INVARIANT MEASURES AND THE SET OF EXCEPTIONS TO LITTLEWOOD'S CONJECTURE

MANFRED EINSIEDLER, ANATOLE KATOK, AND ELON LINDENSTRAUSS

Abstract. We classify the measures on $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$ which are invariant and ergodic under the action of the group A of positive diagonal matrices with positive entropy. We apply this to prove that the set of exceptions to Littlewood's conjecture has Hausdorff dimension zero.

1. Introduction

1.1. **Number theory and dynamics.** There is a long and rich tradition of applying dynamical methods to number theory. In many of these applications, a key role is played by the space $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$ which can be identified as the space of unimodular lattices in $\mathbb{R}^k$. Any subgroup $H < \mathrm{SL}(k,\mathbb{R})$ acts on this space in a natural way, and the dynamical properties of such actions often have deep number theoretical implications.

A significant landmark in this direction is the solution by G.A. Margulis [23] of the long-standing Oppenheim Conjecture through the study of the action of a certain subgroup H on the space of unimodular lattices in three space. This conjecture, posed by A. Oppenheim in 1929, deals with density properties of the values of indefinite quadratic forms in three or more variables. So far there is no proof known of this result in its entirety which avoids the use of dynamics of homogeneous actions.

An important property of the acting group H in the case of the Oppenheim Conjecture is that it is generated by unipotents: i.e. by elements of $\mathrm{SL}(k,\mathbb{R})$ all of whose eigenvalues are 1. The dynamical result proved by Margulis was a special case of a conjecture of M. S. Raghunathan regarding the actions of general unipotents groups. This conjecture (and related conjectures made shortly thereafter) state that for the action of H generated by unipotents by left translations on the homogeneous space G/Γ of an arbitrary connected Lie group G by a lattice Γ the only possible H-orbit closures

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and H-ergodic probability measures are of an algebraic type. Raghunatan's conjecture was proved in full generality by M. Ratner in a landmark series of papers ([38, 39] and others; see also the expository papers [41, 40], and the forthcoming book [49]) which led to numerous applications; in particular, we use Ratner's work heavy in this paper. Ratner's theorems provide the model for the global orbit structure for systems with *parabolic* behavior. See [9] for a general discussion of principal types of orbit behavior in dynamics.

1.2. **Weyl chamber flow and Diophantine approximation.** In this paper we deal with a different homogeneous action, which is not so well understood, namely the action by left multiplication of the group A of positive diagonal $k \times k$ matrices on $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$; A is a split Cartan subgroup of $\mathrm{SL}(k,\mathbb{R})$ and the action of A is also known as a particular case of a *Weyl chamber flow* [15].

For $k=2$ the acting group is isomorphic to $\mathbb{R}$ and and the Weyl chamber flow reduces to the geodesic flow on a surface of constant negative curvature, namely the modular surface. This flow has *hyperbolic* structure; it is Anosov if one makes minor allowances for noncompactness and elliptic points. The orbit structure of such flows is well understood; in particular there is a great variety of invariant ergodic measures and orbit closures. For $k > 2$, the Weyl chamber flow is hyperbolic as an $\mathbb{R}^{k-1}$-action, i.e. transversally to the orbits. Such actions are very different from Anosov flows and display many rigidity properties, see e.g. [15, 17]. One of the manifestations of rigidity concerns invariant measures. Notice that one–parameter subgroups of the Weyl chamber flow are *partially hyperbolic* and each such subgroup still has many invariant measures. However, it is conjectured that A-ergodic measures are rare:

Conjecture 1.1 (Margulis)**.** *Let μ be an A-invariant and ergodic probability measure on $X=\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$ for $k \geq 3$. Then μ is algebraic, i.e. there is a closed, connected group $L > A$ so that μ is the L-invariant measure on a single, closed L-orbit.*

This conjecture is a special case of much more general conjectures in this direction by Margulis [25], and by A. Katok and R. Spatzier [16]. This type of behavior was first observed by Furstenberg [7] for the action of the multiplicative semigroup $\Sigma_{m,n} = \left\{m^k n^l\right\}_{k,l\geq 1}$ on $\mathbb{R}/\mathbb{Z}$, where n, m are two multiplicatively independent integers (i.e. not powers of the same integer), and the action is given by $k.x = kx \bmod 1$ for any $k \in \Sigma_{m,n}$ and $x \in \mathbb{R}/\mathbb{Z}$. Under these assumptions Furstenberg proved that the only infinite closed invariant set under the action of this semigroup is the space $\mathbb{R}/\mathbb{Z}$ itself. He also raised the question of extensions, in particular to the measure theoretic analog as well as to the locally homogeneous context.

There is an intrinsic difference regarding the classification of invariant measures between Weyl chamber flows (e.g. higher rank Cartan actions) and unipotent actions. For unipotent actions, every element of the action already acts in a rigid manner. For Cartan actions, there is no rigidity for the action of individual elements, but only for the full action. In stark contrast to unipotent actions, M. Rees [42][3, Sect. 9] has shown there are lattices $\Gamma < \mathrm{SL}(k, \mathbb{R})$ for which there are non-algebraic A-invariant and ergodic probability measures on $X = \mathrm{SL}(k, \mathbb{R})/\Gamma$ (fortunately, this does not happen for $\Gamma = \mathrm{SL}(k, \mathbb{Z})$, see [21, 25] and more generally [46] for related results). These non-algebraic measures arise precisely because one parameter subactions are not rigid, and come from A invariant homogeneous subspaces which have algebraic factors on which the action degenerates to a one parameter action.

While Conjecture 1.1 is a special case of the general question about the structure of invariant measures for higher rank hyperbolic homogeneous actions, it is of particular interest in view of number theoretic consequences. In particular, it implies the following well-known and long-standing conjecture of Littlewood [24, Sect. 2] :

Conjecture 1.2 (Littlewood (c. 1930))**.** *For every $u, v \in \mathbb{R}$,*

$$\liminf_{n \to \infty} n \langle nu \rangle \langle nv \rangle = 0, \tag{1.1}$$

where $\langle w \rangle = \min_{n \in \mathbb{Z}} |w - n|$ is the distance of $w \in \mathbb{R}$ to the nearest integer.

In this paper we prove the following partial result towards Conjecture 1.1 which has implications toward Littlewood's conjecture:

Theorem 1.3. *Let μ be an A-invariant and ergodic measure on $X = \mathrm{SL}(k, \mathbb{R})/\mathrm{SL}(k, \mathbb{Z})$ for $k \geq 3$. Assume that there is some one parameter subgroup of A which acts on X with positive entropy. Then μ is algebraic.*

In [21] a complete classification of the possible algebraic μ is given. In particular, we have the following:

Corollary 1.4. *Let μ be as in Theorem 1.3. Then μ is not compactly supported. Furthermore, if k is prime μ is the unique $\mathrm{SL}(k, \mathbb{R})$-invariant measure on X.*

Theorem 1.3 and its corollary have the following implication toward Littlewood's conjecture:

Theorem 1.5. *Let*

$$\Xi = \left\{ (u, v) \in \mathbb{R}^2 : \liminf_{n \to \infty} n \langle nu \rangle \langle nv \rangle > 0 \right\}.$$

Then the Hausdorff dimension $\dim_H \Xi = 0$. In fact, Ξ is a countable union of compact sets with box dimension zero.

J. W. S. Cassels and H. P. F. Swinnerton-Dyer [1] showed that (1.1) holds for any u, v which are from the same cubic number field (i.e. any field K with degree $[K : \mathbb{Q}] = 3$).

It is easy to see that for a.e. (u, v) equation (1.1) holds — indeed, for almost every u it is already true that $\liminf_{n \to \infty} n \langle nu \rangle = 0$. However, there is a set of u of Hausdorff dimension 1 for which $\liminf_{n\to\infty} n\langle nu\rangle > 0$; such u are said to be badly approximable. Pollington and Velani [33] showed that for every $u \in \mathbb{R}$, the intersection of the set

$$\{v \in \mathbb{R} : (u, v) \text{ satisfies } (1.1)\} \tag{1.2}$$

with the set of badly approximable numbers has Hausdorff dimension one. Note that this fact is an immediate corollary of our Theorem 1.5 — indeed, Theorem 1.5 implies in particular that the complement of this set (1.2) has Hausdorff dimension zero for all u. We remark that the proof of Pollington and Velani is effective.

Littlewood's conjecture is a special case of a more general question. More generally, for any k linear forms $m_i(x_1, x_2, \dots, x_k) = \sum_{j=1}^{k} m_{ij}x_j$, one may consider the product

$$f_m(x_1, x_2, \dots, x_k) = \prod_{i=1}^{k} m_i(x_1, \dots, x_k),$$

where $m = (m_{ij})$ denotes the $k \times k$ matrix whose rows are the linear forms above. Using Theorem 1.3 we prove the following:

Theorem 1.6. *There is a set $\Xi_k \subset \mathrm{SL}(k,\mathbb{R})$ of Hausdorff dimension $k-1$ so that for every $m \in \mathrm{SL}(k,\mathbb{R}) \setminus \Xi_k$*

$$\inf_{\mathbf{x}\in\mathbb{Z}^k\setminus\{\mathbf{0}\}} |f_m(\mathbf{x})| = 0. \tag{1.3}$$

Indeed, this set Ξ_k is A-invariant, and has zero Hausdorff dimension transversally to the A-orbits.

For more details, see §10 and §11. Note that (1.3) is automatically satisfied if zero is attained by f_m evaluated on $\mathbb{Z}^k \setminus \{0\}$.

We also want to mention another application of our results due to Hee Oh [30], which is related to the following conjecture of Margulis:

Conjecture 1.7. *(Margulis, 1993) Let G be the product of $n \geq 2$ copies of $\mathrm{SL}(2,\mathbb{R})$, $U_1 = \left\{\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix} \times \dots \times \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}\right\}$ and $U_2 = \left\{\begin{pmatrix}1 & 0\\ * & 1\end{pmatrix} \times \dots \times \begin{pmatrix}1 & 0\\ * & 1\end{pmatrix}\right\}$. Let $\Gamma < G$ be a discrete subgroup so that for both $i = 1$ and 2, $\Gamma \cap U_i$ is a lattice in U_i and for any proper connected normal subgroup $N < G$ the intersection $\Gamma \cap N \cap U_i$ is trivial. Then Γ is commensurable with a Hilbert modular lattice[1] up to conjunction in $\mathrm{GL}(2,\mathbb{R}) \times \dots \times \mathrm{GL}(2,\mathbb{R})$.*

Hee Oh [31] has shown that assuming a topological analog to Conjecture 1.1 (which is implied by Conjecture 1.1), Conjecture 1.7 is true for $n \geq 3$. As explained in [30] (and follows directly from [31, Thm. 1.5]), our result Theorem 1.3 implies the following weaker result (also for $n \geq 3$): consider the set $\mathcal{D}$ of possible intersections $\Gamma \cap U_1$ for Γ as in Conjecture 1.7, which is a subset of the space of lattices in U_1. This set $\mathcal{D}$ is clearly invariant

[1] for definition of Hilbert modular lattices, see [31]

under conjugation by the diagonal group in $\mathrm{GL}(2,\mathbb{R})\times\cdots\times\mathrm{GL}(2,\mathbb{R})$; Theorem 1.3 (or more precisely Theorem 10.2 which we prove using Theorem 1.3 in §10) implies that the set $\mathcal{D}$ has zero Hausdorff dimension transversally to the orbit of this n-dimensional group (in particular, this set $\mathcal{D}$ has Hausdorff dimension n; see §7 and §10 for more details regarding Hausdorff dimension and tranversals, and [31, 30] for more details regarding this application).

1.3. **Measure rigidity.** The earliest results for measure rigidity for higher rank hyperbolic actions deal with the Furstenberg problem: [22, 43, 8]. Specifically, Rudolph [43] and Johnson [8] proved that if μ is a probability measure invariant and ergodic under the action of the semigroup generated by $\times m$, $\times n$ (again with m, n not powers of the same integer), and if some element of this semigroup acts with positive entropy, then μ is Lebesgue.

When Rudolph's result appeared, the second author suggested another test model for the measure rigidity: two commuting hyperbolic automorphisms of the three–dimensional torus. Since Rudolph's proof seemed, at least superficially, too closely related to symbolic dynamics, jointly with R. Spatzier, a more geometric technique was developed. This allowed a unified treatment of essentially all the classical examples of higher rank actions for which rigidity of measures is expected [16, 13], and in retrospect, Rudolph's proof can also be interpreted in this framework.

This method (as well as most later work on measure rigidity for these higher rank abelian actions) is based on the study of conditional measures induced by a given invariant measure μ on certain invariant foliations. The foliations considered include stable and unstable foliations of various elements of the actions, as well as intersections of such foliations, and are related to the Lyapunov exponents of the action. For Weyl chamber flows these foliations are given by orbits of unipotent subgroups normalized by the action.

Unless there is an element of the action which acts with positive entropy with respect to μ, these conditional measures are well-known to be δ-measure supported on a single point, and do not reveal any additional meaningful information about μ. Hence this and later techniques are limited to study actions where at least one element has positive entropy. Under ideal situations, such as the original motivating case of two commuting hyperbolic automorphisms of the three torus, no further assumptions are needed, and a result entirely analogous to Rudolph's theorem can be proved using the method of [16].

However, for Weyl chamber flows, an additional assumption is needed for the [16] proof to work. This assumption is satisfied, for example, if the flow along every singular direction in the Weyl chamber is ergodic (though a weaker hypothesis is sufficient). This additional assumption, which unlike the entropy assumption is not stable under weak* limits, precludes applying the results from [16] in many cases.

Recently, two new methods of proofs were developed, which overcome this difficulty.

The first method was developed by the first and second authors [3], following an idea mentioned at the end of [16]. This idea uses the non-commutativity of the above-mentioned foliations (or more precisely, of the corresponding unipotent groups). This paper deals with general $\mathbb{R}$-split semisimple Lie groups; in particular it is shown there that if μ is an A-invariant measure on $X = \mathrm{SL}(k, \mathbb{R})/\Gamma$, and if the entropies of μ with respect to *all* one parameter groups are positive, then μ is the Haar measure. It should be noted that for this method the properties of the lattice do not play any role, and indeed this is true not only for $\Gamma = \mathrm{SL}(k, \mathbb{Z})$ but for every discrete subgroup Γ. An extension to the nonsplit case is forthcoming [4]. Using the methods we present in the second part of the present paper, the results of [3] can be used to show that the set of exceptions to Littlewood's conjecture has Hausdorff dimension at most 1.

A different approach was developed by the third author, and was used to prove a special case of the quantum unique ergodicity conjecture [20]. In its basic form, this conjecture is related to the geodesic flow, which is not rigid, so in order to be able to prove quantum unique ergodicity in certain situations a more general setup for measure rigidity, following Host [11], was needed. A special case of the main theorem of [20] is the following: Let A be an $\mathbb{R}$-split Cartan subgroup of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. Any A-ergodic measure on $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})/\Gamma$ for which some one parameter subgroup of A acts with positive entropy is algebraic. Here Γ is e.g. an irreducible lattice in $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. Since the foliations under consideration in this case do commute, the methods of [3] are not applicable.

The method of [20] can be adapted to quotients of more general groups, and in particular to $\mathrm{SL}(k, \mathbb{R})$. It is noteworthy (and gratifying) that for the space of lattices (and more general quotients of $\mathrm{SL}(k, \mathbb{R})$) these two unrelated methods are completely complementary: measures with "high" entropy (e.g. measures for which many one parameter subgroup have positive entropy) can be handled with the methods of [3], and measures with "low" (but positive) entropy can be handled using the methods of [20]. Together, these methods give Theorem 1.3 (as well as the more general Theorem 2.1 below for more general quotients).

The method of proof in [20], an adaptation of which we use here, is based on studying the behavior of μ along certain unipotent trajectories, using techniques introduced by Ratner in [37, 36] to study unipotent flows, in particular the H-property (these techniques are nicely exposed in §1.5 of the forthcoming book [49]). This is surprising because the techniques are applied on a measure μ which is a priori not even quasi invariant under these (or any other) unipotent flows.

In showing that the high entropy and low entropy cases are complementary we use a variant on the Ledrappier-Young entropy formula [19]. Such use is one of the simplifying ideas in G. Tomanov and Margulis' alternative proof of Ratner's theorem [26].

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Part 1. **Measure Rigidity**

Throughout this paper, let $G = \mathrm{SL}(k, \mathbb{R})$ for some $k \geq 3$, let Γ be a discrete subgroup of G, and let $X = G/\Gamma$. As in the previous section, we let $A < G$ denote the group of $k \times k$ positive diagonal matrices. We shall implicitly identify

$$\Sigma = \{\mathbf{t} \in \mathbb{R}^k : t_1 + \cdots + t_k = 0\}$$

and the Lie algebra of A via the map $(t_1, \ldots, t_k) \mapsto \mathrm{diag}(t_1, \ldots, t_k)$. We write $\alpha^{\mathbf{t}} = \mathrm{diag}(e^{t_1}, \ldots, e^{t_k}) \in A$ and also $\alpha^{\mathbf{t}}$ for the left multiplication by this element on X. This defines an $\mathbb{R}^{k-1}$ flow α on X.

A subgroup $U < G$ is *unipotent* if for every $g \in U$, $g - \mathrm{I}_k$ is nilpotent, i.e. for some n, $(g - \mathrm{I}_k)^n = 0$. A group H is said to be *normalized by* $g \in G$ if $gHg^{-1} = H$; H is normalized by $L < G$ if it is normalized by every $g \in L$; and the *normalizer* $N(H)$ of H is the group of all $g \in G$ normalizing it. Similarly, g *centralizes* H if $gh = hg$ for every $h \in H$, and we set $C(H)$, the *centralizer* of H in G, to be the group of all $g \in G$ centralizing H.

If $U < G$ is normalized by A then for every $x \in X$ and $a \in A$, $a(Ux) = Uax$, so that the *foliation* of X into U orbits is invariant under the action of A. We will say that $a \in A$ expands U if all eigenvalues of $\mathrm{Ad}(a)$ restricted to the Lie algebra of U are greater than one.

For any locally compact metric space Y let $\mathcal{M}_\infty(Y)$ denote the space of Radon measures on Y equipped with the weak* topology, i.e. all locally finite Borel measures on Y with the coarsest topology for which $\rho \mapsto \int_Y f(y) d\rho(y)$ is continuous for every compactly supported continuous f. For two Radon measures ν_1 and ν_2 on Y we write

$$\nu_1 \propto \nu_2 \text{ if } \nu_1 = C\nu_2 \text{ for some } C > 0.$$

and say that ν_1 and ν_2 are proportional.

We let $B_\epsilon^Y(y)$ (or $B_\epsilon(y)$ if Y is understood) denote the ball of radius ϵ around $y \in Y$; if H is a group we set $B_\epsilon^H = B_\epsilon^H(\mathrm{I})$ where I is identity in H; and if H acts on X and $x \in X$ we let $B_\epsilon^H(x) = B_\epsilon^H \cdot x$.

Let $d(\cdot,\cdot)$ be the geodesic distance induced by a right-invariant Riemannian metric on G. This metric on G induces a right-invariant metric on every closed subgroup $H \subset G$, and furthermore a metric on $X = G/\Gamma$. These induced metrics we denote by the same letter.

2. Conditional measures on A-invariant foliations, invariant measures, and shearing

2.1. Conditional measures. A basic construction, which has been introduced in the context of measure rigidity in [16] (and in a sense is already used implicitly in [43]), is the restriction of probability or even Radon measures on a foliated space to the leaves of this foliations. A discussion can be found in [16, Sect. 4], and a fairly general construction is presented in [20, Sect. 3]. Below we consider special cases of this general construction, summarizing its main properties.

Let μ be an A-invariant probability measure on X. For any unipotent subgroup $U < G$ normalized by A, one has a system $\{\mu_{x,U}\}_{x\in X}$ of Radon measures on U and a co-null set $X' \subset X$ with the following properties[2]:

(1) the map $x \mapsto \mu_{x,U}$ is measurable.
(2) for every $\epsilon > 0$ and $x \in X'$ it holds that $\mu_{x,U}(B_\epsilon^U) > 0$.
(3) for every $x \in X'$ and $u \in U$ with $ux \in X'$, we have that $\mu_{x,U} \propto (\mu_{ux,U})u$, where $(\mu_{ux,U})u$ denotes the push forward of the measure $\mu_{ux,U}$ under the map $v \mapsto vu$.
(4) for every $\mathbf{t} \in \Sigma$, and $x, \alpha^{\mathbf{t}}x \in X'$, $\mu_{\alpha^{\mathbf{t}}x,U} \propto \alpha^{\mathbf{t}}(\mu_{x,U})\alpha^{-\mathbf{t}}$.

In general, there is no canonical way to normalize the measures $\mu_{x,U}$; we fix a specific normalization by requiring that $\mu_{x,U}(B_1^U) = 1$ for every $x \in X'$. This implies the next crucial property.

(5) If $U \subset C(\alpha^{\mathbf{t}}) = \{g \in G : g\alpha^{\mathbf{t}} = \alpha^{\mathbf{t}}g\}$ commutes with $\alpha^{\mathbf{t}}$, then $\mu_{\alpha^{\mathbf{t}}x,U} = \mu_{x,U}$ whenever $x, \alpha^{\mathbf{t}}x \in X'$.
(6) μ is U-invariant if, and only if, $\mu_{x,U}$ is a Haar measure on U a.e. (see e.g. [16] or the slightly more general [20, Prop. 4.3]).

The other extreme to U-invariance is when $\mu_{x,U}$ is atomic. If μ is A-invariant then outside some set of measure zero if $\mu_{x,U}$ is atomic then it is supported on the identity $\mathrm{I}_k \in U$, in which case we say that $\mu_{x,U}$ *is trivial*. This follows from Poincaré recurrence for an element $a \in A$ that uniformly expands the U-orbits (i.e. for which the U-orbits are contained in the unstable manifolds). Since the set of $x \in X$ for which $\mu_{x,U}$ is trivial

[2] We are following the conventions of [20] in viewing the conditional measures $\mu_{x,U}$ as measures on U. An alternative approach, which, for example, is the one taken in [16] and [13], is to view the conditional measures as a collection of measures on X supported on single orbits of U; in this approach, however, the conditional measure is not a Radon measure on X, only on the single orbit of U in the topology of this submanifold.

is A-invariant, if μ is A-ergodic then either $\mu_{x,U}$ are trivial a.s. or $\mu_{x,U}$ are nonatomic a.s. Fundamental to us is the following characterization of positive entropy (see [26, Sect. 9] and [16])

(7) If for every $x \in X$ the orbit Ux is the stable manifold through x with respect to $\alpha^{\mathbf{t}}$, then the measure theoretic entropy $\mathsf{h}_\mu(\alpha^{\mathbf{t}})$ is positive if and only if the conditional measures $\mu_{x,U}$ are nonatomic a.e.

So positive entropy implies that the conditional measures are nontrivial a.e., and the goal is to show that this implies that they are Haar measures. Quite often one shows first that the conditional measures are translation invariant under some element up to proportionality, which makes the following observation useful.

(8) Possibly after replacing X' of (1)-(4) by a conull subset, it holds that for any $x \in X'$ and any $u \in U$ with $\mu_{x,U} \propto \mu_{x,U}u$ in fact $\mu_{x,U} = \mu_{x,U}u$ holds.

This has first been shown in [16]. The proof of this fact only uses Poincaré recurrence and (4) above, and for completeness we provide a proof below.

Proof of (8). Let $\mathbf{t}$ be such that $\alpha^{\mathbf{t}}$ uniformly contracts the U-leaves (i.e. for every x the U-orbit Ux is part of the stable manifold with respect to $\alpha^{\mathbf{t}}$). Define for $M > 0$

$$D_M = \left\{ x \in X' : \mu_{x,U}(B_2^U) < M \right\}.$$

We claim that for every $x \in X' \cap \bigcup_M \limsup_{n\to\infty} \alpha^{-n\mathbf{t}} D_M$ (i.e. any $x \in X'$ so that $\alpha^{n\mathbf{t}}$ is in D_M for some M for infinitely many n) if $\mu_{x,U} = c\mu_{x,U}u$ then $c \le 1$.

Indeed, suppose $x \in X' \cap \limsup_{n\to\infty} \alpha^{-n\mathbf{t}} D_M$ and $u \in U$ satisfy $\mu_{x,U} = c\mu_{x,U}u$. Then for any n, k

$$\mu_{\alpha^{n\mathbf{t}}x,U} = c^k \mu_{\alpha^{n\mathbf{t}}x,U}(\alpha^{n\mathbf{t}} u^k \alpha^{-n\mathbf{t}}).$$

Choose $k > 1$ arbitrary. Suppose n is such that $\alpha^{n\mathbf{t}}x \in D_M$ and suppose that n is sufficiently large that $\alpha^{n\mathbf{t}} u^k \alpha^{-n\mathbf{t}} \in B_1^U$, which is possible since $\alpha^{\mathbf{t}}$ uniformly contracts U. Then

$$\begin{aligned} M &\ge \mu_{\alpha^{n\mathbf{t}}x,U}(B_2^U) \ge \mu_{\alpha^{n\mathbf{t}}x,U}(B_1^U \alpha^{n\mathbf{t}} u^k \alpha^{-n\mathbf{t}}) \\ &= (\mu_{\alpha^{n\mathbf{t}}x,U} \alpha^{n\mathbf{t}} u^{-k} \alpha^{-n\mathbf{t}})(B_1^U) \\ &= c^k \mu_{\alpha^{n\mathbf{t}}x,U}(B_1^U) = c^k. \end{aligned}$$

Since k is arbitrary this implies $c \le 1$.

If $\mu_{x,U} = c\mu_{x,U}u$ then $\mu_{x,U} = c^{-1}\mu_{x,U}u^{-1}$, so the above argument applied to u^{-1} shows that $c \ge 1$, hence $\mu_{x,U} = \mu_{x,U}u$.

Thus we see that if we replace X' by $X' \cap \bigcup_M \limsup_{n\to\infty} \alpha^{-n\mathbf{t}} D_M$ — a conull subset of X', then (8) holds for any $x \in X'$. □

Of particular importance to us will be the following one parameter unipotent subgroups of G, which are parameterized by pairs (i,j) of distinct integers in the range $\{1, \dots, k\}$:

$$u_{ij}(s) = \exp(sE_{ij}) = \mathrm{I}_k + sE_{ij}, \qquad U_{ij} = \{u_{ij}(s) : s \in \mathbb{R}\},$$

where E_{ij} denotes the matrix with 1 at the ith row and jth column and zero everywhere else. It is easy to see these groups are normalized by A; indeed, for $\mathbf{t} = (t_1, \dots, t_k) \in \Sigma$

$$\alpha^{\mathbf{t}} u_{ij}(s) \alpha^{-\mathbf{t}} = u_{ij}(e^{t_i - t_j} s).$$

Since these groups are normalized by A, the orbits of U_{ij} form an A-invariant foliation of $X = \mathrm{SL}(k, \mathbb{R})/\Gamma$ with one-dimensional leaves. We will use μ_x^{ij} as a shorthand for $\mu_{x,U_{ij}}$; any integer $i \in \{1, \dots, k\}$ will be called an index; and unless otherwise stated, any pair i, j of indices is implicitly assumed to be distinct.

Note that for the conditional measures μ_x^{ij} it is easy to find a nonzero $\mathbf{t} \in \Sigma$ such that (5) above holds, for this all we need is $t_i = t_j$. Another helpful feature is the one-dimensionality of U_{ij} which also helps to show that μ_x^{ij} are a.e. Haar measures. In particular we have the following:

(9) Suppose there exists a set of positive measure $B \subset X$ such that for any $x \in B$ there exists a nonzero $u \in U_{ij}$ with $\mu_x^{ij} \propto \mu_x^{ij} u$. Then for a.e. $x \in B$ in fact μ_x^{ij} is a Haar measure of U_{ij}, and if α is ergodic then μ is invariant under U_{ij}.

Proof of (9). Recall first that by (8) we can assume $\mu_x^{ij} = \mu_x^{ij} u$ for $x \in B$. Let $K \subset B$ be a compact set of measure almost equal to $\mu(B)$ such that μ_x^{ij} is continuous for $x \in K$. It is possible to find such a K by Luzin's theorem. Note however, that here the target space is the space of Radon measures $\mathcal{M}_\infty(U_{ij})$ equipped with the weak* topology so that a more general version [5, p. 69] of Luzin's theorem is needed. Let $\mathbf{t} \in \Sigma$ be such that U_{ij} is uniformly contracted by $\alpha^{\mathbf{t}}$. Suppose now $x \in K$ satisfies Poincaré recurrence for every neighborhood of x relative to K. Then there is a sequence $x_\ell = \alpha^{n_\ell \mathbf{t}} \in K$ that approaches x with $n_\ell \to \infty$. Invariance of μ_x^{ij} under u implies invariance of μ_{x_ℓ} under the much smaller element $\alpha^{n_\ell \mathbf{t}} u \alpha^{-n_\ell \mathbf{t}}$ and all its powers. However, since $\mu_{x_\ell}^{ij}$ converges to μ_x^{ij} we conclude that μ_x^{ij} is a Haar measure of U_{ij}. The final statement follows from (4) which implies that the set of x where μ_x^{ij} is a Haar measure is α-invariant. □

Even when μ is not invariant under U_{ij} we still have the following maximal ergodic theorem [20, Thm. A.1] proved by the last named author in joint work with D. Rudolph, which is related to a maximal ergodic theorem of Hurewicz [12].

(10) For any $f \in L^1(X, \mu)$ and $\alpha > 0$ we have

$$\mu\Big(\Big\{x : \int_{B_r^{U_{ij}}} f(ux)\, \mathrm{d}\mu_x^{ij} > \alpha \mu_x^{ij}(B_r^{U_{ij}}) \text{ for some } r > 0\Big\}\Big) < \frac{C\|f\|_1}{\alpha}$$

for some universal constant $C > 0$.

2.2. **Invariant measures, high and low entropy cases.** We are now in a position to state the general measure rigidity result for quotients of G:

Theorem 2.1. *Let $X = G/\Gamma$ and A as above. Let μ be an A-invariant and ergodic probability measure on X. For any pair of indices a, b, one of the following three properties must hold.*

(1) *The conditional measures μ_x^{ab} and μ_x^{ba} are trivial a.e.*
(2) *The conditional measures μ_x^{ab} and μ_x^{ba} are Haar a.e., and μ is invariant under left multiplication with elements of $H_{ab} = \langle U_{ab}, U_{ba}\rangle$.*
(3) *Let $A'_{ab} = \{\alpha^{\mathbf{s}} : \mathbf{s} \in \Sigma \text{ and } s_a = s_b\}$. Then a.e. ergodic component of μ with respect to A'_{ab} is supported on a single $C(H_{ab})$-orbit, where $C(H_{ab}) = \{g \in G : gh = hg \text{ for all } h \in H_{ab}\}$ is the centralizer of H_{ab}.*

Remark: If $k = 3$ then (3) is equivalent to the following:

(3') There exists a nontrivial $\mathbf{s} \in \Sigma$ with $s_a = s_b$ and a point $x_0 \in X$ with $\alpha^{\mathbf{s}} x_0 = x_0$ such that the measure μ is supported by the orbit of x_0 under $C(A'_{ab})$. In particular, a.e. point x satisfies $\alpha^{\mathbf{s}} x = x$.

Indeed, in this case $C(H_{ab})$ contains only diagonal matrices, and Poincaré recurrence for A'_{ab} together with (3) imply that a.e. point is periodic under A'_{ab}. However, ergodicity of μ under A implies that the period $\mathbf{s}$ must be the same a.e. Let $x_0 \in X$ be such that every neighborhood of x_0 has positive measure. Then x close to x_0 is fixed under $\alpha^{\mathbf{s}}$ only if $x \in C(A'_{ab})x_0$, and ergodicity shows (3'). The examples of M. Rees [42][3, Sect. 9] of non-algebraic A-ergodic measures in certain quotients of $\mathrm{SL}(3, \mathbb{R})$ (which certainly can have positive entropy) are precisely of this form, and show that case (3) and (3') above are not superfluous.

When $\Gamma = \mathrm{SL}(k, \mathbb{Z})$, however, this phenomenon, which we term *exceptional returns*, does not happen. We will show this in Section 5; similar observations have been made earlier in [25], [21]. We also refer the reader to [46] for a treatment of similar questions for inner lattices in $\mathrm{SL}(k, \mathbb{R})$ (a certain class of lattices in $\mathrm{SL}(k, \mathbb{R})$).

The conditional measures μ_x^{ij} are intimately connected with the entropy. More precisely, μ has positive entropy with respect to $\alpha^{\mathbf{t}}$ if and only if for some i, j with $t_i > t_j$ the measures μ_x^{ij} are not a.s. trivial (see Proposition 3.1 below for more details; this fact was first proved in [16]). Thus (1) in Theorem 2.1 above holds for all pairs of indices i, j if, and only if, the entropy of μ with respect to every one parameter subgroup of A is zero.

In order to prove Theorem 2.1, it is enough to show that for every a, b for which the μ_x^{ab} is a.s. nontrivial either Theorem 2.1.(2) or Theorem 2.1.(3) holds. For each pair of indices a, b, our proof is divided into two cases which we loosely refer to as the high entropy and the low entropy case:

High entropy case: there is some additional pair of indices i, j distinct from a, b such that $i = a$ or $j = b$ for which μ_x^{ij} are nontrivial a.s. In this case we prove:

Theorem 2.2. *If both μ_x^{ab} and μ_x^{ij} are nontrivial a.s., for distinct pairs of indices i, j and a, b with either $i = a$ or $j = b$, then both μ_x^{ab} and μ_x^{ba} are in fact Haar measures a.s. and μ is invariant under H_{ab}.*

The proof in this case, presented in §3 makes use of the non-commutative structure of certain unipotent subgroups of G, and follows closely [3]. However, by careful use of an adaptation of a formula of Ledrappier and Young (Proposition 3.1 below) relating entropy to the conditional measures μ_x^{ab} we are able to extract some additional information. It is interesting to note that Margulis and Tomanov used the Ledrappier-Young theory for a similar purpose in [26], simplifying some of Ratner's original arguments in the classification of measures invariant under the action of unipotent groups.

Low entropy case: for every pair of indices i, j distinct from a, b such that $i = a$ or $j = b$, μ_x^{ij} are trivial a.s. In this case there are two possibilities:

Theorem 2.3. *Assume μ_x^{ab} are a.e. nontrivial, and μ_x^{ij} are trivial a.e. for every pair i, j distinct from a, b such that $i = a$ or $j = b$. Then one of the following properties holds.*

1. *μ is U_{ab} invariant.*
2. *Almost every A'_{ab}-ergodic component of μ is supported on a single $C(H_{ab})$ orbit.*

We will see in Corollary 3.4 that in the low entropy case μ_x^{ba} is also nontrivial, so applying Theorem 2.3 for U_{ba} instead of U_{ab} one sees that either μ is H_{ab}-invariant or almost every A'_{ab}-ergodic component of μ is supported on a single $C(H_{ab}) = C(H_{ba})$ orbit.

In this case we employ the techniques developed by the third named author in [20]. There, one considers invariant measures on irreducible quotients of products of the type $\mathrm{SL}(2, \mathbb{R}) \times L$ for some algebraic group L. Essentially, one tries to prove a Ratner type result (using methods quite similar to Ratner's [36, 37]) for the U_{ab} flow even though μ is not assumed to be invariant or even quasi invariant under U_{ab}. Implicitly in the proof we use a variant of Ratner's H-property (related, but distinct from the one used by Witte in [48, Sect. 6]) together with the maximal ergodic theorem for U_{ab} as in (9) in Section 2.1.

3. More about entropy and the high entropy case

A well-known theorem by Ledrappier and Young [19] relates the entropy, the dimension of conditional measures along invariant foliations, and Lyapunov exponents, for a general C^2 map on a compact manifold, and in [26, Sect. 9] an adaptation of the general results to flows on locally homogeneous spaces is provided. In the general context, the formula giving the entropy in terms of the dimensions of conditional measures along invariant foliations requires consideration of a sequence of subfoliations, starting from the foliation of the manifold into stable leaves. However, because the measure μ is invariant under the full A-action one can relate the entropy to the conditional measures on the one-dimensional foliations into orbits of U_{ij} for all pairs of indices i, j.

We quote the following from [3]; in that paper, this proposition is deduced from the fine structure of the conditional measures on full stable leaves for A-invariant measure; however, it can also be deduced from a more general result of Hu regarding properties of commuting diffeomorphisms [10]. It should be noted that the constants $s_{ij}(\mu)$ that appear below have explicit interpretation in terms of the pointwise dimension of μ_x^{ij} [19].

Proposition 3.1 ([3, Lemma 6.2])**.** *Let μ be an A-invariant and ergodic probability measure on $X = G/\Gamma$ with $G = \mathrm{SL}(k, \mathbb{R})$ and $\Gamma < G$ discrete. Then for any pair of indices i, j there are constants $s_{ij}(\mu) \in [0, 1]$ so that:*

1. *$s_{ij}(\mu) = 0$ if and only if for a.e. x, μ_x^{ij} are atomic and supported on a single point.*
2. *if a.s. μ_x^{ij} are Haar (i.e. μ is U_{ij} invariant), then $s_{ij}(\mu) = 1$*
3. *for any $\mathbf{t} \in \Sigma$*

$$\mathsf{h}_\mu(\alpha^{\mathbf{t}}) = \sum_{i,j} s_{ij}(\mu)(t_i - t_j)^+. \tag{3.1}$$

Here $(r)^+ = \max(0, r)$ denotes the positive part of $r \in \mathbb{R}$.

We note that the converse to (2) is also true. A similar proposition holds for more general semisimple groups G. In particular we get the following (which is also proved in a somewhat different way in [16]):

Corollary 3.2. *For any $\mathbf{t} \in \Sigma$, the entropy $\mathsf{h}_\mu(\alpha^{\mathbf{t}})$ is positive if and only if there is a pair of indices i, j with $t_i - t_j > 0$ for which μ_x^{ij} are nontrivial a.s.*

A basic property of the entropy is that for any $\mathbf{t} \in \Sigma$,

$$\mathsf{h}_\mu(\alpha^{\mathbf{t}}) = \mathsf{h}_\mu(\alpha^{-\mathbf{t}}). \tag{3.2}$$

As we will see this gives nontrivial identities between the $s_{ij}(\mu)$.

The following is a key lemma from [3], see Figure 1.

Lemma 3.3 ([3, Lemma 6.1])**.** *Suppose μ is an A-invariant and ergodic probability measure, i, j, k distinct indices such that both μ_x^{ij} and μ_x^{jk} are nonatomic a.e. Then μ is U_{ik}-invariant.*

Proof of Theorem 2.2. For $\ell = a, b$ we define the sets

$$\begin{aligned} C_\ell =& \{i \in \{1, \dots, k\} \setminus \{a, b\} : s_{i\ell}(\mu) > 0\}, \\ R_\ell =& \{j \in \{1, \dots, k\} \setminus \{a, b\} : s_{\ell j}(\mu) > 0\}, \\ C_\ell^L =& \{i \in \{1, \dots, k\} \setminus \{a, b\} : \mu \text{ is } U_{i\ell}\text{-invariant}\} \\ R_\ell^L =& \{j \in \{1, \dots, k\} \setminus \{a, b\} : \mu \text{ is } U_{\ell j}\text{-invariant}\}. \end{aligned}$$

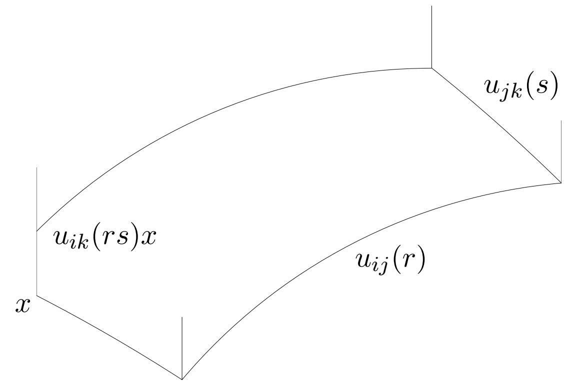

FIGURE 1. One key ingredient of the proof of Lemma 3.3 in [3] is the translation produced along U_{ik} when going along U_{ij} and U_{jk} and returning to the same leaf $U_{ik}x$.

Suppose $i \in C_a$, then the conditional measures μ_x^{ia} are nontrivial a.e. by Proposition 3.1. Since by assumption μ_x^{ab} are nontrivial a.e., Lemma 3.3 shows that μ_x^{ib} are Lebesgue a.e. This shows that $C_a \subset C_b^L$, and $R_b \subset R_a^L$ follows similarly.

Let $\mathbf{t} = (t_1, \dots, t_k)$ with $t_i = -1/k$ for $i \neq a$ and $t_a = 1 - 1/k$. For the following expression set $s_{aa} = 0$. By Proposition 3.1 the entropy of $\alpha^{\mathbf{t}}$ equals

$$\mathsf{h}_\mu(\alpha^{\mathbf{t}}) = s_{a1}(\mu) + \dots + s_{ak}(\mu) = \\ s_{ab}(\mu) + |R_a^L| + \sum_{j \in R_a \setminus R_a^L} s_{aj}(\mu) > |R_a^L|, \quad (3.3)$$

where we used our assumption that $s_{ab}(\mu) > 0$. Applying Proposition 3.1 for $\alpha^{-\mathbf{t}}$ we see similarly that

$$\mathsf{h}_\mu(\alpha^{-\mathbf{t}}) = s_{1a}(\mu) + \dots + s_{ka}(\mu) = s_{ba}(\mu) + \sum_{i \in C_a} s_{ia}(\mu) \leq (1 + |C_a|), \quad (3.4)$$

where we used that $s_{ia}(\mu) \in [0, 1]$ for $a = 2, \dots, k$. However, since the entropies of $\alpha^{\mathbf{t}}$ and of $\alpha^{-\mathbf{t}}$ are equal, we get $|R_a^L| \leq |C_a|$.

Using $\mathbf{t}' = (t_1', \dots, t_k')$ with $t_i' = -1/k$ for $i \neq b$ and $t_b' = 1 - 1/k$ instead of $\mathbf{t}$ in the above paragraph shows similarly $|C_b^L| \leq |R_b|$. Recall that $C_a \subset C_b^L$ and $R_b \subset R_a^L$. Combining these inequalities we conclude that

$$|R_a^L| \leq |C_a| \leq |C_b^L| \leq |R_b| \leq |R_a^L|,$$

and so all of these sets have the same cardinality. However, from (3.3)–(3.4) we see that $s_{ab}(\mu) + |R_a^L| \leq \mathsf{h}_\mu(\alpha^{\mathbf{t}}) \leq s_{ba}(\mu) + |C_a|$. Together we see that

$$s_{ba}(\mu) \geq s_{ab}(\mu) > 0. \quad (3.5)$$

From this we conclude as before that $C_a \subset C_b^L \subset C_a^L$, and so $C_a = C_a^L$. Similarly, one sees $R_b = R_b^L$.

This shows that if $s_{ab}(\mu) > 0$ and $s_{ij}(\mu) > 0$ for some other pair i, j with either $i = a$ or $j = b$, then in fact μ is U_{ij}-invariant. If there was at least one such pair of indices i, j we could apply the previous argument to i, j instead of a, b and get that μ is U_{ab}-invariant. □

In particular, we have seen in the proof of Theorem 2.2 that $s_{ab} > 0$ implies (3.5). We conclude the following symmetry.

Corollary 3.4. *We have $s_{ab} = s_{ba}$ for any pair of indices (a, b). In particular, μ_x^{ab} are nontrivial a.s., if and only if, μ_x^{ba} are nontrivial a.s.*

4. The low entropy case

We let $A'_{ab} = \{\alpha^{\mathbf{s}} \in A : s_a = s_b\}$, and let $\alpha^{\mathbf{s}} \in A'_{ab}$. Then $\alpha^{\mathbf{s}}$ commutes with U_{ab}, which implies that $\mu_x^{ab} = \mu_{\alpha^{\mathbf{s}}x}^{ab}$ a.e.

For a given pair of indices a, b, we define the following subgroups of G:

$$\begin{aligned} L_{(ab)} =& C(U_{ab}) \\ U_{(ab)} =& \langle U_{ij} : i = a \text{ or } j = b \rangle \\ C_{(ab)} =& C(H_{ab}) = C(U_{ab}) \cap C(U_{ba}). \end{aligned}$$

Recall that the metric on X is induced by a right-invariant metric on G. So for every two $x, y \in X$ there exists a $g \in G$ with $y = gx$ and $d(x, y) = d(\mathrm{I}_k, g)$.

4.1. Exceptional returns.

Definition 4.1. We say for $K \subset X$ that the *A'_{ab}-returns to K are exceptional (strong exceptional)* if there exists a $\delta > 0$ so that for all $x, x' \in K$, and $\alpha^{\mathbf{s}} \in A'_{ab}$ with $x' = \alpha^{\mathbf{s}} x \in B_\delta(x) \cap K$ we have that every $g \in B_\delta^G$ with $x' = gx$ satisfies $g \in L_{(ab)}$ ($g \in C_{(ab)}$ respectively).

Lemma 4.2. *There exists a null set $N \subset X$ such that for any compact $K \subset X \setminus N$ with exceptional A'_{ab}-returns to K the A'_{ab}-returns to K are in fact strong exceptional.*

Proof. To simplify notations, assume without loss of generality that $a = 1, b = 2$, and write A', U, L, C for A'_{12}, $U_{(12)}$, $L_{(12)}$, $C_{(12)}$ respectively. We write, for a given matrix $g \in G$,

$$g = \begin{pmatrix} a_1 & g_{12} & g_{1*} \\ g_{21} & a_2 & g_{2*} \\ g_{*1} & g_{*2} & a_* \end{pmatrix}, \tag{4.1}$$

with the understanding that $a_1, a_2, g_{12}, g_{21} \in \mathbb{R}$, g_{1*}, g_{2*} (resp. g_{*1}, g_{*2}) are row (resp. column) vectors with $k - 2$ components, and $a_* \in \mathrm{Mat}(k - 2, \mathbb{R})$. (For $k = 3$ of course all of the above are real numbers, and we can write 3 instead of the symbol $*$.) Then $g \in L$ if and only if $a_1 = a_2$ and g_{21}, g_{*1}, g_{2*} are all zero. $g \in C$ if in addition g_{12}, g_{1*}, g_{*2} are zero.

For $\ell \geq 1$ let D_ℓ be the set of $x \in X$ with the property that for all $z \in B_{1/\ell}(x)$ there exists a unique $g \in B^G_{1/\ell}$ with $z = gx$. Note that $\bigcup_{\ell=1}^\infty D_\ell = X$, and that for every compact set $K \subset D_\ell$ for some $\ell > 0$.

Let first $\alpha^{\mathbf{s}} \in A'$ be a fixed element, and let $E_{\ell,\mathbf{s}} \subset D_\ell$ be the set of points x for which $x' = \alpha^{\mathbf{s}} x \in B_{1/\ell}(x)$ and $x' = gx$ with $g \in B^G_{1/\ell} \cap L = B^L_{1/\ell}$. Since $g \in B^G_{1/\ell}$ is uniquely determined by x (for a fixed $\mathbf{s}$), we can define (in the notation of (4.1)) the measurable function

$$f(x) = \max(|g_{12}|, \|g_{1*}\|, \|g_{*2}\|) \text{ for } x \in E_{\ell,\mathbf{s}}.$$

Let $\mathbf{t} = (-1, 1, 0, \ldots, 0) \in \Sigma$. Then conjugation with $\alpha^{\mathbf{t}}$ contracts U. In fact for g as in (4.1) the entries of $\alpha^{\mathbf{t}} g \alpha^{-\mathbf{t}}$ corresponding to g_{12}, g_{1*} and g_{2*} are $e^{-2} g_{12}, e^{-1} g_{1*}$ and $e^{-1} g_{2*}$, and those corresponding to g_{21}, g_{*1} and g_{*2} are $e^2 g_{21}, e g_{*1}$ and $e g_{*2}$. Notice that the latter are assumed to be zero. This shows that for $x \in E_{\ell,\mathbf{s}}$ and $\alpha^{-n\mathbf{t}} x \in D_\ell$, in fact $\alpha^{-n\mathbf{t}} x \in E_{\ell,\mathbf{s}}$. Furthermore $f(\alpha^{-n\mathbf{t}} x) \leq e^{-n} f(x)$. Poincaré recurrence shows that $f(x) = 0$ for a.e. $x \in E_{\ell,\mathbf{s}}$ – or equivalently $\alpha^{\mathbf{s}} x \in B^C_{1/\ell}(x)$ for a.e. $x \in D_\ell$ with $\alpha^{\mathbf{s}} x \in B^L_{1/\ell}(x)$.

Varying $\mathbf{s}$ over all elements of Σ with rational coordinates and $\alpha^{\mathbf{s}} \in A'$, we arrive at a nullset $N_\ell \subset D_\ell$ so that $\alpha^{\mathbf{s}} x \in B^L_{1/\ell}(x)$ implies $\alpha^{\mathbf{s}} x \in B^C_{1/\ell}(x)$ for all such rational $\mathbf{s}$. Let N be the union of N_ℓ for $\ell = 1, 2, \ldots$. We claim that N satisfies the lemma.

So suppose $K \subset X \setminus N$ has A'-exceptional returns. Choose $\ell \geq 1$ so that $K \subset D_\ell$, and furthermore so that $\delta = 1/\ell$ can be used in the definition of A'-exceptional returns to K. Let $x \in K$, $x' = \alpha^{\mathbf{s}} x \in B_{1/\ell}(x)$ for some $\mathbf{s} \in \Sigma$ with $\alpha^{\mathbf{s}} \in A'$, and $g \in B^G_{1/\ell}$ with $x' = gx$. By assumption on K, we have that $g \in L$. Choose a rational $\tilde{\mathbf{s}} \in \Sigma$ close to $\mathbf{s}$ with $\alpha^{\tilde{\mathbf{s}}} \in A'$ so that $\alpha^{\tilde{\mathbf{s}}} x \in B_{1/\ell}(x)$. Clearly $\tilde{g} = \alpha^{\tilde{\mathbf{s}} - \mathbf{s}} g$ satisfies $\alpha^{\tilde{\mathbf{s}}} x = \tilde{g} x$ and so $\tilde{g} \in B^L_{1/\ell}$. Since $x \in K \subset D_{1/\ell} \setminus N_{1/\ell}$, it follows that $\tilde{g} \in C$. Going back to $x' = \alpha^{\mathbf{s}} x$ and g it follows that $g \in C$. $\square$

Our interest in exceptional returns is explained by the following proposition. Note that condition (1) below is exactly Theorem 2.3(2).

Proposition 4.3. *For any pair of indices a, b the following two conditions are equivalent.*

1. *A.e. ergodic component of μ with respect to A'_{ab} is supported on a single $C_{(ab)}$-orbit.*
2. *For every $\epsilon > 0$ there exists a compact set K with measure $\mu(K) > 1 - \epsilon$ so that the A'_{ab}-returns to K are strong exceptional.*

The ergodic decomposition of μ with respect to A'_{ab} can be constructed in the following manner; Let $\mathcal{E}'$ denote the σ-algebra of Borel sets which are A'_{ab} invariant. For technical purposes, we use the fact that $(X, \mathcal{B}_X, \mu)$ is a Lebesgue space to replace $\mathcal{E}'$ by an equivalent countably generated sub-sigma algebra $\mathcal{E}$. Let $\mu^{\mathcal{E}}_x$ be the family of conditional measures of μ with respect to the σ-algebra $\mathcal{E}$. Since $\mathcal{E}$ is countably generated the atom $[x]_{\mathcal{E}}$ is well defined for all x, and it can be arranged that for *all* x and y with $y \in [x]_{\mathcal{E}}$ the conditional measures $\mu_x^{\mathcal{E}} = \mu_y^{\mathcal{E}}$, and that for all x, $\mu_x^{\mathcal{E}}$ is a probability measure.

Since $\mathcal{E}$ consists of A'_{ab}-invariant sets, a.e. conditional measure is A'_{ab}-invariant, and can be shown to be ergodic. So the decomposition of μ into conditionals

$$\mu = \int_X \mu_x^{\mathcal{E}} \, \mathrm{d}\mu \tag{4.2}$$

gives the ergodic decomposition of μ with respect to A'_{ab}.

Proof. For simplicity, we write $A' = A'_{ab}$ and $C = C_{(ab)}$.

(1) $\Longrightarrow$ (2)**:**

Suppose a.e. A' ergodic component is supported on a single C-orbit. Let $\epsilon > 0$. For any fixed $r > 0$ we define

$$f_r(x) = \mu_x^{\mathcal{E}}(B_r^C(x)).$$

By the assumption $f_r(x) \nearrow 1$ for $r \to \infty$ and a.e. x. Therefore, there exists a fixed $r > 0$ with $\mu(C_r) > 1 - \epsilon$, where $C_r = \{x : f_r(x) > 1/2\}$.

Fix some $x \in X$. We claim that for every small enough $\delta > 0$

$$B_{2r}^C(x) \cap B_\delta(x) = B_\delta^C(x). \tag{4.3}$$

Indeed, by the choice of the metric on X there exists $\delta' > 0$ so that the map $g \mapsto gx$ from $B_{3\delta'}^G$ to X is an isometry. Every $g \in \overline{B_{2r}^C}$ satisfies that either $d(B_{\delta'}^C(x), B_{\delta'}^C(gx)) > 0$, or that there exists $h \in \overline{B_{\delta'}^C}$ with $hx \in \overline{B_{\delta'}^C(gx)}$. In the latter case $B_{\delta'}^C(gx) \subset B_{3\delta'}^C(x)$. The sets $B_{\delta'}^C(g)$ for $g \in B_{2r}^C$ cover the compact set $\overline{B_{2r}^C}$. Taking a finite subcover, we find some $\eta > 0$ so that $d(gx, x) > \eta$ or $gx \in B_{3\delta'}^C(x)$ for every $g \in B_{2r}^C$. It follows that (4.3) holds with $\delta = \min(\eta, \delta')$. In other words, $C_r = \bigcup_{\delta > 0} D_\delta$, where

$$D_\delta = \{x \in C_r : B_{2r}^C(x) \cap B_\delta(x) \subset B_\delta^C(x)\},$$

and there exists $\delta > 0$ with $\mu(D_\delta) > 1 - \epsilon$.

Let $K \subset D_\delta$ be compact. We claim that the A'-returns to K are strongly exceptional. So suppose $x \in K$ and $x' = \alpha^{\mathbf{s}} x \in K$ for some $\alpha^{\mathbf{s}} \in A'$. Then since x and x' are in the same atom of $\mathcal{E}$, the conditional measures satisfy $\mu_x^{\mathcal{E}} = \mu_{x'}^{\mathcal{E}}$. By definition of C_r we have $\mu_x^{\mathcal{E}}(B_r^C(x)) > 1/2$ and the same for x'. Therefore $B_r^C(x)$ and $B_r^C(x')$ cannot be disjoint, and $x' \in B_{2r}^C(x)$ follows. By definition of D_δ it follows that $x' \in B_\delta^C(x)$. Thus the A'-returns to K are indeed strongly exceptional.

(2) $\Longrightarrow$ (1)**:**

Suppose that for every $\ell \geq 1$ there exists a compact set K_ℓ with $\mu(K_\ell) > 1 - 1/\ell$ so that the A'-returns to K are strong exceptional. Then $N = X \setminus \bigcup_\ell K_\ell$ is a nullset. It suffices to show that (1) holds for every A' ergodic $\mu_x^{\mathcal{E}}$ which satisfies $\mu_x^{\mathcal{E}}(N) = 0$.

For any such x there exists $\ell > 0$ with $\mu_x^{\mathcal{E}}(K_\ell) > 0$. Choose some $z \in K_\ell$ with $\mu_x^{\mathcal{E}}(B_{1/m}(z) \cap K_\ell) > 0$ for all $m \geq 1$. We claim that $\mu_x^{\mathcal{E}}$ is supported on

Cz, i.e. that $\mu_x^{\mathcal{E}}(Cz) = 1$. Let δ be as in the definition of strong exceptional returns. By ergodicity there exists for $\mu_x^{\mathcal{E}}$-a.e. $y_0 \in X$ some $\alpha^{\mathbf{s}} \in A'$ with $y_1 = \alpha^{\mathbf{s}} y_0 \in B_\delta(z) \cap K_\ell$. Moreover, there exists a sequence $y_n \in A' y_0 \cap K_\ell$ with $y_n \to z$. Since $y_n \in B_\delta(y_1)$ for large enough n and since the A'-returns to K_ℓ are strong exceptional, we conclude that $y_n \in \overline{B_\delta^C(y_1)}$. Since y_n approaches z and $d(z, y_1) < \delta$, we have furthermore $z \in \overline{B_\delta^C(y_1)}$. Therefore $y_1 \in Cz$, $y_0 = \alpha^{-\mathbf{s}} y_1 \in Cz$, and the claim follows. □

Lemma 4.4. (1) *Under the assumptions of the low entropy case (i.e. $s_{ab}(\mu) > 0$ but $s_{ij}(\mu) = 0$ for all i, j with either $i = a$ or $j = b$), there exists a μ-nullset $N \subset X$ such that for $x \in X \setminus N$ it holds that*

$$U_{(ab)} x \cap X \setminus N \subset U_{ab} x.$$

(2) *Furthermore, unless μ is U_{ab}-invariant, one can also arrange that*

$$\mu_x^{ab} \neq \mu_y^{ab}$$

for any $x \in X \setminus N$ and any $y \in U_{(ab)} x \setminus N$ which is different from x.

Proof. Set $U = U_{(ab)}$ and let $\mu_{x,U}$ be the conditional measures for the foliation into U-orbits. By [3, Prop. 8.3] the conditional measure $\mu_{x,U}$ is a.e. – say for $x \notin N$ – a product measure of the conditional measures μ_x^{ij} over all i, j for which $U_{ij} \subset U$. Clearly, by the assumptions of the low entropy case, μ_x^{ab} is the only one of these which is nontrivial. Therefore, $\mu_{x,U}$ – as a measure on U – is supported on the one dimensional group U_{ab}.

By (3) in §2.1 the conditional measures satisfy furthermore that there is a null set – enlarge N accordingly – such that for $x, y \notin N$ and $y = ux \in Ux$ the conditionals $\mu_{x,U}$ and $\mu_{y,U}$ satisfy that $\mu_{x,U} \propto \mu_{y,U} u$. However, since $\mu_{x,U}$ and $\mu_{y,U}$ are both supported by U_{ab}, it follows that $u \in U_{ab}$. This shows Lemma 4.4.(1).

In order to show Lemma 4.4.(2), we note that we already know that $y \in U_{ab} x$. So if $\mu_x^{ab} = \mu_y^{ab}$, then μ_x^{ab} is again by (3) in §2.1 invariant (up to proportionality) under multiplication by some nontrivial $u \in U_{ab}$. If this were to happen on a set of positive measure, then by (9) in §2.1 μ_x^{ab} are in fact Haar a.e. – a contradiction to our assumption. □

4.2. **Sketch of proof of Theorem 2.3.** We assume that the two equivalent conditions in Proposition 4.3 fail (the first of which is precisely the condition of Theorem 2.3 (2)). From this we will deduce that μ is U_{ab}-invariant which is precisely the statement in Theorem 2.3 (1).

For the following we assume without loss of generality that $a = 1$ and $b = 2$. Write A' and $u(r) = \mathrm{I}_k + rE_{12} \in U_{12}$ for $r \in \mathbb{R}$ instead of A'_{12} and $u_{12}(r)$. Also, we shall at times implicitly identify μ_x^{12} (which is a measure on U_{12}) with its push forward under the map $u(r) \mapsto r$, e.g. write $\mu_x^{12}([a, b])$ instead of $\mu_x^{12}(u([a, b]))$.

By Poincaré recurrence we have for a.e. $x \in X$ and every $\delta > 0$ that

$$d(\alpha^{\mathbf{s}} x, x) < \delta \text{ for some large } \alpha^{\mathbf{s}} \in A'.$$

For a small enough δ there exists a unique $g \in B_\delta^G$ such that $x' = \alpha^{\mathbf{s}} = gx$.

Since $\alpha^{\mathbf{s}}$ preserves the measure and since $A' \subset L_{12} = C(U_{12})$ the conditional measures satisfy

$$\mu_x^{12} = \mu_{x'}^{12}. \tag{4.4}$$

by (5) in §2.1. Since μ_x^{12} is nontrivial, we can find many $r \in \mathbb{R}$ so that $x(r) = u(r)x$ and $x'(r) = u(r)x'$ are again typical. By (3) in §2.1 the conditionals satisfy

$$\mu_{x(r)}^{12} u(r) \propto \mu_x^{12} \tag{4.5}$$

and similar for $x'(r)$ and x'. Together with (4.4) and the way we have normalized the conditional measures this implies that

$$\mu_{x(r)}^{12} = \mu_{x'(r)}^{12}.$$

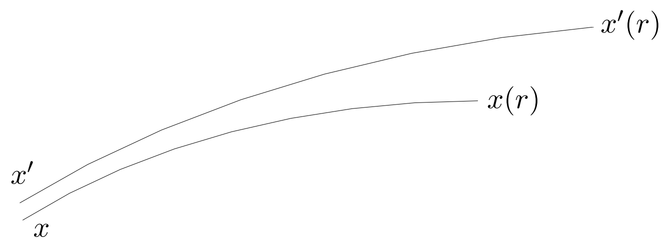

FIGURE 2. Ratner's H-property: When moving along the unipotent $u(r)$, the points $x(r)$ and $x'(r)$ noticably differ first only along $U_{(12)}$.

The key to the low entropy argument, and this is also the key to Ratner's seminal work on rigidity of unipotent flows, is how the unipotent orbits $x(r)$ and $x'(r)$ diverge for r large (see Figure 2). Ratner's H-property (which was introduced and used in her earlier works on rigidity of unipotent flows [36, 37] and was generalized by D. Morris-Witte in [48]) says that this divergence occurs only gradually and in prescribed directions. We remark that in addition to our use of the H-property, the general outline of our argument for the low entropy case is also quite similar to [36, 37].

We shall use a variant of this H-property in our paper, which at its heart is the following simple matrix calculation (cf. [36, Lemm. 2.1] and [37, Def. 1]). Let the entries of $g \in B_\delta^G$ be labelled as in (4.1). A simple calculation shows that $x'(r) = g(r)x(r)$ for

$$g(r) = u(r)gu(-r) = \begin{pmatrix} a_1 + g_{21}r & g_{12} + (a_2 - a_1)r - g_{21}r^2 & g_{1*} + g_{2*}r \\ g_{21} & a_2 - g_{21}r & g_{2*} \\ g_{*1} & g_{*2} - g_{*1}r & a_* \end{pmatrix}. \tag{4.6}$$

Since the return is not exceptional, $g \notin L_{12} = C(U_{12})$ and one of the following holds; $a_2 - a_1 \neq 0$, $g_{21} \neq 0$, $g_{*1} \neq 0$, or $g_{2*} \neq 0$. From this it is immediate that there exists some r so that $g(r)$ is close to I_k in all entries except at least one entry corresponding to the subgroup $U_{(12)}$. More precisely, there is an absolute constant C so that there exists r with

$$C^{-1} \leq \max(|(a_2 - a_1)r - g_{21}r^2|, \|g_{2*}r\|, \|g_{*1}r\|) \leq C, \tag{4.7}$$

$$|g_{21}r| \leq C\delta^{3/8}. \tag{4.8}$$

With some care we will arrange it so that $x(r), x'(r)$ belong to a fixed compact set $X_1 \subset X \setminus N$. Here N is as in Lemma 4.4 and X_1 satisfies that μ_z^{12} depends continuously on $z \in X_1$, which is possible by Luzin's theorem.

If we can indeed find for every $\delta > 0$ two such points $x(r), x'(r)$ with (4.7)–(4.8), we let δ go to zero and conclude from compactness that there are two different points $y, y' \in X_1$ with $y' \in U_{(12)}y$ which are limits of a sequence of points $x(r), x'(r) \in X_1$. By continuity of μ_z^{12} on X_1 we get that $\mu_y^{12} = \mu_{y'}^{12}$. However, this contradicts Lemma 4.4 unless μ is invariant under U_{12}.

The main difficulty consists in ensuring that $x(r), x'(r)$ belong to the compact set X_1 and satisfy (4.7)-(4.8). For this we will need several other compact sets with large measure and various properties.

Our proof follows closely the methods of [20, Sect. 8]. The arguments can be simplified if one assumes additional regularity for the conditional measures μ_z^{12} — see [20, Sect. 8.1] for more details.

4.3. **The construction of a nullset and three compact sets.** As mentioned before we will work with two main assumptions: that μ satisfies the assumptions of the low entropy case and that the equivalent conditions in Proposition 4.3 fail. By the former there exists a nullset N so that all statements of Lemma 4.4 are satisfied for $x \in X \setminus N$. By the latter we can assume that for small enough ϵ and for any compact set with $\mu(K) > 1 - \epsilon$ the A'-returns to K are not strong exceptional.

We enlarge N so that $X \setminus N \subset X'$ where X' is as in §2.1. Furthermore, we can assume that N also satisfies Lemma 4.2. This shows that for every compact set $K \subset X \setminus N$ with $\mu(K) > 1 - \epsilon$ the A'-returns (which exist due to Poincaré recurrence) are not exceptional, i.e. for every $\delta > 0$ there exists $z \in K$ and $\mathbf{s} \in A'$ with $z' = \alpha^{\mathbf{s}} z \in B_\delta(z) \setminus B_\delta^L(z)$.

Construction of X_1: The map $x \mapsto \mu_x^{12}$ is a measurable map from X to a separable metric space. By Luzin's theorem [5, p. 76] there exists a compact $X_1 \subset X \setminus N$ with measure $\mu(X_1) > 1 - \epsilon^4$, and the property that μ_x^{12} depends continuously on $x \in X_1$.

Construction of X_2: To construct this set, we use the maximal inequality (10) in §2.1 from [20, Appendix A]. Therefore, there exists a set $X_2 \subset X \setminus N$ of measure $\mu(X_2) > 1 - C_1\epsilon^2$ (with C_1 some absolute constant) so that for any $R > 0$ and $x \in X_2$

$$\int_{[-R,R]} 1_{X_1}(u(r)x) \, \mathrm{d}\mu_x^{12}(r) \geq (1 - \epsilon^2)\mu_x^{12}([-R, R]). \tag{4.9}$$

Construction of $K = X_3$: Since μ_x^{12} is assumed to be nontrivial a.e., we have $\mu_x^{12}(\{0\}) = 0$ and $\mu_x^{12}([-1,1]) = 1$. Therefore, we can find $\rho \in (0, 1/2)$ so that

$$\mathcal{X}(\rho) = \{x \in X \setminus N : \mu_x^{12}([-\rho, \rho]) < 1/2\} \tag{4.10}$$

has measure $\mu(\mathcal{X}(\rho)) > 1 - \epsilon^2$. Let $\mathbf{t} = (1, -1, 0, \dots, 0) \in \Sigma$ be fixed for the following. By the (standard) maximal inequality we have that there exists a compact set $X_3 \subset X \setminus N$ of measure $\mu(X_3) > 1 - C_2\epsilon$ so that for every $x \in X_3$ and $T > 0$ we have

$$\begin{aligned} \frac{1}{T}\int_0^T 1_{X_2}(\alpha^{-\tau\mathbf{t}}x)\,\mathrm{d}\tau &\geq (1-\epsilon), \\ \frac{1}{T}\int_0^T 1_{\mathcal{X}(\rho)}(\alpha^{-\tau\mathbf{t}}x)\,\mathrm{d}\tau &\geq (1-\epsilon). \end{aligned} \tag{4.11}$$

4.4. **The construction of $z, z' \in X_3$, $x, x' \in X_2$.** Let $\delta > 0$ be very small (later δ will approach zero). In particular, the matrix $g \in B_\delta^G$ (with entries as in (4.1)) is uniquely defined by $z' = gz$ whenever $z, z' \in X_3$ and $d(z, z') < \delta$. Since the A'-returns to X_3 are not exceptional, we can find $z \in X_3$ and $\alpha^{\mathbf{s}} \in A'$ with $z' = \alpha^{\mathbf{s}} z \in B_\delta(z) \cap X_3$ so that

$$\kappa(z, z') = \max(|a_2 - a_1|, |g_{21}|^{1/2}, \|g_{*1}\|, \|g_{2*}\|) \in (0, c\delta^{1/2}), \tag{4.12}$$

where c is an absolute constant allowing us to change from the metric $d(\cdot,\cdot)$ to the norms we used above.

For the moment let $x = z$, $x' = z'$, and $r = \kappa(z, z')^{-1}$. Obviously $\max(|(a_2 - a_1)r|, |g_{21}|^{1/2}r, \|g_{2*}r\|, \|g_{*1}r\|) = 1$. If the maximum is achieved in one of the last two expressions, then (4.7)-(4.8) is immediate with $C = 1$. However, if the maximum is achieved in either of the first two expressions, it is possible that $(a_2 - a_1)r - g_{21}r^2$ is very small. In this case we could set $r = 2\kappa^{-1}(z, z')$, then $(a_2 - a_1)r$ is about 2 and $g_{21}r^2$ is about 4. Now (4.7)-(4.8) hold with $C = 10$. The problem with this naive approach is that we do not have any control on the position of $x(r), x'(r)$. For all we know these points could belong to the null set N constructed in the last section.

To overcome this problem we want to use the conditional measure μ_x^{12} to find a working choice of r in a some interval I containing $\kappa(z, z')^{-1}$. Again, this is not immediately possible since a priori this interval could have very small μ_x^{12}-measure, or even be a nullset. To fix this, we use $\mathbf{t} = (1, -1, 0, \dots, 0)$ and the flow along the $\alpha^{\mathbf{t}}$-direction in Lemma 4.6. However, note that $x = \alpha^{-\tau\mathbf{t}} z$ and $x' = \alpha^{-\tau\mathbf{t}} z'$ differ by $\alpha^{-\tau\mathbf{t}} g \alpha^{\tau\mathbf{t}}$. This results possibly in a difference of $\kappa(x, x')$ and $\kappa(z, z')$ as in Figure 3, and so we might have to adjust our interval along the way. The way $\kappa(x, x')$ changes for various values of τ depends on which terms give the maximum.

Lemma 4.5. *For $z, z' \in X_3$ as above let $T = \frac{1}{4}|\ln \kappa(z, z')|$, $\eta \in \{0, 1\}$, and $\theta \in [4T, 6T]$. There exists subsets $P, P' \subset [0, T]$ of density at least $1 - 9\epsilon$ such that for any $\tau \in P$ ($\tau \in P'$) we have*

(1) *$x = \alpha^{-\tau\mathbf{t}} z \in X_2$ ($x' = \alpha^{-\tau\mathbf{t}} z' \in X_2$) and*

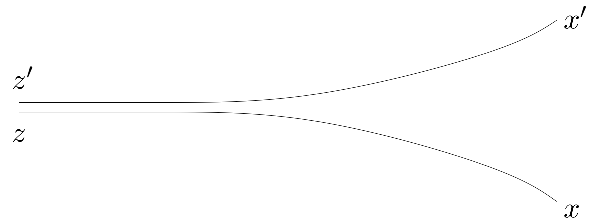

FIGURE 3. The distance function $\kappa(x, x')$ might be constant for small τ and increase exponentially later.

(2) *the conditional measure* μ_x^{12} *satisfies the estimate*

$$\mu_x^{12}([-\rho S, \rho S]) < \frac{1}{2}\mu_x^{12}([-S, S]) \tag{4.13}$$

where $S = S(\tau) = e^{\theta - \eta\tau}$ *(and similarly for* $\mu_{x'}^{12}$*).*

Proof. By the first line in (4.11) there exists a set $Q_1 \subset [0, T]$ of density at least $1 - \epsilon$ (with respect to the Lebesgue measure) such that $x = \alpha^{-\tau\mathbf{t}}z$ belongs to X_2 for every $\tau \in Q_1$.

By the second line in (4.11) there exists a set $Q_2 \subset [0, 4T]$ of density at least $1 - \epsilon$ such that $\alpha^{-v\mathbf{t}}z \in \mathcal{X}(\rho)$ for $v \in Q_2$. Let

$$Q_3 = \Big\{\tau \in [0, T] : \frac{1}{2}(\theta + (2 - \eta)\tau) \in Q_2\Big\}.$$

A direct calculations shows that Q_3 has density at least $1 - 8\epsilon$ in $[0, T]$, and for $\tau \in Q_3$ and $v = \frac{1}{2}(\theta + (2 - \eta)\tau)$ we have $y = \alpha^{-v\mathbf{t}}z \in \mathcal{X}_\rho$.

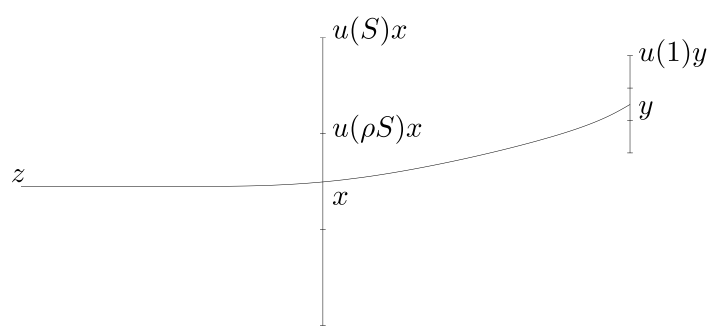

FIGURE 4. From the way the leaf $U_{12}x$ is contracted along $\alpha^{-\mathbf{t}}$ we can ensure (4.13) if $y = \alpha^{-w\mathbf{t}}x \in \mathcal{X}_\rho$

We claim the set $P = Q_1 \cap Q_3 \subset [0, T]$ satisfies all assertions of the lemma, see Figure 4. First P has at least density $1 - 9\epsilon$. Now suppose $\tau \in P$, then $x = \alpha^{-\tau\mathbf{t}}z \in X_2$ by definition of Q_1. Let $w = \frac{1}{2}(\theta - \eta\tau)$, then

$$y = \alpha^{-w\mathbf{t}}x = \alpha^{-v\mathbf{t}}z \in \mathcal{X}_\rho$$

by the last paragraph. By (4.10)

$$\mu_y^{12}([-\rho, \rho]) < \frac{1}{2}\mu_y^{12}([-1, 1]) = \frac{1}{2}.$$

By property (4) in §2 of the conditional measures we get that

$$\frac{\mu_y^{12}([-\rho,\rho])}{\mu_y^{12}([-1,1])} = \frac{(\alpha^{-w\mathbf{t}}\mu_x^{12}\alpha^{w\mathbf{t}})([-\rho,\rho])}{(\alpha^{-w\mathbf{t}}\mu_x^{12}\alpha^{w\mathbf{t}})([-1,1])} = \frac{\mu_x^{12}([-\rho e^{2w},\rho e^{2w}])}{\mu_x^{12}([-e^{2w},e^{2w}])}$$

This implies (4.13) for $S = e^{2w} = e^{\theta-\eta\tau}$. The construction of P' for z' is similar. □

The next lemma uses Lemma 4.5 to construct x and x' with the property that certain intervals containing $\kappa(x,x')^{-1}$ have μ_x^{12}-measure which is not too small. This will allow us in §4.5 to find r so that both $x(r)$ and $x'(r)$ have all the desired properties.

Lemma 4.6. *Let $z,z' \in X_3$ and $T = \frac{1}{4}|\ln\kappa(z,z')|$ be as above. If $\epsilon < \frac{1}{100}$, then there exists $\tau \in [0,T]$ such that*

1. *both $x = \alpha^{-\tau\mathbf{t}}z$ and $x' = \alpha^{-\tau\mathbf{t}}z'$ are in X_2,*
2. *$\kappa(x,x') < c\delta^{3/8}$, and*
3. *for $R = \kappa(x,x')^{-1}$ (as well as $R' = \rho^{-5}R$)*

$$\begin{aligned}
&\mu_x^{12}([-\rho R,\rho R]) < \frac{1}{2}\mu_x^{12}([-R,R]) \text{ and}\\
&\mu_{x'}^{12}([-\rho R,\rho R]) < \frac{1}{2}\mu_{x'}^{12}([-R,R]).
\end{aligned} \tag{4.14}$$

Proof. Let

$$\begin{aligned}
\kappa_a(z,z') &= |a_2 - a_1|,\\
\kappa_u(z,z') &= \max(|g_{21}|^{1/2}, \|g_{*1}\|, \|g_{2*}\|) \in (0, c\delta^{1/2}).
\end{aligned}$$

The corresponding quantities for x,x' are defined similarly. The number T is chosen so that the two points $x = \alpha^{-\tau\mathbf{t}}z$ and $x' = \alpha^{-\tau\mathbf{t}}z'$ are still close together for $\tau \in [0,T]$. In fact,

$$\tilde g = \alpha^{-\tau\mathbf{t}}\begin{pmatrix} a_1 & g_{12} & g_{1*}\\ g_{21} & a_2 & g_{2*}\\ g_{*1} & g_{*2} & a_* \end{pmatrix}\alpha^{\tau\mathbf{t}} = \begin{pmatrix} a_1 & e^{-2\tau}g_{12} & e^{-\tau}g_{1*}\\ e^{2\tau}g_{21} & a_2 & e^{\tau}g_{2*}\\ e^{\tau}g_{*1} & e^{-\tau}g_{*2} & a_* \end{pmatrix} \tag{4.15}$$

satisfies $x' = \tilde g x$, and so

$$\begin{aligned}
&\kappa_a(x,x') = \kappa_a(z,z'),\ \kappa_u(x,x') = e^{\tau}\kappa_u(z,z') \le \kappa(z,z')^{\frac{3}{4}} < c\delta^{\frac{3}{8}}\\
&\qquad\text{and } \kappa(x,x') = \max(\kappa_a(x,x'),\kappa_u(x,x')) < c\delta^{\frac{3}{8}},
\end{aligned} \tag{4.16}$$

see also Figure 3. Hence the second statement of the lemma holds.

For the other two statements of the lemma we will use Lemma 4.5 to define four subsets $P_a, P_u, P'_a, P'_u \subset [0,T]$, each of density at least $1-9\epsilon$, so that for every τ in the intersection of these four sets both (1) and (3) hold.

Definition of P_a: if $\kappa(x,x') > \kappa_a(x,x')$ for all $\tau \in [0,T]$ (recall that x,x' depend implicitly on τ) we set $P_a = [0,T]$.

Otherwise, it follows from (4.16) that $\kappa(z,z') = \kappa_a(z,z')$. We apply Lemma 4.5 for $\eta = 0$ and $\theta = -\log\kappa_a(z,z') = 4T$, and see that (4.13) holds for $\tau \in P_a$, where $P_a \subset [0,T]$ has density at least $1-9\epsilon$, and $S_a = \kappa_a(z,z')^{-1} = \kappa_a(x,x')^{-1}$.

Definition of P_u: if $\kappa(x,x') > \kappa_u(x,x')$ for all $\tau \in [0,T]$ we set $P_u = [0,T]$.

Otherwise, it follows from (4.16) that $\kappa(x,x') = e^{\tau}\kappa_u(z,z') \geq \kappa(z,z')$ for some $\tau \in [0,T]$, hence $\kappa_u(z,z') \in [\kappa(z,z')^{5/4}, \kappa(z,z')]$. This time, we apply Lemma 4.5 with $\eta = 1$ and $\theta = -\log \kappa_u(z,z') \in [4T, 5T]$. We conclude that in this case (4.13) holds for $\tau \in P_u$, where $P_u \subset [0,T]$ is a set of density $1-9\epsilon$, and $S_u = \kappa_u(x,x')^{-1} = e^{\theta-\tau}$.

Clearly, since $\kappa(x,x')$ is either $\kappa_u(x,x')$ or $\kappa_a(x,x')$ at least one of the sets P_a or P_u is constructed using Lemma 4.5, so in particular if $\tau \in P_a \cap P_u$ then $x \in X_2$. Furthermore, if $\tau \in P_a \cap P_u$ we have that (4.13) holds for $S = R = \kappa(x,x')^{-1} = \min(\kappa_a(x,x'), \kappa_u(x,x'))$.

The sets P_a' and P_u' are defined similarly using z'.

The set $P_a \cap P_a' \cap P_u \cap P_u' \subset [0,T]$ has density at least $(1-36\epsilon)$, so in particular if ϵ is small it is nonempty. For any τ in this intersection, $x, x' \in X_2$ and (4.14) holds for $R = \kappa(x,x')^{-1}$.

The additional statement in the parenthesis follows similarly, the only difference being the use of a slightly different value for θ in both cases, and then taking the intersection of $P_a \cap P_a' \cap P_u \cap P_u'$ with four more subsets of $[0,T]$ with similar estimates on their densities. □

4.5. **Construction of $x(r)$, $x'(r)$ and the conclusion of the proof.** Recall that we found $z, z' \in X_3$ using Poincaré recurrence and the assumption that the A'-returns to X_3 are not exceptional. In the last section we constructed $x = \alpha^{-\tau\mathbf{t}}z, x' = \alpha^{-\tau\mathbf{t}}z' = \alpha^{\mathbf{s}} \in X_2$ using the properties of X_3 to ensure (4.14). Since $\alpha^{\mathbf{s}}$ acts isometrically on the U_{12}-leaves, it follows from property (4) of the conditional measures in §2 that $\mu_z^{12} = \mu_{z'}^{12}$ and $\mu_x^{12} = \mu_{x'}^{12}$.

Let

$$P = \{r \in [-R,R] : u(r)x \in X_1\} \text{ and}$$
$$P' = \{r \in [-R,R] : u(r)x' \in X_1\}.$$

By (4.9) we know that P and P' both have density at least $(1-\epsilon^2)$ with respect to the measure $\mu_x^{12} = \mu_{x'}^{12}$. By (4.14) we know that $[-\rho R, \rho R]$ contains less than one half of the μ_x^{12}-mass of $[-R,R]$. Therefore, if ϵ is small enough there exists $r \in P \cap P' \setminus [-\rho R, \rho R]$. We define $x(r) = u(r)x$ and $x'(r) = u(r)x'$, and conclude that $x(r), x'(r) \in X_1$ satisfy $\mu_{x(r)}^{12} = \mu_{x'(r)}^{12}$ by property (3) in §2.

Let $\tilde{g}$ be defined as in (4.15) and write $\tilde{g}_{12}, \ldots$ for the matrix entries. With $\tilde{g}(r) = u(r)\tilde{g}u(-r)$ we have $x'(r) = \tilde{g}(r)x(r)$ and

$$\tilde{g}(r) = \begin{pmatrix} a_1 + \tilde{g}_{21}r & \tilde{g}_{12} + (a_2 - a_1)r - \tilde{g}_{21}r^2 & \tilde{g}_{1*} + \tilde{g}_{2*}r \\ \tilde{g}_{21} & a_2 - \tilde{g}_{21}r & \tilde{g}_{2*} \\ \tilde{g}_{*1} & \tilde{g}_{*2} - \tilde{g}_{*1}r & a_* \end{pmatrix}.$$

We claim it is possible to achieve

$$C^{-1} \le \max(|(a_2-a_1)r - \tilde g_{21} r^2|, \|\tilde g_{2*} r\|, \|\tilde g_{*1} r\|) \le C, \tag{4.17}$$

$$|\tilde g_{21} r| \le C\delta^{3/8}. \tag{4.18}$$

for some constant C, see Figure 2.

We proceed to the proof of (4.17)–(4.18). For (4.18) we first recall that $|\tilde g_{21}| \le \kappa(x,x')^2$, and then use (4.12) and (4.16) to get

$$|\tilde g_{21} r| \le \kappa(x,x')^2 R = \kappa(x,x') \le \kappa(z,z')^{3/4} \le (c\delta^{1/2})^{3/4} \le c^{3/4}\delta^{3/8}.$$

We now turn to prove (4.17). It is immediate from the definition of R that

$$\rho \le \max(|(a_2-a_1)r|, |\tilde g_{21}|^{1/2}|r|, \|\tilde g_{2*} r\|, \|\tilde g_{*1} r\|) \le 1. \tag{4.19}$$

There are two differences of this estimate to the one in (4.17); first we need to take the square of the second term – this replaces the lower bound ρ by its square, secondly we looked above at $(a_2-a_1)r$ and $\tilde g_{21} r^2$ separately – taking the difference as in (4.7) might produce a too small a number (almost cancellation). So (4.17) follows with $C = \rho^{-2}$, unless

$$\max(|(a_2-a_1)r|, |\tilde g_{21}|^{1/2}|r|) \ge \rho \tag{4.20}$$

$$|(a_2-a_1)r - \tilde g_{21} r^2| < \rho^2 < \frac{\rho}{2}. \tag{4.21}$$

This is a minor problem, and we can overcome it using the last statement in Lemma 4.6. Assume that for some $r \in [-R,R] \setminus [-\rho R, \rho R]$ this problem occurs. We deduce a lower estimate on $|\tilde g_{21}|$. If the maximum in (4.20) is achieved at $|\tilde g_{21}|^{1/2}|r| \ge \rho$, then $|\tilde g_{21}| \ge \rho^2 R^{-2}$. If the maximum is achieved at $|(a_2-a_1)r| \ge \rho$, (4.21) shows that $|\tilde g_{21} r^2| \ge \rho/2 \ge \rho^2$ (since $\rho < 1/2$) and so in both cases

$$|\tilde g_{21}| \ge \rho^2 R^{-2}. \tag{4.22}$$

Now we go through the construction of r again, only this time using the last statement in Lemma 4.6, and find $r \in [\rho^{-5}R, \rho^{-5}R] \setminus [\rho^{-4}R, \rho^{-4}R]$. The equivalent to (4.19) is now the estimate

$$\rho^{-4} \le \max(|(a_2-a_1)r|, |\tilde g_{21}|^{1/2}|r|, \|\tilde g_{2*} r\|, \|\tilde g_{*1} r\|) \le \rho^{-5}.$$

This shows that $|(a_2-a_1)r| \le \rho^{-5}$, and (4.22) shows that

$$|\tilde g_{21} r^2| \ge \rho^2 R^{-2} (\rho^{-4} R)^2 = \rho^{-6}.$$

Together, we find a lower bound for

$$|(a_2-a_1)r - \tilde g_{21} r^2| \ge \rho^{-6} - \rho^{-5} > 0,$$

i.e. the problem of almost cancellation cannot happen again.

Starting with the non-exceptional return $z' \in X_3$ of $z \in X_3$ we have found two points $x_r, x_r' \in X_1$ which satisfy (4.17)-(4.18). Since we assume to have non-exceptional returns to X_3 for every $\delta = \frac{1}{n} > 0$, we get two sequences y_n and y_n' of points in X_1 with the same conditional measures

$$\mu^{12}_{y_n} = \mu^{12}_{y_n'}.$$

Compactness shows that we can find convergent subsequences with limits $y, y' \in X_1$. It follows from (4.17)-(4.18) that $y' \in Uy$, and from (4.17) that $y' \neq y$. By continuity of μ_x^{12} for $x \in X_1$ the conditional measures $\mu_y^{12} = \mu_{y'}^{12}$ agree. However, this contradicts Lemma 4.4, unless μ is invariant under U_{12}.

5. Proof that exceptional returns are not possible for $\Gamma = \mathrm{SL}(k, \mathbb{Z})$

If case (3) in Theorem 2.1 holds, then this gives some restriction on Γ. In other words, for some lattices in $\mathrm{SL}(k, \mathbb{R})$, exceptional returns cannot occur. As will be shown below, such is the case for $\Gamma = \mathrm{SL}(k, \mathbb{Z})$.

We recall that $H_{ab} \subset \mathrm{SL}(k, \mathbb{R})$ is an A-normalized subgroup isomorphic to $\mathrm{SL}(2, \mathbb{R})$, and $A' = A \cap C(H_{ab})$. If case (3) of Theorem 2.1 holds then any A'-ergodic component of μ is supported on a single $C(H_{ab})$-orbit. In particular, we have an abundance of A'-invariant probability measures supported on single $C(H_{ab})$ orbits. Merely the existence of such measures is a restriction on Γ.

Theorem 5.1. *Suppose that ν is an A' invariant probability measure on $X = \mathrm{SL}(k, \mathbb{R})/\Gamma$, and that $\operatorname{supp} \nu \subset C(H_{ab})x$ for some $x \in X$. Then there is a $\gamma \in \Gamma$ which is*

1. *diagonalizable over $\mathbb{R}$*
2. *± 1 is not an eigenvalue of γ*
3. *all eigenvalues of γ are simple except precisely one which has multiplicity two.*

Before we prove this theorem, we note the following:

Proposition 5.2. *There is no $\gamma \in \mathrm{SL}(k, \mathbb{Z})$ satisfying the three conditions of Theorem 5.1.*

In particular, case (3) of Theorem 2.1 cannot occur for $\mathrm{SL}(k, \mathbb{R})/\mathrm{SL}(k, \mathbb{Z})$.

Proof of Proposition 5.2. Suppose $\gamma \in \mathrm{SL}(k, \mathbb{Z})$ is diagonalizable over $\mathbb{R}$. Then its eigenvalues (with the correct multiplicities) are roots of the characteristic polynomial of γ, a polynomial with integer coefficients and both leading term and constant term equal to one. If there is some eigenvalue which is not equal to ± 1 and which occurs with multiplicity greater than one then necessarily this eigenvalue is not rational, and its Galois conjugates would also have multiplicity greater than one, contradicting (3). □

To prove Theorem 5.1, we need the following standard estimate:

Lemma 5.3. *There is a neighborhood U_0 of the identity in $\mathrm{SL}(m, \mathbb{R})$ so that for any $\lambda_1, \lambda_2, \dots, \lambda_m$ with $|\lambda_i - \lambda_j| > 1$ and $h \in U_0$ one has that $h \operatorname{diag}(e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_m})$ is diagonalizable over $\mathbb{R}$ with positive eigenvalues and the eigenvalues $e^{\lambda'_1}, e^{\lambda'_2}, \dots, e^{\lambda'_m}$ satisfy $|\lambda'_i - \lambda_i| < \frac{1}{2}$.*

Proof. Without loss of generality, suppose $\lambda_1 > \lambda_2 > \cdots > \lambda_m$. Let $f = \operatorname{diag}(e_1^\lambda, e_2^\lambda, \ldots, e_m^\lambda)$ and $\eta_1, \eta_2, \ldots$ be the eigenvalues of $f' = hf$ ordered according to descending absolute value. Set for $1 \leq i \leq m$, $\lambda_i' = \log|\eta_i|$.

Clearly, $\lambda_1' = \lim_{n\to\infty} \frac{\log\|f'^n\|}{n}$. Since f is self adjoint, $\|f\| = e^{\lambda_1}$ so

$$\lambda_1' \leq \log\|h\| + \lambda_1.$$

Let $\delta > 0$ be small (it will be chosen later and will be independent of f). Consider the cones

$$K = \{(x_1, \ldots, x_m) : |x_l| \leq \delta\,|x_1| \text{ for every } l \neq 1\}$$
$$K' = \left\{(x_1, \ldots, x_m) : |x_l| \leq \delta e^{-1}\,|x_1| \text{ for every } l \neq 1\right\}.$$

Then $fK \subset K'$, and for every $x \in K$

$$\|fx\| \geq (1 - c\delta)e^{\lambda_1}\,\|x\|,$$

for some c depending only on m. Suppose now that h is close enough to the identity so that $hK' \subset K$. Then $f'K = hfK \subset hK' \subset K$, and again assuming that h is in some fixed neighborhood of the identity, for any $x \in K$,

$$\|f'x\| \geq (1 - 2c\delta)e^{\lambda_1}\,\|x\|$$

so

$$\|f'^n\| \geq \|f'^n e_1\| \geq \left((1 - 2c\delta)e^{\lambda_1})\right)^n.$$

In other words, if h is in some fixed neighborhood of the identity (independently of f) then

$$|\lambda_1' - \lambda_1| < C_1\delta.$$

Similarly, $e^{\lambda_1+\lambda_2}$ is the dominating eigenvalue of $f \wedge f$, i.e. the natural action of f on the space $\mathbb{R}^n \wedge \mathbb{R}^n$. Applying the same logic as before, $\lambda_1' + \lambda_2' = \lim_{n\to\infty} \frac{\log\|(f'\wedge f')^n\|}{n}$, and as long as h is in some fixed neighborhood of the identity, independently of f

$$|\lambda_1' + \lambda_2' - \lambda_1 - \lambda_2| < C_2\delta$$

and more generally

$$\left|\sum_{i=1}^{k}(\lambda_i' - \lambda_i)\right| < C_i\delta. \tag{5.1}$$

Clearly, (5.1) implies that there is some C depending only on m, and a neighborhood of the identity in $\mathrm{SL}(m, \mathbb{R})$ depending only on δ so that if h is in that neighborhood

$$|\lambda_i' - \lambda_i| < C\delta \qquad \text{for all } 1 \leq i \leq m.$$

In particular, if $\lambda_i > \lambda_{i-1} + 1$ for every i then if $C\delta < \frac{1}{2}$ all λ_i' are distinct. Since this holds for all h in a connected neighborhood of the identity, all the eigenvalues of f' are real and also positive, so $\eta_i = e^{\lambda_i'}$. ∎

Proof of Theorem 5.1. Without loss of generality, take $a = 1, b = 2$. Let $a(t) = \operatorname{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_k t}) \in A'$ with $\lambda_1 = \lambda_2$ and for every other pair i, j we have $\lambda_i \neq \lambda_j$.

Take U_0 to be a symmetric neighborhood of the identity in $\mathrm{SL}(k-2, \mathbb{R})$ as in Lemma 5.3, and

$$U_1 = \left\{ \begin{pmatrix} e^r & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & e^{-\frac{2r}{k-2}} h \end{pmatrix} : r \in (-1/8, 1/8) \text{ and } h \in U_0 \right\},$$

and let $t_0 = 2 \max_{\lambda_i \neq \lambda_j} |\lambda_i - \lambda_j|^{-1}$. Note that U_1 is also symmetric, i.e. $U_1^{-1} = U_1$.

By Poincaré recurrence, for ν-almost every $x = g\Gamma \in \mathrm{SL}(k, \mathbb{R})/\Gamma$ there is a $t > t_0$ so that $a(t)x \in U_1 x$, so in particular $U_1 a(t) \cap g\Gamma g^{-1} \neq \varnothing$. Let

$$\tilde{\gamma} = \begin{pmatrix} e^s & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & f' \end{pmatrix} \in U_1 a(t) \cap g\Gamma g^{-1}$$

be any element from this intersection. By assumption, for every pair i, j except $1, 2$ we have

$$|\lambda_i - \lambda_j| \, t > 2, \tag{5.2}$$

and we can apply Lemma 5.3 to deduce that the eigenvalues of

$$f' = h e^{-\frac{2s}{k-2}} \operatorname{diag}(e^{\lambda_3 t}, e^{\lambda_4 t}, \dots, e^{\lambda_k t})$$

for some $h \in U_0$ are of the form $e^{\lambda'_3}, \dots, e^{\lambda'_k}$ with $|\lambda'_i - \lambda_i t| < 3/4$ for $i = 3, \dots, k$. Finally $|s - \lambda_1 t| = |s - \lambda_2 t| \leq 1/8$.

In view of (5.2) it is clear that $\tilde{\gamma}$ and hence $g^{-1}\tilde{\gamma} g \in \Gamma$ satisfy all the conditions of Theorem 5.1. □

6. Conclusion of the proof of Theorem 1.3

In this section, we conclude the derivation of Theorem 1.3, and its corollary, Corollary 1.4, from Theorem 2.1. Throughout this section, X will denote the quotient space $\mathrm{SL}(k, \mathbb{R}) / \mathrm{SL}(k, \mathbb{Z})$, and μ be a A-ergodic and invariant probability measure on X. For every pair a, b of distinct indices in $\{1, \dots, k\}$, one of the three possibilities of Theorem 2.1 holds. However, in view of the results of the previous section, in particular Theorem 5.1 and Proposition 5.2, Theorem 2.1.(3), i.e. the case of exceptional returns, cannot occur for the lattice $\mathrm{SL}(k, \mathbb{Z})$. Therefore, for every pair a, b of distinct indices one of the following two mutually exclusive posibilities hold:

(1) The conditional measures μ_x^{ab} and μ_x^{ba} are trivial a.e.
(2) The conditional measures μ_x^{ab} and μ_x^{ba} are Haar and μ is invariant under left multiplication with elements of $H_{ab} = \langle U_{ab}, U_{ba} \rangle$.

Define a relation $a \sim b$ if μ is U_{ab}-invariant. By (2) above it follows that $a \sim b$ if and only if $b \sim a$. Furthermore, since the group generated by U_{ab} and U_{bc} contains U_{ac}, it is clear that $\sim$ is in fact an equivalence relation on $\{1, \dots, k\}$. Let H be the group generated by all U_{ab} with $a \sim b$. Let r denotes the number of equivalence classes for $\sim$ which contain more than one element, and $k_1, k_2, \dots, k_r$ be their sizes, so in particular $\sum_{i=1}^{r} k_i \leq k$. By permuting the indices if necessary we can assume these equivalence classes are consecutive indices and $H = \prod_{i=1}^{r} \mathrm{SL}(k_i, \mathbb{R})$. By definition, H leaves the measure μ invariant, is normalized by A, and is generated by unipotent one parameter subgroups of $\mathrm{SL}(n, \mathbb{R})$ — indeed, H is precisely the maximal subgroup of $\mathrm{SL}(n, \mathbb{R})$ satisfying these three conditions.

Measures invariant under groups generated by unipotent one parameter groups are well understood. In particular, in a seminal series of papers culminating in [38], M. Ratner showed that if H is such a group the only H-ergodic and invariant probability measures are the algebraic measures: L-invariant measures supported on a closed L-orbit for some $L > H$ (here and throughout, we use the notation $L > H$ to denote that H is a subgroup and $L \supset H$; specifically H may be equal to L) . For the A-invariant measure μ and H as above we only know that μ is A-ergodic and H-invariant, but not necessarily H-ergodic; we shall use the following version of these measure rigidity results by Margulis and Tomanov [27][3] (similar techniques were used also in [28, proof of Thm. 1]; see also [18, Sect. 4.4] and [44]). For any connected real Lie group G, we shall say that $g \in G$ is in element of class $\mathcal{A}$ if $\mathrm{Ad}\, g$ is semi simple, with all eigenvalues integer powers of some $\lambda \in \mathbb{R} \backslash \{\pm 1\}$, and g is contained in a maximal reductive subgroup of G.

Theorem 6.1 ([27, Theorems (a) and (b)]). *Let G be a connected real Lie group, $\Gamma < G$ a discrete subgroup, and $\tilde{H}$ generated by unipotent one parameter groups and elements of class $\mathcal{A}$, with $H < \tilde{H}$ the subgroup generated by unipotent one parameter groups. Let μ be an $\tilde{H}$-invariant and ergodic probability measure on G/Γ. Then there is an $L \geq H$ so that almost every H-ergodic component of μ is the L-invariant probability measure on a closed L-orbit. Furthermore, if*

$$SN_G(L) = \{g \in N_G(L) : \text{conjunction by } g \text{ preserves Haar measure on } L\}$$

then $\tilde{H} < SN_G(L)$ and μ is supported on a single $SN_G(L)$-orbit. In particular, L is normalized by $\tilde{H}$.

Lemma 6.2. *Let $H = \prod_{i=1}^{r} \mathrm{SL}(k_i, \mathbb{R})$ with $\sum_{i=1}^{r} k_i < k$. Let*

$$X_H = \{x \in X : Hx \text{ is closed and of finite volume}\} .$$

Then there is a one parameter subgroup $a(t)$ of A so that for every $x \in X_H$ its trajectory $a(t)x \to \infty$ as $t \to \infty$.

Proof. Suppose Hx is closed and of finite volume, with $x = g\,\mathrm{SL}(k, \mathbb{Z})$ and $g = (g_{ij})$. Let $k' = \sum_i k_i < k$. Let $\Lambda = g^{-1} H g \cap \Gamma$. Since Λ is Zariski dense

[3]The main theorem of [27] was substantially more general than what we quote here. In particular, in their theorem Γ can be any closed subgroup of G, and the group G can be a product of real and p-adic Lie groups (satisfying some mild additional conditions).

in $g^{-1}Hg$ there is a $\gamma = g^{-1}h_0 g \in \Lambda$ with $h_0 = \begin{pmatrix} h_0' & 0 \\ 0 & I_{k-k'} \end{pmatrix}$ so that

$$V_g := \left\{ y \in \mathbb{R}^k : y^T g^{-1} h g = y^T \quad \text{for all } h \in H \right\} = \left\{ y \in \mathbb{R}^k : y^T \gamma = y^T \right\}. \tag{6.1}$$

Notice that since $g(g^{-1}h_0 g) = h_0 g$ (the transpose of) the last $k-k'$ rows of g are in V_g.

Clearly $\dim V_g = k - k'$, and using the right hand side of (6.1) it is clear that V_g is a rational subspace of $\mathbb{R}^n$ (i.e. has a basis consisting of rational vectors). Since V_g is rational, there is an integer vector $m \in \mathbb{Z}^n \cap (V_g)^\perp$. In particular, the last $k-k'$ entries in the vector gm (which is a vector in the lattice in $\mathbb{R}^k$ corresponding to $g\,\mathrm{SL}(k,\mathbb{Z})$) are zero. For any $t \in \mathbb{R}$ set $\mathbf{t} = (t_1, \ldots, t_k)$ with $t_1 = \cdots = t_{k'} = k't$ and $t_{k'+1} = \cdots = t_k = (k'-k)t$ and $a(t) = \alpha^{\mathbf{t}}$. Then since the last $k-k'$ entries in the vector gm are zero

$$a(t)(gm) \to 0 \qquad \text{as } t \to \infty$$

so by Mahler's criterion $a(t)x = a(t)g\,\mathrm{SL}(k,\mathbb{Z}) \to \infty$. □

We are finally in a position to finish the proof of Theorem 1.3:

Proof of Theorem 1.3. Let $H = \prod_{i=1}^r \mathrm{SL}(k_i, \mathbb{R})$ be the maximal group fixing μ, generated by unipotent one parameter subgroups, and normalized by A as above. By Theorem 6.1, applied to μ with $\tilde{H} = AH$, we know that there is some $L > H$ which is normalized by A so that almost every H-ergodic component of μ is the L-invariant measure on a closed L orbit. In particular μ is L-invariant, which unless $L < AH$ contradicts the definition of H as the maximal group with the above properties. Let now $x = g\,\mathrm{SL}(k,\mathbb{Z})$ have a closed L-orbit Lx of finitely volume. Then $\Lambda_L = g^{-1}Lg \cap \mathrm{SL}(k,\mathbb{Z})$ is a lattice in $g^{-1}Lg$, and so the latter is defined over $\mathbb{Q}$. Therefore, the same is true for the semi-simple $g^{-1}Hg = [g^{-1}Lg, g^{-1}Lg]$, $\Lambda_H = g^{-1}Hg \cap \mathrm{SL}(k,\mathbb{Z})$ is a lattice in $g^{-1}Hg$, and Hx is closed with finite volume. However, this implies $H = L$.

Thus we conclude that almost every H-ergodic component of μ is supported on a single H-orbit; in other words, in the notations of Lemma 6.2, the support of μ is contained in X_H.

By Lemma 6.2, this implies that the sum $\sum_i k_i = k$ since otherwise there is a one parameter subgroup $a(t)$ of A so that for every $x \in X_H$ its trajectory $a(t)x \to \infty$ as $t \to \infty$, in contradiction to Poincaré recurrence.

But if $\sum_i k_i = k$, the set $SN_G(H)$ of Theorem 6.1 satisfies

$$SN_G(H) = N_G(H) = AH$$

so by this theorem μ is supported on a single AH-orbit. But μ is also AH-invariant. This show that μ is algebraic: an AH-invariant probability measure on a single AH-orbit. Note that this AH-orbit has finite volume, hence is closed in X. □

Proof of Corollary 1.4. Let μ be an A-ergodic probability measure on X with positive entropy. By Theorem 1.3, μ is algebraic, i.e. there is a subgroup $A < L < G$ and a point $x = g\,\mathrm{SL}(k,\mathbb{Z}) \in X$ so that Lx is closed and μ is the L-invariant measure on Lx.

Since μ is a probability measure, this implies that $g^{-1}Lg \cap \mathrm{SL}(k,\mathbb{Z})$ is a lattice in $g^{-1}Lg$, which, in turn, implies that $g^{-1}Lg$ is defined over $\mathbb{Q}$. Moreover, the fact that L has any lattice implies it is unimodular, which in view of $A < L < G$ (and since A is the maximal torus in G) implies L is reductive (this can also be seen directly from the proof of Theorem 1.3).

We conclude that $g^{-1}Lg$ is a reductive group defined over $\mathbb{Q}$, and $g^{-1}Ag$ is a maximal torus in this group. By [34, Thm. 2.13], there is a $h \in L$ so that $g^{-1}h^{-1}Ahg$ is defined over $\mathbb{Q}$ and is

$\mathbb{Q}$-anisotropic. This implies that Ahx is closed and of finite volume (i.e., since $A \cong \mathbb{R}^{k-1}$, compact), so that Lx contains a compact A orbit.

By [21, Thm. 1.3], it follows that (possibly after conjugating by a permutation matrix) L is the subgroup of $g = (g_{ij}) \in \mathrm{SL}(k,\mathbb{R})$ with $g_{ij} = 0$ unless i is congruent to j mod m for some $1 \neq m|k$ (by the Moore ergodicity theorem it is clear that A acts ergodicaly on Lx, hence the condition in that theorem that Lx contains a relatively dense A orbit is satisfied), and that Lx is not compact. Note that if k is prime this implies that $L = \mathrm{SL}(k,\mathbb{R})$. □

Part 2. Positive entropy and the set of exceptions to Littlewood's Conjecture

7. Definitions

We recall the definition of Hausdorff dimension, box dimension, topological and metric entropy. In the following let Y be a metric space with metric $d_Y(\cdot,\cdot)$.

7.1. Notions of dimension.

For $D \geq 0$ the *D-dimensional Hausdorff measure* of a set $B \subset Y$ is defined by

$$\mathcal{H}^D(B) = \lim_{\epsilon \to 0} \inf_{\mathcal{C}_\epsilon} \sum_i (\operatorname{diam} C_i)^D,$$

where $\mathcal{C}_\epsilon = \{C_1, C_2, \ldots\}$ is any countable cover of B with sets C_i of diameter $\operatorname{diam}(C_i)$ less than ϵ. Clearly, for $D > m$ any set in the Euclidean space $\mathbb{R}^m$ has Hausdorff measure zero. The *Hausdorff dimension* $\dim_H(B)$ is defined by

$$\dim_H(B) = \inf\{D : \mathcal{H}^D(B) = 0\} = \sup\{D : \mathcal{H}^D(B) = \infty\}. \tag{7.1}$$

For every $\epsilon > 0$ a set $F \subset B$ is ϵ-separated if $d_Y(x,y) \geq \epsilon$ for every two different $x, y \in F$. Let $b_\epsilon(B)$ be the cardinality of the biggest ϵ-separated subset of B, then the *(upper) box dimension* (*upper Minkowski dimension*) is defined by

$$\dim_{\mathrm{box}}(B) = \limsup_{\epsilon \to 0} \frac{\log b_\epsilon(B)}{|\log \epsilon|}. \tag{7.2}$$

Note that $b_{\epsilon_2}(B) \geq b_{\epsilon_1}(B)$ if $\epsilon_2 < \epsilon_1$. Therefore, it is sufficient to consider a sequence ϵ_n in (7.2) if $\log \epsilon_{n+1} / \log \epsilon_n \to 1$ for $n \to \infty$.

We recall some elementary properties. First Hausdorff dimension and box dimension do not change when we use instead of the metric $d_Y(\cdot,\cdot)$ a different but Lipschitz equivalent metric $d'_Y(\cdot,\cdot)$. The Hausdorff dimension of a countable union is given by

$$\dim_H\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sup_i \dim_H(B_i). \tag{7.3}$$

(This follows easily from the fact that the measure $\mathcal{H}^D$ is subadditive.) For any B we have

$$\dim_H(B) \leq \dim_{\mathrm{box}}(B). \tag{7.4}$$

If $Y = Y_1 \times Y_2$ is nonempty and

$$d_Y((x_1,x_2),(y_1,y_2)) = \max(d_{Y_1}(x_1,x_2), d_{Y_2}(y_1,y_2)),$$

then $\dim_{\mathrm{box}} Y \leq \dim_{\mathrm{box}} Y_1 + \dim_{\mathrm{box}} Y_2$.

7.2. **Entropy and the variational principle.** Let T be an endomorphism of a *compact* metric space Y. For $\epsilon > 0$ and a positive integer N we say that a set $E \subset Y$ is (N,ϵ)-separated (with respect to T) if for any two different $x, y \in E$ there exists an integer $0 \leq n < N$ with $d(T^n x, T^n y) \geq \epsilon$. Let $s_{N,\epsilon}(T)$ be the cardinality of the biggest (N,ϵ)-separated set, then the *topological entropy* of T is defined by

$$\mathsf{h}_{\mathrm{top}}(T) = \lim_{\epsilon \to 0} \limsup_{N\to\infty} \frac{1}{N} \log s_{N,\epsilon}(T) = \sup_{\epsilon>0} \limsup_{N\to\infty} \frac{1}{N} \log s_{N,\epsilon}(T). \tag{7.5}$$

Let μ be a T-invariant measure on Y, and let $\mathcal{P}$ be a finite partition of Y into measurable sets. Then

$$\mathsf{H}_\mu(\mathcal{P}) = -\sum_{P\in\mathcal{P}} \mu(P)\log\mu(P)$$

is the *entropy of the finite partition* $\mathcal{P}$. (Here $0\log 0 = 0$.) For two such partitions $\mathcal{P}$ and $\mathcal{Q}$ let $\mathcal{P} \vee \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$ be the common refinement. The *metric entropy* of T with respect to μ and $\mathcal{P}$ is defined by

$$\mathsf{h}_\mu(T,\mathcal{P}) = \lim_{N\to\infty} \frac{1}{N} \mathsf{H}_\mu\Big(\bigvee_{i=0}^{N-1} T^{-i}\mathcal{P}\Big)$$

and the *metric entropy* of T with respect to μ is

$$\mathsf{h}_\mu(T) = \sup_{\mathcal{P}} \mathsf{h}_\mu(T,\mathcal{P}), \tag{7.6}$$

where the supremum is taken over all finite partitions $\mathcal{P}$ of Y into measurable sets.

Topological and metric entropy are linked: For a compact metric space Y, a continuous map $T : Y \to Y$, and a T-invariant measure μ on Y the entropies satisfy

$$\mathsf{h}_\mu(T) \leq \mathsf{h}_{\mathrm{top}}(T).$$

Furthermore, the variational principle [47, Thm. 8.6] states that

$$\mathsf{h}_{\mathrm{top}}(T) = \sup_{\mu} \mathsf{h}_{\mu}(T), \tag{7.7}$$

where the supremum is taken over all T-invariant measures μ on Y.

8. Box dimension and topological entropy

We return the study of the left action of the positive diagonal subgroup A on $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$. We fix an element $a \in A$ and study the multiplication from the left by a on X, in particular we are interested in the dynamical properties of the restriction $a|_K$ of this map to a compact subset $K \subset X$. This will lead to a close connection between topological entropy and box dimension in an *unstable manifold*.

The following easy lemma shows that the dimensions can be defined using the right invariant metric or a norm on $\mathrm{Mat}(k,\mathbb{R})$.

Lemma 8.1. *For every $r > 0$ there exists a constant $c_0 \geq 1$ such that*

$$c_0^{-1}\|g-h\| \leq d(g,h) \leq c_0\|g-h\| \text{ for all } g,h \in B_r^G,$$

where $\|A\| = \max_{i,j}|a_{ij}|$ for $A = (a_{ij}) \in \mathrm{Mat}(k,\mathbb{R})$.

X is locally isomorphic to $\mathrm{SL}(k,\mathbb{R})$; more specifically, for every $x \in X$ there exists some $r = r(x) > 0$ such that B_r^G and $B_r(x)$ are isomorphic by sending g to gx. For small enough r this is an isometry. For a compact set $K \subset X$ we can choose $r = r(K) > 0$ uniformly with this property for all $x \in K$.

Let $x \in X$, $g \in B_r^G$, $y = gx$, $\mathbf{t} \in \Sigma$, and $a = \alpha^{\mathbf{t}}$. Then $ay = (aga^{-1})ax$. In other words, when we use the local description of X as above at x and ax, left multiplication by a acts in this local picture like conjugation by a on B_r^G. For this reason we define the subgroups

$$\begin{aligned} U &= \{g \in \mathrm{SL}(k,\mathbb{R}) : a^n g a^{-n} \to 0 \text{ for } n \to -\infty\}, \\ V &= \{g \in \mathrm{SL}(k,\mathbb{R}) : a^n g a^{-n} \to 0 \text{ for } n \to \infty\}, \text{ and} \\ C &= \{g \in \mathrm{SL}(k,\mathbb{R}) : aga^{-1} = g\}, \end{aligned}$$

which are the *unstable*, *stable*, and *central* subgroup (for conjugation with a). Let $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$, so that $\mathrm{I}_k = (\delta_{ij})$. It is easy to check that $g \in C$ if $g_{ij} = 0$ for all i,j with $a_{ii} \neq a_{jj}$, $g \in U$ if $g_{ij} = \delta_{ij}$ for all i,j with $a_{ii} \geq a_{jj}$, and similarly $g \in V$ if $g_{ij} = \delta_{ij}$ for all i,j with $a_{ii} \leq a_{jj}$. Furthermore, there exists a neighborhood $U_0 \subset \mathrm{SL}(k,\mathbb{R})$ of the identity so that every $g \in U_0$ can be written uniquely as $g = g_C g_U g_V$ for some small $g_C \in C$, $g_U \in U$, and $g_V \in V$. If similarly $h = h_C h_U h_V$, then

$$c_1^{-1} d(g,h) \leq \max\bigl(d(g_C,h_C), d(g_U,h_U), d(g_V,h_V)\bigr) \leq c_1 d(g,h) \tag{8.1}$$

for some constant $c_1 \geq 1$.

Since A is commutative, we have $A \subset C$. The map $T(x) = ax$ on X is *partially hyperbolic:* T is not hyperbolic (since the identity is not an isolated point of C), but part of the local description has hyperbolic structure as follows.

Lemma 8.2. *Let $K \subset X$ be compact with $aK \subset K$ and let $r = r(K)$ be as above. There exists $\lambda > 1$ and $c_2 > 0$ so that for any small enough $\epsilon > 0$, any $z \in K$ and $f \in B_r^U$, and any integer $N \geq 1$ with $d(fz, z) \geq \lambda^{-N}\epsilon$, there exists a non-negative integer $n < N$ with $d(a^n fz, a^n z) \geq c_2\epsilon$.*

Proof. By continuity there exists $\epsilon \in (0, r)$ such that $d(afa^{-1}, \mathrm{I}_k) < r$ whenever $d(f, \mathrm{I}_k) < \epsilon$. This will be the only requirement on ϵ. On the other hand, since U is expanded by conjugation with a, there exists some $\lambda > 1$ so that $\|afa^{-1} - \mathrm{I}_k\| \geq \lambda \|f - \mathrm{I}_k\|$ for all $f \in B_r^U$. By Lemma 8.1 $d(a^n fa^{-n}, \mathrm{I}_k) \geq c_0^{-2}\lambda^n d(f, \mathrm{I}_k)$ for all $f \in B_r^U$ and all n for which

$$\max(d(f, \mathrm{I}_k), \ldots, d(a^n fa^{-n}, \mathrm{I}_k)) < r.$$

By assumption $\lambda^{-N}\epsilon \leq d(fz, z) = d(f, Id) < r$. It follows that there exists $n < N$ with $c_0^{-2}\lambda^{-1}\epsilon < d(a^n fa^{-n}, \mathrm{I}_k) < r$. Since $a^n z \in K$ we get

$$d(a^n fz, a^n z) = d((a^n fa^{-n})a^n z, a^n z) = d(a^n fa^{-n}, \mathrm{I}_k) > c_0^{-2}\lambda^{-1}\epsilon. \qquad \square$$

We are ready to give a close connection between box dimension and entropy.

Proposition 8.3. *Let $a \in A$ and $K \subset X$ be compact with $aK \subset K$. Then one of the following properties holds.*

1. *The intersection $Ux \cap K$ of the* unstable manifold *Ux with K is a countable union of compact sets of box dimension zero for every $x \in X$.*
2. *The restriction $a|_K$ of the multiplication operator a to K has positive topological entropy.*

Proof. Note, that the first possibility follows if there exists some $\epsilon > 0$ such that

$$P_y = K \cap (B_\epsilon^U y) \text{ has box dimension zero for every } y \in K. \tag{8.2}$$

To see this suppose $K \cap Ux$ for $x \in X$ is non-empty, and cover $K \cap Ux$ by countably many sets P_y as in (8.2). Taking the union for every such x shows the first statement of the proposition.

So it suffices to show that if (8.2) fails for ϵ as in Lemma 8.2, then the topological entropy $\mathsf{h}_{\mathrm{top}}(a|_K) > 0$ is positive. Assume $2\epsilon \leq r$ and that (8.2) fails for $y \in K$. We use this to construct a sequence of (N, ϵ)-separated sets $F_N \subset K$. Let $b \in (0, \dim_{\mathrm{box}}(P_y))$. For every $N > 0$ let $F_N \subset P_y$ be a maximal (finite) $\epsilon\lambda^{-N}$-separated set. By choice of b and the definition of box dimension in (7.2) there are infinitely many integers N with $|F_N| \geq \lambda^{bN}\epsilon^{-b}$.

We claim that F_N is an $(N, c_2\epsilon)$-separated set for a restricted to K. Let $gx, hx \in F_N$ be two different points with $g, h \in B_\epsilon^U$. By construction $\epsilon\lambda^{-N} \leq d(gx, hx) < 2\epsilon \leq r$. By Lemma 8.2 applied to $z = hx$ and $f = gh^{-1}$ there exists a non-negative $n < N$ with $d(a^n gx, a^n hx) \geq c_2\epsilon$. Therefore

F_N is $(N, c_2\epsilon)$-separated as claimed, and for infinitely many N we have $s_N(a) \geq |F_N| > \lambda^{bN}$. Finally, the definition of topological entropy in (7.5) implies that $\mathsf{h}_{\mathrm{top}}(\alpha^{\mathbf{t}}) \geq b \log \lambda > 0$. □

The remainder of this section is only needed for Theorem 1.6 and Theorem 10.2. For a compact set which is invariant in both directions we can also look at the stable and unstable subgroup simultaneously. Note however, that the set UV is not a subgroup of $\mathrm{SL}(k, \mathbb{R})$.

Lemma 8.4. *Let $K \subset X$ be compact with $aK = K$. Then $B_r^U B_r^V \subset B_{2r}^G$ and there exists $\lambda > 1$ and $c_3 \geq 1$ so that for any small enough $\epsilon > 0$, any $x \in X$ and $g, h \in B_r^U B_r^V$ with $hx \in K$, and any integer $N \geq 1$ with $d(gx, hx) \geq \lambda^{-N}\epsilon$, there exists an integer n with $d(a^n gx, a^n hx) \geq c_3\epsilon$ and $|n| < N$.*

Proof. Recall that we use the right invariant metric d to define the balls B_r^U, B_r^V, and B_{2r}^G. Therefore, if $g_U \in B_r^U$ and $g_V \in B_r^V$, then

$$d(g_U g_V, \mathrm{I}_k) \leq d(g_U g_V, g_V) + d(g_V, \mathrm{I}_k) = d(g_U, \mathrm{I}_k) + d(g_V, \mathrm{I}_k) < 2r$$

and so $B_r^U B_r^V \subset B_{2r}^G$.

If necessary we reduce the size of r such that (8.1) holds for every $g, h \in B_{3r}^G$. Assume ϵ is small enough so that $aB_\epsilon^G a^{-1} \subset B_{c_1^{-1} r}^G$. Let $\lambda > 1$ be such that $\|afa^{-1} - \mathrm{I}_k\| \geq \lambda \|f - \mathrm{I}_k\|$ for $f \in U$ and $\|a^{-1}fa - \mathrm{I}_k\| \geq \lambda \|f - \mathrm{I}_k\|$ for $f \in V$.

Let $g, h \in B_r^U B_r^V$ and $x \in X$ be as in the lemma. Define $f = gh^{-1}$, so that $d(f, \mathrm{I}_k) = d(g, h) \geq \lambda^{-N}\epsilon$. Write $f = f_C f_U f_V$ and $w = \max\bigl(d(f_U, \mathrm{I}_k), d(f_V, \mathrm{I}_k)\bigr)$. By (8.1)

$$\max\bigl(d(f_C, \mathrm{I}_k), w\bigr) \geq c_1^{-1} d(f, \mathrm{I}_k) \geq c_1^{-1}\lambda^{-N}\epsilon. \tag{8.3}$$

We need to rule out the case that $d(f_C, \mathrm{I}_k)$ is the only big term in this maximum. Clearly $f_C h = f_C h_U h_V$ and $g = g_U g_V$ are the correct decompositions in the sense of (8.1), and so $d(f_C, \mathrm{I}_k) \leq c_1 d(f_C h, g)$. By right invariance of the metric d we get $d(f_C h, g) = d(f_C, f)$ and again by (8.1) we get that $d(f_C, f) \leq c_1 w$. We conclude that $d(f_C, \mathrm{I}_k) \leq c_1^2 w$, which allows us to improve (8.3) to $w \geq c_1^{-3}\lambda^{-N}\epsilon$.

Depending on which term in $w = \max\bigl(d(f_U, \mathrm{I}_k), d(f_V, \mathrm{I}_k)\bigr)$ achieves the maximum, we find either a positive or a negative n with $|n| < N$ so that $\tilde{f} = a^n f a^{-n} = f_C \tilde{f}_U \tilde{f}_V$ satisfies $d(\tilde{f}, \mathrm{I}_k) \in (c_3\epsilon, r)$ for some absolute constant c_3. Since $hx, a^n hx \in K$, it follows that

$$d(a^n gx, a^n hx) = d(\tilde{f} a^n hx, a^n hx) = d(\tilde{f}, \mathrm{I}_k) \geq c_3\epsilon. \qquad \square$$

Lemma 8.5. *Let $a \in A$ and $K \subset X$ be compact with $aK = K$. Then one of the following properties holds.*

1. *The intersection $B_r^U B_r^V x \cap K$ has box dimension zero for every $x \in X$.*
2. *The restriction $a|_K$ of the multiplication operator a to K has positive topological entropy.*

Proof. Suppose that $\dim_{\text{box}}(B_r^U B_r^V x \cap K) > b > 0$ for some $x \in X$. By (7.2) there exists a $\lambda^{-N}\epsilon$-separated set F_N for $N \geq 1$, which satisfies $|F_N| \geq \lambda^{bN}\epsilon^{-b}$ for infinitely many N.

Let $gx, hx \in F_N$ with $g, h \in H$ and $g \neq h$. By Lemma 8.4 there exists an integer n with $|n| < N$ such that $d(a^n gx, a^n hx) \geq c_2\epsilon$. This shows that $a^{-N+1}F_N \subset K$ is $(2N-1, c_2\epsilon)$-separated with respect to a. It follows that $s_{2N-1}(a) \geq |F_N| \geq \lambda^{bN}\epsilon^{-b}$ for infinitely many N, and so $\mathsf{h}_{\text{top}}(a|_K) \geq \frac{1}{2}d\log\lambda > 0$. □

9. Upper semi-continuity of the metric entropy

For the construction of an A-ergodic measure μ as in Theorem 1.3 we need one more property of the metric entropy namely *upper semi-continuity* with respect to the measure. More specifically we consider the metric entropy $\mathsf{h}_\mu(a)$ as a function of the a-invariant measure μ, where we use the weak* topology on the space of probability measures supported on a fixed compact a-invariant set K. We will show that $\limsup_{\ell\to\infty}\mathsf{h}_{\mu_\ell}(a) \leq \mathsf{h}_\mu(a)$ whenever μ_ℓ is a sequence of a-invariant measures satisfying $\lim_{\ell\to\infty}\mu_\ell = \mu$. This is well known to hold for expansive maps [47, Thm. 8.2] and also for C^∞ automorphisms of compact manifolds [29, Thm. 4.1]. Strictly speaking neither of the two results applies to our case, the left multiplication by a is not expansive, $X = \mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$ is a noncompact manifold, and there is no reason why the compact subsets $K \subset X$ we study should be manifolds at all. However, the proof for the expansive case in [47, Sect. 8.1] can be adapted to our purposes – which we will do here for the sake of completeness. We will need a few more facts about entropy and conditional entropy, see [47, Ch. 4] and [32, Ch. 2 and 4].

Let μ be a probability measure on a compact metric space Y. Let $\mathcal{A} \subset \mathcal{B}_Y$ be a σ-algebra, which is countably generated by $A_1, \ldots, A_i, \ldots$. Then the atom of x is defined by

$$[x]_\mathcal{A} = \bigcap_{i:x\in A_i} A_i \cap \bigcap_{i:x\notin A_i} X \setminus A_i,$$

and the conditional measure $\mu_x^\mathcal{A}$ is a probability measure supported on $[x]_\mathcal{A}$ a.s. Let $\mathcal{P}$ be a finite partition. We will need the notion of conditional entropy

$$\mathsf{H}_\mu(\mathcal{P}|\mathcal{A}) = \int \mathsf{H}_{\mu_x^\mathcal{A}}(\mathcal{P})\,\mathrm{d}\mu$$

and the following basic properties.

For the trivial σ-algebra $\mathcal{N} = \{\varnothing, Y\}$ the conditional entropy equals the entropy $\mathsf{H}_\mu(\mathcal{P}|\mathcal{N}) = \mathsf{H}_\mu(\mathcal{P})$. For two partitions $\mathcal{P}$ and $\mathcal{Q}$ we have the addition formula

$$\mathsf{H}_\mu(\mathcal{P} \vee \mathcal{Q}|\mathcal{A}) = \mathsf{H}_\mu(\mathcal{P}|\mathcal{A}) + \mathsf{H}_\mu(\mathcal{Q}|\mathcal{P} \vee \mathcal{A}).$$

If $\mathcal{P}$ is finer than $\mathcal{Q}$ and $\mathcal{C} \subseteq \mathcal{A}$ is another countably generated σ-algebra, then

$$\mathsf{H}_\mu(\mathcal{Q}|\mathcal{A}) \leq \mathsf{H}_\mu(\mathcal{P}|\mathcal{C}).$$

Finally, the conditional entropy $\mathsf{H}_\mu(\mathcal{P}|\mathcal{A})$ vanishes if and only if there exists a nullset N such that $[x]_\mathcal{A} \setminus N$ is contained in one of the elements of $\mathcal{P}$ for every $x \in Y \setminus N$.

Suppose $T : Y \to Y$ is measure preserving and invertible. Then the metric entropy (7.6) of T with respect to a finite partition $\mathcal{Q}$ can also be written as

$$\mathsf{h}_\mu(T, \mathcal{Q}) = \mathsf{H}_\mu\Big(\mathcal{Q}| \bigvee_{n=1}^{\infty} T^{-n}\mathcal{Q}\Big),$$

see [47, Thm. 4.14].

We will also need the dynamical version of relative entropy. Suppose $\mathcal{A}$ is a countably generated σ-algebra that satisfies $T\mathcal{A} = \mathcal{A}$. We define

$$\mathsf{h}_\mu(T, \mathcal{Q}|\mathcal{A}) = \mathsf{H}_\mu\Big(\mathcal{Q}| \bigvee_{n=1}^{\infty} T^{-n}\mathcal{Q} \vee \mathcal{A}\Big), \tag{9.1}$$

then

$$\mathsf{h}_\mu(T, \mathcal{P} \vee \mathcal{Q}) = \mathsf{h}_\mu(T, \mathcal{P}) + \mathsf{h}_\mu\Big(T, \mathcal{Q}| \bigvee_{i=-\infty}^{\infty} T^i\mathcal{P}\Big). \tag{9.2}$$

The entropy with respect to an invariant measure is defined as a supremum over all finite partitions, see (7.6). For this reason the following general principle will be helpful.

Lemma 9.1. *Let $a \in A$ and $K \subset X$ be compact with $aK \subset K$. Let μ be an a-invariant measure supported on K. There exists a sequence of finite partitions $\mathcal{Q}_m$ of K which satisfies for all m that $\mathcal{Q}_{m+1}$ is finer than $\mathcal{Q}_m$. The boundaries of the elements of $\mathcal{Q}_m$ are μ-null sets, and the σ-algebra $\bigvee_{m=1}^{\infty} \mathcal{Q}_m$ equals the Borel σ-algebra $\mathcal{B}_K$ of K. Furthermore, $\mathsf{h}_\mu(a) = \lim_{m\to\infty} \mathsf{h}_\mu(a, \mathcal{Q}_m)$.*

Proof. Let $x \in X$ and define $f(y) = d(x, y)$. Then the measure $f_*\mu$ is a probability measure on $\mathbb{R}^+$, there exist arbitrarily small $\epsilon > 0$ such that $f_*\mu(\{\epsilon\}) = 0$, and so $\mu(\partial B_\epsilon(x)) = 0$.

For $m > 0$ we can cover K with finitely many ϵ-balls with $\epsilon < 1/m$ whose boundaries are null sets. Let $\mathcal{P}_m$ be the partition generated by these balls. For $P \in \mathcal{P}_m$ the boundary ∂P is contained in the union of the boundaries of the balls, thus it is a null set. To ensure that the sequence of partitions is getting finer we define $\mathcal{Q}_m = \bigvee_{i=1}^{m} \mathcal{P}_m$. It follows that every $Q \in \mathcal{Q}_m$ has a null set as boundary, and that $\mathcal{Q}_m \nearrow \mathcal{B}_K$ for $m \to \infty$.

The last statement follows from [32, Sect. 4, Thm. 3]. □

Proposition 9.2. *Let $a \in A$, and $K \subset X$ be compact with $aK \subset K$. For every finite partition $\mathcal{P}$ of K into measurable sets with small enough diameters and for any a-invariant measure μ supported on K we have $\mathsf{h}_\mu(a) = \mathsf{h}_\mu(a, \mathcal{P})$.*

Proof. Every T-invariant measure μ which is supported by K is in fact supported on the set $K' = \bigcap_{n\geq 0} a^n K$. Clearly, K' is compact and satisfies $aK' = K'$. Since a partition of K into small sets induces a partition of K' into small sets, we can assume without loss of generality that K satisfies $aK = K$.

Recall that there exists $r = r(K) > 0$ with $d(x, gx) = d(\mathrm{I}_k, g)$ whenever $d(\mathrm{I}_k, g) < r$ and $x \in K$. Let $\epsilon < r$ be as in Lemma 8.4, let $\mathcal{P}$ be a finite partition into measurable sets with diameter less than δ (to be specified later), and define the σ-algebra $\mathcal{A} = \bigvee_{i=-\infty}^{\infty} a^{-i}\mathcal{P}$ generated by the orbit of $\mathcal{P}$.

Let $x, y \in K$ belong to the same atom of $\mathcal{A}$, in other words suppose that for all $i \in \mathbb{Z}$ the images $a^i x, a^i y \in P_i$ belong to the same partition element of $\mathcal{P}$. We claim that (for small enough δ) this implies $x = f_C y$ for some small $f_C \in C$.

Let $x = fy$ with $f \in B_\delta^G$ and suppose $f \notin C$. Let $f = f_C f_U f_V$ with $f_C \in C$, $f_U \in U$, and $f_V \in V$. For small enough $\delta > 0$ we have $d(f_C, \mathrm{I}_k) < c_3\epsilon/2$, $d(f_U, \mathrm{I}_k) < r$ and $d(f_V, \mathrm{I}_k) < r$. Let $z = f_U f_V y = f_C^{-1} x$, then $z \in B_r^U B_r^V y$. Since $af_C = f_C a$ we have $d(a^n x, a^n z) = d(f_C, \mathrm{I}_k) < c_3\epsilon/2$ for all n. By Lemma 8.4 there exists some integer n with $d(a^n z, a^n y) \geq c_3\epsilon$. We assume $\delta < c_3\epsilon/2$, then

$$d(a^n x, a^n y) \geq d(a^n z, a^n y) - d(a^n x, a^n z) > c_3\epsilon/2$$

shows that $a^n x$ and $a^n y$ cannot belong to the same partition element of $\mathcal{P}$. This contradiction shows the claim.

Suppose $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ is one of the partitions of Lemma 9.1. We remove all the boundaries of the elements of the partition and obtain a partition modulo μ into open sets of small diameter.

By (9.2) we have

$$\mathsf{h}_\mu(T, \mathcal{Q}) \leq \mathsf{h}_\mu(T, \mathcal{P} \vee \mathcal{Q}) = \mathsf{h}_\mu(T, \mathcal{P}) + \mathsf{h}_\mu(T, \mathcal{Q}|\mathcal{A}),$$

where $\mathsf{h}_\mu(T, \mathcal{Q}|\mathcal{A})$ is the relative entropy as in (9.1). We will show that this last term vanishes, which together with Lemma 9.1 will conclude the proof of the proposition.

Let $B \subset X$ be measurable. By Poincaré recurrence, there exists a null set N such that every $x \notin N$ and $x \in B$ there exists some $n \geq 1$ with $a^n x \in B$. We apply this simultaneously to the countable family of sets

$$B_{i,j,g_C,\ell} = \{x : B_{1/\ell}(x) \subset Q_i \cap g_C^{-1} Q_j\}$$

for $g_C \in C \cap \mathrm{SL}(k, \mathbb{Q})$, $Q_i, Q_j \in \mathcal{Q}$ and $\ell \geq 1$. To show that the relative entropy

$$\mathsf{h}_\mu(T, Q|\mathcal{A}) = \mathsf{H}_\mu(\mathcal{Q}|\tilde{\mathcal{A}}) \text{ with } \tilde{\mathcal{A}} = \bigvee_{n=1}^{\infty} T^{-n}\mathcal{Q} \vee \mathcal{A} (\operatorname{mod} \mu)$$

vanishes, we have to show that for $x, y \notin N$ which are in the same atom with respect to $\tilde{\mathcal{A}}$ and satisfy $x \in Q_i \in \mathcal{Q}$ and $y \in Q_j \in \mathcal{Q}$ in fact $i = j$

holds. Since x and y belong to the same atom with respect to $\mathcal{A}$, we know from the above claim that $y = f_C x$ for some small $f_C \in C$. Therefore, $x \in Q_i \cap f_C^{-1} Q_j$ and there exists some rational g_C close to f_C with $x \in Q_i \cap g_C^{-1} Q_j$. Furthermore, we can ensure that $B_{1/\ell}(x) \subset Q_i \cap g_C^{-1} Q_j$, $d(g_C, f_C) < 1/\ell$, and $1/\ell < r$. It follows that $x \in B_{i,j,g_C,\ell}$. By construction of N there exists $n > 0$ with $a^n x \in B_{i,j,g_C,\ell}$. Therefore $a^n x \in Q_i$ and $B_{1/\ell}(a^n x) \subset g_C^{-1} Q_j$. From

$$d(g_C^{-1} f_C a^n x, a^n x) = d(g_C^{-1} f_C, \mathrm{I}_k) < 1/\ell < r$$

we see that $g_C^{-1} f_C a^n x \in g_C^{-1} Q_j$. Since a commutes with f_C, $f_C a^n x = a^n y \in Q_j$. We have shown that $x \in a^{-n} Q_i$ and $y \in a^{-n} Q_j$. Since $a^{-n} Q_i, a^{-n} Q_j$ belong to $\tilde{\mathcal{A}}$ and x, y are assumed to belong to the same atom with respect to $\tilde{\mathcal{A}}$, it follows that $i = j$ as claimed. □

The above proposition has the following important consequence.

Corollary 9.3. *Let $a \in A$ and $K \subset X$ be compact with $aK \subset K$. Then the metric entropy $\mathsf{h}_\mu(a|_K)$ is upper semi-continuous with respect to the measure μ, i.e. for every a-invariant μ and every $\epsilon > 0$ there is a neighborhood U of μ in the weak* topology of probability measures on K such that $\mathsf{h}_\nu(a) \leq \mathsf{h}_\mu(a) + \epsilon$ for every a-invariant $\nu \in U$.*

Proof. As in the proof of Lemma 9.1 we can find a partition $\mathcal{P}$ of K whose elements have small enough diameter to satisfy Proposition 9.2 and whose boundaries are null sets with respect to μ. Therefore $\mathsf{h}_\nu(a) = \mathsf{h}_\nu(a, \mathcal{P})$ for every a-invariant measure ν supported on K. Let $\epsilon > 0$. By the definition of entropy there exists $N \geq 1$ with

$$\frac{1}{N} \mathsf{H}_\mu\Big(\bigvee_{n=0}^{N-1} a^{-n} \mathcal{P} \Big) < \mathsf{h}_\mu(a, \mathcal{P}) + \epsilon/2.$$

Since the sets in the partition $\mathcal{Q} = \bigvee_{n=0}^{N-1} a^{-n} \mathcal{P}$ all have boundaries which are null sets with respect to μ, there exists a weak* neighborhood U of μ such that $\nu(Q)$ is very close to $\mu(Q)$ for every $Q \in \mathcal{Q}$. The entropy of the partition $\mathcal{Q}$ depends only on the measures of the elements of $\mathcal{Q}$, therefore we can make sure that

$$\frac{1}{N} \big| \mathsf{H}_\nu(\mathcal{Q}) - \mathsf{H}_\mu(\mathcal{Q}) \big| < \epsilon/2.$$

For any a-invariant $\nu \in U$ it follows that

$$\mathsf{h}_\nu(a) = \mathsf{h}_\nu(a, \mathcal{P}) \leq \frac{1}{N} \mathsf{H}_\nu(\mathcal{Q}) \leq \frac{1}{N} \mathsf{H}_\mu(\mathcal{Q}) + \epsilon/2 < \mathsf{h}_\mu(a, \mathcal{P}) + \epsilon,$$

where we used Proposition 9.2 for ν and μ, and furthermore that $\mathsf{h}_\nu(a, \mathcal{P})$ is the infimum over $\frac{1}{M} \mathsf{H}_\nu(\bigvee_{n=0}^{M-1} a^n \mathcal{P})$ by subadditivity [47, Thm. 4.10]. □

10. Transversal Hausdorff dimension for the set of points with bounded orbits

In this section we apply Theorem 1.3 to prove two theorems about sets with bounded orbits.

For a unimodular lattice $\Lambda \subset \mathbb{R}^k$ we define

$$\delta_{\mathbb{R}^k}(\Lambda) = \min_{\mathbf{y} \in \Lambda \setminus \{0\}} \|y\|.$$

Clearly, every point $x = m\,\mathrm{SL}(k,\mathbb{Z})$ with $m \in \mathrm{SL}(k,\mathbb{R})$ can be identified with the unimodular lattice generated by the columns of m. By this identification $\delta_{\mathbb{R}^k}$ becomes a positive continuous function on X with the property that the preimages $K_\rho = \delta_{\mathbb{R}^k}^{-1}[\rho,\infty)$ are compact sets for every $\rho > 0$ by Mahler's criterion. In other words $B \subset X$ is bounded if and only if $\inf_{x\in B} \delta_{\mathbb{R}^k}(x) > 0$.

A nonempty subset $\Sigma' \subset \Sigma$ is a cone if Σ' is convex and satisfies $r\mathbf{t} \in \Sigma'$ whenever $r > 0$ and $\mathbf{t} \in \Sigma'$.

Theorem 10.1. *Let $X = \mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$ with $k \geq 3$, and let Σ' be an open cone in Σ. Define*

$$D = \{x \in X : \inf_{\mathbf{t}\in\Sigma'} \delta_{\mathbb{R}^k}(\alpha^{\mathbf{t}} x) > 0\}$$

to be the set of points with bounded Σ'-orbits. Then for every $\mathbf{t} \in \Sigma'$ and $x \in X$ the $\alpha^{\mathbf{t}}$-unstable manifold Ux through x intersects D in a set $D \cap Ux$ of Hausdorff dimension zero. In fact, $D \cap Ux$ is a countable union of sets with upper box dimension zero.

Proof. For $\rho > 0$ we define the compact set

$$D_\rho = \{x \in X : \inf_{\mathbf{t}\in\Sigma'} \delta_{\mathbb{R}^k}(\alpha^{\mathbf{t}} x) \geq \rho\}. \tag{10.1}$$

Clearly $D = \bigcup_{n=1}^\infty D_{1/n}$. Let $\mathbf{t} \in \Sigma'$, $a = \alpha^{\mathbf{t}}$, and $x \in X$. Then $aD_\rho \subset D_\rho$. By Proposition 8.3 there are two possibilities; $D_\rho \cap Ux$ is a countable union of compact sets of box dimension zero, or $a|_{D_\rho}$ has positive topological entropy. If the first possibility takes place for all $\rho > 0$, the theorem follows from (7.3) and (7.4).

We will show that the second possibility cannot happen ever. Suppose $a|_{D_\rho}$ has positive topological entropy. By the variational principle (Section 7.2 and [47, Thm. 8.6]) there exists an a-invariant measure ν supported on D_ρ with positive metric entropy $h_\nu(a) > 0$. However, we need to find an A-ergodic measure with this property in order to get a contradiction to Theorem 1.3.

Since $\Sigma' \subseteq \Sigma$ is open we can find a basis $\mathbf{t}_1, \ldots, \mathbf{t}_{k-1} \in \Sigma'$ of Σ. By construction $K = D_\rho$ is compact and satisfies $\alpha^{\mathbf{s}} K \subset K$ for all $\mathbf{s} \in \mathbb{R}^+\mathbf{t}_1 + \cdots + \mathbb{R}^+\mathbf{t}_{k-1}$. For $N > 0$ the measure

$$\nu_N = \frac{1}{N^{k-1}} \int_0^N \cdots \int_0^N (\alpha^{s_1\mathbf{t}_1 + \cdots s_{k-1}\mathbf{t}_{k-1}})_* \nu \, \mathrm{d}s_1 \cdots \mathrm{d}s_{k-1}$$

is supported on K and a-invariant. Since entropy is affine [47, Thm. 8.1] and upper semi-continuous by Corollary 9.3 with respect to the measure, entropy with respect to a generalized convex combination of measures is the integral of the entropies. In particular $\mathsf{h}_{\nu_N}(T) = \mathsf{h}_\nu(T)$.

Let μ be a weak* limit of a subsequence of ν_N. From the definition of ν_N it follows that μ is A-invariant. It is also clear that μ is supported on K. From upper semi-continuity it follows that the entropy $\mathsf{h}_\mu(a) \geq \mathsf{h}_\nu(a) > 0$ is positive. The ergodic decomposition (4.2) of μ writes μ as a generalized convex combination of A-ergodic measures μ_τ, which have almost surely support contained in K. Since $\mathsf{h}_\mu(T) > 0$, there exists some A-ergodic measure μ_τ with $\mathsf{h}_{\mu_\tau}(T) > 0$ and support in K. This contradicts Theorem 1.3 and concludes the proof of Theorem 10.1. □

Let $D \subseteq X$ be A-invariant. We say D has *transversal box dimension zero* if $\{g \in B_r^G : gx \in D\}$ and $g_{ii} = 1$ for $i = 1, \dots, k$ has box dimension zero for all $x \in D$. (Note that the particular shape of the set used here does not matter as long as this set is still transversal to the subgroup A.) It is easy to check, that an A-invariant set D with transversal box dimension zero has box dimension $k-1$ (unless D is empty).

Theorem 10.2. *Let $X = \mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$ with $k \geq 3$, let $A \subset \mathrm{SL}(k,\mathbb{R})$ be the subgroup of positive diagonal matrices. Define*

$$D = \{x \in X : \inf_{a \in A} \delta_{\mathbb{R}^k}(ax) > 0\}$$

to be the set of points with bounded A-orbits. Then D is a countable union of sets with transversal box dimension zero and has Hausdorff dimension $k-1$.

Clearly D is A-invariant, and non-empty since it contains every periodic A-orbit.

Proof. As before we define the A-invariant compact sets D_ρ as in (10.1) with $\Sigma' = \Sigma$. Pick an element $a = \alpha^{\mathbf{t}} \in A$ with $\mathbf{t} \in \Sigma$, $t_i \neq t_j$ for $i \neq j$. Then the corresponding central subgroup equals $C = A$ and $B_r^U B_r^V$ is transversal to A. Let $x \in X$ and $\rho > 0$.

We give some conditions on $r > 0$. Our first restriction is that $B_{3r}(x)$ and B_{3r}^G should be isometric. Let $O = B_r^A \times B_r^U \times B_r^V$ and use the metric

$$d_O\big((f_C, f_U, f_V), (g_C, g_U, g_V)\big) = \max\big(d(f_C, g_C), d(f_U, g_U), d(f_V, g_V)\big).$$

Furthermore, define $\psi : O \to \mathrm{SL}(k,\mathbb{R})$ by $\psi(f_C, f_U, f_V) = f_C f_U f_V$ and assume ψ is invertible and Lipschitz in both directions (as in (8.1)).

Let $P = \{(f_C, f_U, f_V) \in O : \psi(f_C, f_U, f_V)x \in D_\rho\}$. Since D_ρ is A-invariant, the set $P' = P \cap (\{\mathrm{I}_k\} \times B_r^U \times B_r^V)$ determines $P = \{(f_C, f_U, f_V) \in O : (\mathrm{I}_k, f_U, f_V) \in P'\}$. Clearly $\psi(P')x = (B_r^U B_r^V x) \cap D_\rho$. By Lemma 8.5 there are two possibilities; P' has box dimension zero or a has positive topological entropy when restricted to D_ρ. As in the proof of Theorem

10.1 the latter contradicts Theorem 1.3. Therefore D_ρ has transversal box dimension zero. □

11. The set of exceptions to Littlewood's Conjecture

For any $u, v \in \mathbb{R}$, define $\tau_{u,v}$ to be the point

$$\tau_{u,v} = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ v & 0 & 1 \end{pmatrix} \mathrm{SL}(3,\mathbb{Z});$$

in other words, $\tau_{u,v}$ is the point in X corresponding to the lattice in $\mathbb{R}^3$ generated by $(1,u,v),(0,1,0)$, and $(0,0,1)$. The following well-known proposition gives the reduction of Littlewood's conjecture to the dynamical question which we studied in Section 10, see also [24, Sect. 2] and [45, Sect. 30.3]. We include the proof for completeness.

Proposition 11.1. *The tuple (u, v) satisfy*

$$\liminf_{n\to\infty} n\langle nu\rangle\langle nv\rangle = 0, \tag{11.1}$$

if and only if the orbit $A^+\tau_{u,v}$ is unbounded where A^+ is the semigroup

$$A^+ = \left\{ \begin{pmatrix} e^{-r-s} & & \\ & e^r & \\ & & e^s \end{pmatrix} : \text{ for } r, s \in \mathbb{R}^+ \right\}.$$

Proof. By the the properties of $\delta_{\mathbb{R}^k}$ we have to show for $(u, v) \in \mathbb{R}^2$ that (11.1) holds if and only if $\inf_{a\in A^+} \delta_{\mathbb{R}^k}(a\tau_{u,v}) = 0$.

Suppose $\epsilon > 0$ and there exists $a \in A^+$ with $\delta_{\mathbb{R}^k}(a\tau_{u,v}) < \epsilon$. Then

$$a\tau_{u,v} = \begin{pmatrix} e^{-r-s} & 0 & 0 \\ e^r u & e^r & 0 \\ e^s v & 0 & e^s \end{pmatrix} \mathrm{SL}(3,\mathbb{Z})$$

and by definition of $\delta_{\mathbb{R}^k}$ there exists nonzero $(n, m_1, m_2) \in \mathbb{Z}^3$ with

$$\left\| \begin{pmatrix} ne^{-r-s} \\ ne^r u + m_1 e^r \\ ne^s v + m_2 e^s \end{pmatrix} \right\| < \epsilon.$$

Taking the product of all three entries of this vector we find that

$$|ne^{-r-s}(ne^r u + m_1 e^r)(ne^s v + m_2 e^s)| = |n(nu + m_1)(nv + m_2)| < c\epsilon^3$$

is small (c depends only on the norm used in $\mathbb{R}^3$), and so (11.1) follows. Note that $n \neq 0$ since otherwise the lower two entries in the vector cannot be small.

Suppose now (11.1) holds for (u, v). Let $\epsilon > 0$ and find $n > 0$ and $(m_1, m_2) \in \mathbb{Z}^2$ with $|n(nu + m_1)(nv + m_2)| < \epsilon^5$. We would like to have additionally that

$$\max(|nu + m_1|, |nv + m_2|) < \epsilon. \tag{11.2}$$

Suppose this is not true, and assume without loss of generality that $|nv + m_2| \geq \epsilon$ and $|n(nu + m_1)| < \epsilon^4$. Then by Dirichlet's theorem there exists an integer $q < 1/\epsilon$ so that $\langle qnv \rangle < \epsilon$. It follows that $|qn(qnu + qm_1)| < \epsilon^2$, and $|qnv + m_2'| < \epsilon$ for some $m_2' \in \mathbb{Z}$. In other words when we replace n by nq and m_1, m_2 by qm_1 and m_2' respectively, we see that (11.2) and $|n(nu + m_1)(nv + m_2)| < \epsilon^3$ hold simultaneously. Therefore we can find $r > 0$ and $s > 0$ with $e^r|nu + m_1| = \epsilon$ and $e^s|nv + m_2| = \epsilon$. (If one of the expressions vanishes, we use some large r resp. s instead.) Then $e^{-r-s}n < \epsilon$ and $\delta_{\mathbb{R}^k}(a\tau_{u,v}) < c\epsilon$ follows. □

Proof of Theorem 1.5. By Proposition 11.1 the set Ξ is embedded by the map $(u, v) \mapsto \tau_{u,v}$ to the set D with A^+-bounded orbits. We apply Theorem 10.1 with $\Sigma' = \{(-r - s, r, s) : r, s > 0\}$. Therefore D intersects every unstable manifold of $\alpha^{\mathbf{t}}$ in a set of Hausdorff dimension zero where $\mathbf{t} = (-2, 1, 1)$. Note that the unstable manifold of $\alpha^{\mathbf{t}}$ through $\mathrm{I}_k \,\mathrm{SL}(3, \mathbb{Z})$ is the image of τ. It follows that Ξ has Hausdorff dimension zero, and similarly that Ξ is a countable union of sets with box dimension zero. □

Proof of Theorem 1.6. We apply Theorem 10.2 and set $\Xi_k = D$. Suppose $m \notin \Xi_k$, then $\delta_{\mathbb{R}^k}(am) < \epsilon$ for some $a \in A$. By definition of $\delta_{\mathbb{R}^k}$ there exists some $\mathbf{n} \in \mathbb{Z}^k$ such that $\left\| \begin{pmatrix} a_{11} m_1(\mathbf{n}) \\ \vdots \\ a_{kk} m_k(\mathbf{n}) \end{pmatrix} \right\| < \epsilon$ and (1.3) follows. □

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Manfred Einsiedler, Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195
E-mail address: einsiedl@math.washington.edu

Anatole Katok, Department of Mathematics, Penn State University, State College, PA 16802
E-mail address: katok_a@math.psu.edu

Elon Lindenstrauss, Department of Mathematics, Stanford University, Stanford, CA 94305 *and* Courant Institute of Mathematical Sciences, 251 Mercer St., New York, NY 10012
Current address: Department of Mathematics, Princeton University, Princeton, NJ 08544
E-mail address: elonl@princeton.edu