}{\mathrm{vol}(M)}\int_M f(x)\,d\,\mathrm{vol}(x) \qquad \forall f \in C^\infty(M). \tag{4.2}$$

There is also a slightly stronger form of this conjecture for test functions in phase space. Both versions of the conjecture are open, and there does not seem to be strong evidence for it in high dimensions. However in the special case of $M = \mathbb{H}/\Gamma$ with Γ an arithmetic lattice of congruence type (either congruence sublattices of $\mathrm{PGL}(2,\mathbb{Z})$ or of $\mathrm{PGL}(1,\mathcal{O})$ for $\mathcal{O}$ an order in an indefinite quaternion algebra over $\mathbb{Q}$; in the latter case M is compact) we have a lot of extra symmetry that aids the analysis: an infinite commuting ensemble of self-adjoint operators, generated by the Laplacian and, for each prime p outside a possible finite set P of "bad" primes, a corresponding Hecke operators T_p.

Theorem 7 (Brooks and L. [BL10,Lin06]). *Let $M = \mathbb{H}/\Gamma$ be as above, and $p \notin P$, with M compact. Then any orthonormal sequence ϕ_i of joint eigenfunctions of the Laplacian and T_p on M satisfies* (4.2).

This theorem refines a previous theorem that relied on work by Bourgain and myself [BL03]. When Γ is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, i.e. M is not compact, there is an extra complication in that one needs to show that no mass escapes to the cusp in the limit. Under the assumption of ϕ_i being joint eigenfunctions of all Hecke operators this has been established by Soundararajan [Sou09].

4.9. The proof of Theorem 7 does not quite use multiparameter diagonalizable flows but rather the following theorem (generalized in [EL08]) of similar but somewhat more general flavor:

Definition 2. Let X be locally compact space, H a locally compact group acting continuously on X, and μ any σ-finite measure on X (not necessarily H invariant). Then μ is H-recurrent if for every set $B \subset X$ with $\mu(B) > 0$ for almost every $x \in X$ the set $\{h \in H : h.x \in B\}$ is unbounded (has noncompact closure).

Theorem 8 ([Lin06]). *Let $G = \mathrm{PGL}(2,\mathbb{R}) \times \mathrm{PGL}(2,\mathbb{Q}_p)$, $H = \mathrm{PGL}(2,\mathbb{Q}_p)$ considered as a subgroup of G, A_1 the diagonal subgroup of $\mathrm{SL}(2,\mathbb{R})$ (also considered as a subgroup of G), and $\Lambda < G$ an irreducible lattice. Let μ be a probability measure on G/Λ which is (i) A_1-invariant (ii) H-recurrent (iii) a.e. A_1-ergodic component of μ has positive entropy (with respect to A_1). Then μ is the uniform measure on G/Λ.*

Note that if μ as in Theorem 8 were invariant under any unbounded subgroup of H, by Poincaré recurrence it would be H-recurrent.

The connection to Theorem 7 uses the fact that for $\Gamma < \mathrm{PGL}(2,\mathbb{R})$ of congruence type as above and $p \notin P$, $\mathbb{H}/\Gamma$ can be identified with $K\backslash G/\Lambda$ for G as in Theorem 8 and $K < G$ the compact subgroup $\mathrm{PO}(2,\mathbb{R}) \times \mathrm{PGL}(2,\mathbb{Z}_p)$; let $\pi : G/\Lambda \to \mathbb{H}/\Gamma$ be the projection corresponding to this identification. The Hecke operator T_p is related to this construction as follows: for $f \in L^2(\mathbb{H}/\Gamma)$ and $\tilde{x} \in G/\Lambda$

$$\big[T_p f\big](\pi(\tilde{x})) = p^{-1/2} \int_{\mathrm{PGL}(2,\mathbb{Z}_p)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathrm{PGL}(2,\mathbb{Z}_p)} \tilde{f} \circ \pi((e,h).\tilde{x})\, dh.$$

The crux of both [BL03] and [BL10] is the verification of the entropy assumption (iii) above, which can be rephrased in terms of decay rates of measures of small tubes in G/Λ.

4.10. Note that though Theorems 4 and 5 are clearly partial results, in Theorem 6 and Theorem 7 one essentially obtains unconditionally full equidistribution statements using only the partial measure classification results currently available.

4.11. A more detailed discussion of quantum unique ergodicity in the arithmetic context can be found in Soundararajan's contribution to these proceedings [Sou10], which also include a discussion of some recent exciting results of Holowinsky and Soundararajan [HS09] regarding an analoguous question for holomorphic forms.

5. Zariski dense subgroups of groups generated by unipotents

5.1. An important difference between groups generated by unipotent subgroups and diagonalizable groups is the size of norm balls in these groups. Given a closed subgroup $H < G$ with large norm balls, i.e. for which

$$\lambda_H\left(\{h \in H : \|\mathrm{Ad}(h)\| < T\}\right) \geq CT^{\alpha} \qquad \text{for some } \alpha > 0 \tag{5.1}$$

the discussion in §3 might lead us to hope that we may be able to understand the behavior of individual H-orbits for the action of H on a quotient space G/Γ for a lattice $\Gamma < G$.

5.2. A natural class of groups which satisfy the thickness condition (5.1) are Zariski dense discrete subgroups Λ of semisimple algebraic groups. For instance, one may look at the action of a subgroup $\Lambda < \mathrm{SL}(d, \mathbb{Z})$ with a large Zariski closure on $\mathbb{T}^d$, or at the action of a subgroup $\Lambda < G$ with large Zariski closure (in the simplest case, G) on G/Γ where G is a simple real algebraic group. Two substantial papers addressing this question appeared in the same Tata Institute Studies volume by Furstenberg [Fur98] and by N. Shah [Sh98], the latter paper addressing this question when Λ is generated by unipotent elements.

5.3. In the context of actions of subgroups $\Lambda < \mathrm{SL}(d, \mathbb{Z})$ on $\mathbb{T}^d$, under the assumption of strong irreducibility of the Λ-action and that the identity component of the Zariski closure of Λ is semisimple, Muchnik [Muc05] and Guivarc'h and Starkov [GS04] show that for any $x \in \mathbb{T}^d$ the orbit $\Lambda.x$ is either finite or dense, in analogy with theorems of Furstenberg (cf. §2.8) and Berend [Ber84] who address this question in the context of the action of two or more commuting automorphisms of $\mathbb{T}^d$.

5.4. Groups Λ as above with a large Zariski closure are not amenable[8], and hence in general there is no reason why the behavior of individual orbits in a continuous action of Λ on a compact (or locally compact) space X should be governed by Λ-invariant measures, even to the more limited extent manifest by actions of diagonalizable groups. A natural substitute for invariant measures in this context was suggested by Furstenberg (e.g. in [Fur98]): choose an arbitrary auxiliary probability measure ν on Λ whose support generates Λ, subject to an integrability condition, e.g. the finite moment condition $\int \|g\|^{\delta} d\nu(g) < \infty$ for some $\epsilon > 0$ (if Λ is finitely generated one can take ν to be finitely supported). A measure μ on X is said to be ν-stationary if

$$\nu * \mu := \int g_* \mu \, d\nu(g) = \mu.$$

Unlike invariant measures, even in the nonamenable setting, if X is compact then for every $x \in X$ there is a ν-stationary probability measure supported on $\overline{\Lambda.x}$.

5.5. In analogy with Conjecture 3, one may conjecture that if ν is a measure on $\mathrm{SL}(d, \mathbb{Z})$ whose support generates a subgroup Λ acting strongly irreducibly on $\mathbb{T}^d$ and whose Zariski closure is semisimple, in particular if Λ is Zariski dense in $\mathrm{SL}(d, \mathbb{R})$, any ν-stationary probability measure on $\mathbb{T}^d$ is a linear combination of Lebesgue measure $\lambda_{\mathbb{T}^d}$ and finitely supported measures each on a finite Λ-orbit. In particular, one may hope that any ν-stationary measure is in fact Λ-invariant, a phenomenon Furstenberg calls *stiffness*. Guivarc'h posed the following question, suggesting that a much stronger statement might be true: whether under the conditions above, for any $x \in \mathbb{T}^d$ with at least one irrational component,

$$\nu^{*k} * \delta_x := \underbrace{\nu * \cdots * \nu}_{k} * \delta_x \to \lambda_{\mathbb{T}^d} \qquad \text{as } k \to \infty. \tag{5.2}$$

[8]See footnote on p. 9 for a definition.

Equation (5.2) clearly implies that if μ is any nonatomic measure, $\nu^{*k} * \mu \to \lambda_{\mathbb{T}^d}$, hence it implies the above classification of ν-stationary measures.

5.6. In joint work with Bourgain, Furman and Mozes, a positive quantitative answer to Guivarc'h question is given under the assumption that Λ acts totally irreducibly on $\mathbb{T}^d$ and has a proximal element[9], in particular, if Λ is Zariski dense in $\mathrm{SL}(d, \mathbb{R})$:

Theorem 9 (Bourgain, Furman, Mozes and L. [BFLM10]). *Let $\Lambda < \mathrm{SL}_d(\mathbb{R})$ satisfy the assumptions above, and let ν be a probability measure supported on a set of generators of Λ satisfying the moment condition of §5.4. Then there are constants $C, c > 0$ so that if for a point $x \in \mathbb{T}^d$ the measure $\mu_n = \nu^{*n} * \delta_x$ satisfies that for some $a \in \mathbb{Z}^d \setminus \{0\}$*

$$|\widehat{\mu}_n(a)| > t > 0, \qquad \text{with} \qquad n > C \cdot \log(\frac{2\|a\|}{t}),$$

then x admits a rational approximation p/q for $p \in \mathbb{Z}^d$ and $q \in \mathbb{Z}_+$ satisfying

$$\left\| x - \frac{p}{q} \right\| < e^{-cn} \qquad \text{and} \qquad |q| < \left(\frac{2\|a\|}{t} \right)^C. \tag{5.3}$$

This proof uses in an essential way the techniques of arithmetic combinatorics, particularly a nonstandard projections theorem by Bourgain [Bou10].

5.7. A purely ergodic theoretic approach to classifying Λ-stationary measures, as well as Λ-orbit closures, has been developed by Y. Benoist and J. F. Quint. Their approach has a considerable advantage that it is significantly more general in scope, though the analytic approach of [BFLM10] where applicable gives much more precise and quantitative information. In particular, in [BQ09] the following is proved for homogeneous quotients G/Γ:

Theorem 10 (Benoist and Quint). *Let G be the connected component of a simple real algebraic group, Γ a lattice in G. Let ν be a finitely supported probability measure G whose support generates a Zariski dense subgroup $\Lambda < G$ then*

1. *Any non-atomic ν-stationary measure on G/Γ is the uniform measure on G/Γ.*
2. *For any $x \in G/\Gamma$, the orbit $\Lambda.x$ is either finite or dense. Moreover, in the latter case the Cesàro averages $\frac{1}{n}\sum_{k=1}^{n} \nu^{*n} * \delta_x$ converge weak* to the uniform measure on G/Γ.*

It is not known in this case if the sequence $\nu^{*n} * \delta_x$ converges to the uniform measure. A technique introduced by Eskin and Margulis [EM04] to establish nondivergence of the sequence of measures $\nu^{*k} * \delta_x$ on G/Γ and further developed by Benoist and

[9] An element $g \in \mathrm{SL}(d, \mathbb{R})$ is said to be proximal if it has a simple real eigenvalue strictly larger in absolute value than all other eigenvalues.

Quint is used crucially in this work, and in particular gives a useful substitute in this context for the linearization techniques for unipotent flows discussed in §3.5. Some of the ideas of Ratner's Measure Classification Theorem (see §2.3) are used in the proof of Theorem 10, as well as the result itself.

6. Quantitative aspects

6.1. As we have seen, dynamical techniques applied in the context of homogeneous spaces are extremely powerful, and have many applications in number theory and other subjects. However they have a major deficiency, in that they are quite hard to quantify. For example, Margulis' proof of the Oppenheim conjecture (*cf.* §2.1) does not give any information about the size of the smallest $v \in \mathbb{Z}^3 \smallsetminus \{0\}$ satisfying $|Q(v)| < \epsilon$ for a given indefinite ternary quadratic form Q not proportional to a rational one (note that necessarily any quantitative statement of this type needs to be somewhat involved as the qualitative statement fails for integral Q, and any quantitative statement has to take into account how well Q can be approximated by forms proportional to rational forms of a given height.)

Contrast this with the proof by Davenport and Heilbronn [DH46] of the Oppenheim Conjecture for diagonal forms with $d \geq 5$ variables (forms of the type $Q(x_1, \ldots, x_d) = \sum_i \lambda_i x_i^2$ where not all λ_i have the same sign) using a variant of the Hardy-Littlewood circle method, from which it can be deduced[10] that the shortest vector v with $|Q(v)| < \epsilon$ is $O(\epsilon^{-C})$, and the much more recent work of Götze and Margulis [GM10] who treat the general $d \geq 5$ case using substantially more elaborate analytic tools and obtain a similar quantitative estimate.

6.2. Overcoming this deficiency is an important direction of research within the theory of flows on homogeneous spaces. There is one general class in which at least in principle it had long been known that fairly sharp quantitative equidistribution statements can be given, and that is for the action of horocyclic groups. Recall that $U < G$ is said to be *horocyclic* if there is some $g \in G$ for which $U = \{u \in G : g^n u g^{-n} \to e \text{ as } n \to \infty\}$; the prototypical example is $U = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ in $\mathrm{SL}(2, \mathbb{R})$. Such quantitative equidistribution results have been given by Sarnak [Sar81] and Burger [Bur90, Thm. 2] and several other authors since. Even in this well-understood case, quantitative equidistribution results have remarkable applications such as in the work of Michel and Venkatesh on subconvex estimates of L-functions [Ven05, MV09].

6.3. Another case which is well understood, particularly thanks to the work of Green and Tao [GT07], is the action of a subgroup of G on $G \backslash \Gamma$ when G is *nilpotent*; these nilsystems appear naturally in the context of combinatorial ergodic theory,

[10] At least for forms that are not too well approximated by forms proportional to rational ones, though by Meyer's Theorem for $d \geq 5$ rational forms should not cause any significant complication.

and have a different flavor from the type of dynamics we consider here, e.g. when G is a semisimple group or a solvable group of exponential growth.

6.4. We list below several nonhorospherical quantitative equidistribution results closer to the main topics of this note:

(a) Using deep results from the theory of automorphic forms, and under some additional assumptions that are probably not essential, Einsiedler, Margulis and Venkatesh were able to give a quantitative analysis of equidistribution of *periodic* orbits of semisimple groups on homogeneous spaces [EMV09] with a polynomial rate of convergence — a result that I suspect should have many applications.

(b) Let ν be a probability measure on $\mathrm{SL}(d, \mathbb{Z})$ as in §5.6. Theorem 9 quoted above from [BFLM10] gives a quantitative equidistribution statement for successive convolutions $\nu^{*n} * \delta_x$ for $x \in \mathbb{T}^d$, which in particular gives quantitative information on the random walk associated with ν on $(\mathbb{Z}/N\mathbb{Z})^d$ as $N \to \infty$ irrespective of the prime decomposition of N. This has turned out to be useful in the recent work of Bourgain and P. Varjú [BV10] that show that the Cayley graphs of $\mathrm{SL}(d, \mathbb{Z}/N\mathbb{Z})$ with respect to a finite set S of elements in $\mathrm{SL}(d, \mathbb{Z})$ generating a Zariski dense subgroup of $\mathrm{SL}(d, \mathbb{R})$ are a family of expanders as $N \to \infty$ as long as N is not divisible by some fixed set of prime numbers depending on S.

(c) In joint work with Margulis we give an effective dynamical proof of the Oppenheim Conjecture, i.e. one that does give bounds on the minimal size of a nonzero integral vector v for which $|Q(v)| < \epsilon$. The bound obtained is of the form $\|v\| \ll \exp(\epsilon^{-C})$. Nimish Shah has drawn my attention to a paper of Dani [Dan94] which has a proof of the Oppenheim conjecture that in principle is quantifiable, i.e. without the use of minimal sets or the axiom of choice, though it is not immediately apparent what quality of quantification may be obtained from his method.

(d) In work with Bourgain, Michel and Venkatesh [BLMV09] we have given an effective version of Furstenberg's Theorem (cf. §2.8), giving in particular that if a, b are multiplicatively independent integers, for sufficiently large C depending on a, b and some $\theta > 0$, for all $N \in \mathbb{N}$ and m relatively prime to N,

$$\left\{ \frac{a^n b^k m}{N} : 0 \leq n, k \leq C \log N \right\}$$

intersects any interval in $\mathbb{R}/\mathbb{Z}$ of length $\gg \log \log \log N^{\theta}$. This has been generalized by Z. Wang [Wan10] in the context of commuting actions of toral automorphisms.

Clearly, there is ample scope for further research in this direction, particularly regarding the quality of these quantitative results and their level of generality. In particular, I think any improvement on the quality of the estimate obtained in (d) above would be quite interesting.

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