RAMSEY PROPERTIES OF RANDOM DISCRETE STRUCTURES

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Abstract. We study thresholds for Ramsey properties of random discrete structures. In particular, we determine the threshold for Rado's theorem for solutions of partition regular systems of equations in random subsets of the integers and we prove the 1-statement of the conjectured threshold for Ramsey's theorem for random hypergraphs. Those results were conjectured by Rödl and Ruciński and similar results were obtained independently by Conlon and Gowers.

1. Introduction

Ramsey theory is an important branch of combinatorics. Roughly speaking, a Ramsey type result asserts for some given configuration F and some integer r the existence of a configuration G such that any partition (or coloring) of G into r classes has the property that a copy of F is completely contained in one of the r partition classes. For example, one of the first results of this type can be found in the work of Hilbert [14], where it was shown that for every ℓ and for every finite partition of the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ there exists a partition class which contains an affine cube of dimension ℓ, i.e., a set of the form $\{x_0 + \sum_{i=1}^{\ell} \varepsilon_i x_i \colon \varepsilon_i \in \{0, 1\}\}$ for some $x_0, x_1, \dots, x_\ell \in \mathbb{N}$. Classical results of that type include the work of Schur [33], van der Waerden [36], Rado [24], Ramsey [25], Erdős and Szekeres [3], Hales and Jewett [13], Graham, Leeb, and Rothschild [11], and others (see the, e.g., [12] for more details).

Applications of probabilistic arguments to obtain bounds in Ramsey theory have a long tradition. On the other hand, the study of Ramsey type properties of random structures was initiated only more recently by Łuczak, Ruciński, and Voigt [19] and further studied by Rödl and Ruciński with their collaborators [7, 9, 10, 26, 27, 28, 29, 30, 31] (for more related results by others see [8, 17, 18, 20, 21, 22, 23]). The aim of this paper is to establish a general result which yields Ramsey type results for random discrete structures (see Theorem 2.5). As a consequence, combined with the work from [29] we establish the threshold for Rado's theorem for random subsets of the integers (see Theorem 1.1) and we obtain the 1-statement for the conjectured threshold of Ramsey's theorem for random hypergraphs (see Theorem 1.2). Similar results were obtained independently by Conlon and Gowers [2].

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1.1. **Random subsets of the integers.** Ramsey type results for the integers embody the following pattern. For every finite coloring of $\mathbb{N}$ there exist integers $x_1, \dots, x_k$ all of the same color, which satisfy some prescribed condition. For the condition $x_1 + x_2 = x_3$ such a result was proved by Schur [33] and for $x_1, \dots, x_k$ forming an arithmetic progression of length k this is the result of van der Waerden [36]. In 1933 Rado [25] published a far-reaching generalization of these results. For an $\ell \times k$ matrix $A = (a_{ij})$ of integers consider the system $\mathcal{L}(A)$ of homogeneous linear equations

$$\sum_{j=1}^{k} a_{ij} x_j = 0 \quad \text{for } 1 \le i \le \ell\,.$$

We say that a matrix A is *partition regular* if for any finite coloring of $\mathbb{N}$ there is always a solution $(x_1, \dots, x_k)$ of $\mathcal{L}(A)$ with all x_i having the same color. Rado characterized partition regular matrices and it follows directly from that characterization that $k \ge \operatorname{rank}(A) + 2$ is a necessary condition (see, e.g., [12] for details). We note that the single equation $x_1 + x_2 - x_3 = 0$ is partition regular due to Schur's theorem while the same follows for $x_1 + x_2 - 2x_3 = 0$ by van der Waerden's theorem. On the other hand, the equation $x_1 + x_2 - 3x_3 = 0$ fails to have that property.

We say a partition regular matrix A is *irredundant* if there exists a solution $(x_1, \dots, x_k)$ of $\mathcal{L}(A)$ such that $x_i \ne x_j$ for all $1 \le i < j \le k$ and otherwise we say A is redundant. It is easy to show that for every redundant $\ell \times k$ matrix A there exists an irredundant $\ell' \times k'$ matrix A' for some $\ell' < \ell$ and $k' < k$ with the same family of solutions (viewed as sets). More precisely, $(y_1, \dots, y_{k'})$ is a solution of $\mathcal{L}(A')$ if and only if there exists a solution $(x_1, \dots, x_k)$ for $\mathcal{L}(A)$ with

$$\{x_1, \dots, x_k\} = \{y_1, \dots, y_{k'}\}$$

(see, e.g., [29, Section 1] for details). Due to this consideration it is natural to restrict to irredundant, partition regular matrices A.

We denote by $[n] = \{1, \dots, n\}$ the first n positive integers and for a subset $Z \subseteq [n]$, a positive integer $r \in \mathbb{N}$, and an irredundant, $\ell \times k$ integer matrix A we write

$$Z \to (A)_r \tag{1}$$

if for every coloring of Z with r colors, there exists a solution $(x_1, \dots, x_k)$ of $\mathcal{L}(A)$ such that all x_i are distinct and contained in Z and have the same color. A standard compactness argument combined with Rado's theorem yields that for any $r \in \mathbb{N}$ and every

partition regular matrix A we have $[n] \to (A)_r$ for every $n \ge n(A, r)$ sufficiently large. Our first main result determines the density required by random subsets of $[n]$ to satisfy the same property.

For $p \in (0, 1]$ let $[n]_p$ denote the binomial random subset of $[n]$ with integers from $[n]$ included independently, each with probability p. In other words, we consider the finite probability space on all subsets of $[n]$, where

$$\mathbb{P}\left([n]_p = Z\right) = p^{|Z|}(1-p)^{n-|Z|}$$

holds for all $Z \subseteq [n]$. In [10, 28, 29] the question when $[n]_p \to (A)_r$ holds with probability close to 1 was investigated. To characterize the sequences of probabilities $\boldsymbol{p} = (p_n)_{n \in \mathbb{N}}$ with that property we consider the following parameter introduced in [29].

Let A be an $\ell \times k$ integer matrix and let the columns be indexed by $[k]$. For a partition $W\dot\cup\overline{W} \subseteq [k]$ of the columns of A, we denote by $A_{\overline{W}}$ the matrix obtained from A by restricting to the columns indexed by $\overline{W}$. Let $\operatorname{rank}(A_{\overline{W}})$ be the rank of $A_{\overline{W}}$, where $\operatorname{rank}(A_{\overline{W}}) = 0$ for $\overline{W} = \emptyset$. We set

$$m_A = \max_{\substack{W\dot\cup\overline{W}=[k]\\ |W|\geq 2}} \frac{|W|-1}{|W|-1+\operatorname{rank}(A_{\overline{W}})-\operatorname{rank}(A)}\,. \tag{2}$$

It was shown in [29, Proposition 2.2 ii)] that for irredundant, partition regular matrices A the denominator of (2) is always at least 1.

For example, if A consists of the single equation $x_1 + x_2 - x_3 = 0$ considered by Schur, then $m_A = 2$. Moreover, if A corresponds to an irredundant, partition regular matrix with the property that the solutions of $\mathcal{L}(A)$ form an arithmetic progression of length k, then $m_A = k - 1$.

One of the main results in [29] asserts that for every irredundant, partition regular matrix A there exists some $c > 0$ such that if $\boldsymbol{p} = (p_n)$ satisfies $p_n \leq cn^{-1/m_A}$, then

$$\lim_{n\to\infty} \mathbb{P}\left([n]_{p_n} \to (A)_2\right) = 0\,. \tag{3}$$

Note that by definition $\mathbb{P}\left([n]_{p_n} \to (A)_r\right) \leq \mathbb{P}\left([n]_{p_n} \to (A)_2\right)$ for every $r \geq 2$. Moreover, extending a result from [28] in [29] the complementing result for $p \gg n^{-1/m_A}$ was obtained for a special subclass of partition regular matrices, which we consider below.

We say an irredundant, partition regular $\ell \times k$ matrix A is *density regular* if any subset $Z \subseteq \mathbb{N}$ with positive upper density, i.e.,

$$\limsup_{n\to\infty} \frac{|Z \cap [n]|}{n} > 0\,,$$

contains a solution $(x_1, \dots, x_k)$ of $\mathcal{L}(A)$ with all x_i distinct. For example, Szemerédi's famous theorem on arithmetic progressions [34] shows that if the solutions of $\mathcal{L}(A)$ form an arithmetic progression, then A is density regular. More generally, it was shown in [4] that an irredundant, partition regular matrix is density regular if and only if $(1, \dots, 1)$ is a solution of $\mathcal{L}(A)$.

Complementing (3), Rödl and Ruciński showed in [29] that for every irredundant, density regular matrix A and every integer $r \geq 2$ there exists $C > 0$ such that if $\boldsymbol{p} = (p_n)$ satisfies $p_n \geq Cn^{-1/m_A}$, then

$$\lim_{n\to\infty} \mathbb{P}\left([n]_{p_n} \to (A)_r\right) = 1\,. \tag{4}$$

For the special case, when solutions of $\mathcal{L}(A)$ form an arithmetic progression the same result appeared already in [28]. In other words, combining (3) and (4) it follows that $p_n = n^{-1/m_A}$ is the *threshold* for the property $[n]_{p_n} \to (A)_r$ for irredundant, density regular matrices A. It was conjectured in [29] that this extends to all irredundant, partition regular matrices A. For the special case, when A consists only of the equation $x_1 + x_2 - x_3 = 0$ (considered by Schur) and $r = 2$ this was verified in [10]. Our first main result addresses the general case.

Theorem 1.1. *Let A be an irredundant, partition regular integer matrix and let $r \in \mathbb{N}$. There exist constants $0 < c < C$ such that for any sequence of probabilities*

$\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ *we have*

$$\lim_{n\to\infty} \mathbb{P}\left([n]_{p_n} \to (A)_r\right) = \begin{cases} 1, & \text{if } p_n \geq Cn^{-1/m_A} \\ 0, & \text{if } p_n \leq cn^{-1/m_A}\,. \end{cases}$$

Due to (3) it suffices to show the 1-statement in the theorem above. This statement will follow from a more general result presented in Section 2. We deduce the 1-statement of Theorem 1.1 in Section 3.

1.2. **Ramsey properties for random hypergraphs.** The second main result concerns partition properties of random hypergraphs. An ℓ-uniform hypergraph H is a pair (V, E), where the vertex set V is some finite set and the edge set $E \subseteq [V]^\ell$ is a subfamily of the ℓ-element subsets of V. As usual we call 2-uniform hypergraphs simply graphs. For some hypergraph H we denote by $V(H)$ and $E(H)$ its vertex set and its edge set and we denote by $v(H)$ and $e(H)$ the cardinalities of those sets. For an integer n we denote by $K_n^{(\ell)}$ the complete ℓ-uniform hypergraph on n vertices, i.e., $v(K_n^{(\ell)}) = n$ and $e(K_n^{(\ell)}) = \binom{n}{\ell}$. For a subset $U \subseteq V(H)$ we denote by $E(U)$ the edges of H contained in U and we set $e(U) = |E(U)|$. Moreover, we write $H[U]$ for the subhypergraph induced on U, i.e., $H[U] = (U, E(U))$.

Ramsey's theorem [25] asserts that for every ℓ-uniform hypergraph F and every $r \in \mathbb{N}$ there exists some $n(F, r)$ such that for every $n \geq n(F, r)$ we have

$$K_n^{(\ell)} \to (F)_r\,,$$

i.e., every r-coloring of the edges of $K_n^{(\ell)}$ yields a monochromatic copy of F. More generally, for ℓ-uniform hypergraphs F and G and $r \in \mathbb{N}$ we write $G \to (F)_r$, if for every partition $E^1\dot\cup\ldots\dot\cup E^r = E(G)$ there exists some $s \in [r]$ and an injective mapping $\varphi\colon\ V(F) \to V(G)$ such that $\varphi(e) \in E^s$ for every $e \in E(F)$.

Similarly as in the context of Rado's theorem we are interested in random versions of Ramsey's theorem. Here we study the binomial model $G^{(\ell)}(n, p)$ of ℓ-uniform hypergraphs, where edges of the complete hypergraph $K_n^{(\ell)}$ are included independently with probability p. More precisely, we consider the finite probability space with ground set $E(K_n^{(\ell)})$ where for any ℓ-uniform hypergraph H with vertex set $V(K_n^{(\ell)})$ we have

$$\mathbb{P}\left(G^{(\ell)}(n, p) = H\right) = p^{e(H)}(1-p)^{\binom{n}{\ell}-e(H)}\,.$$

For a fixed ℓ-uniform hypergraph F and $r \in \mathbb{N}$ we are interested in the asymptotic growth of the smallest sequence of probabilities $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ such that $G^{(\ell)}(n, p_n) \to (F)_r$ holds *asymptotically, almost surely (a.a.s.)*, i.e.,

$$\lim_{n\to\infty} \mathbb{P}\left(G^{(\ell)}(n, p_n) \to (F)_r\right) = 1\,. \tag{5}$$

This question was first studied in [19] and there it was shown that (5) holds for $F = K_3$ being a graph triangle, $r = 2$, and $p = p_n \gg 1/\sqrt{n}$ (as noted in [19] this also follows implicitly from an earlier result in [5]). The result from [19] was generalized for the same condition on p to an arbitrary number of colors by Rödl and Ruciński in [27]. Finally in [28] Rödl and Ruciński solved the problem for arbitrary graphs F and any number of colors $r \in \mathbb{N}$ by showing that (5) is valid as long as $p \geq Cn^{-1/m_F}$ for some $C = C(F, r)$, where (in general for an ℓ-uniform hypergraph F with $e(F) \geq 1$) we set

$$m_F = \max_{\substack{F' \subseteq F \\ e(F') \geq 1}} d(F') \quad \text{with} \quad d(F') = \begin{cases} \frac{e(F')-1}{v(F')-\ell}\,, & \text{if } v(F') > \ell \\ 1/\ell\,, & \text{if } v(F') = \ell\,. \end{cases} \tag{6}$$

If follows from the definition of m_F, that $p = \Omega(n^{-1/m(F)})$ then a.a.s. the number of copies of every subhypergraph graph $F' \subseteq F$ in the random hypergraph $G^{(\ell)}(n, p)$ has at least the same order of magnitude, as the number of edges. This property seems to be a necessary condition for (5) to hold. This belief was indeed verified for graphs in [26], where it was shown that for "most" graphs F there exists some $c > 0$ such that for any $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ with $p_n \leq cn^{-1/m_F}$ we have

$$\lim_{n\to\infty} \mathbb{P}\left(G^{(2)}(n, p_n) \to (F)_2\right) = 0$$

Here "most" means all graphs F with the exception of forests consisting of stars and paths of length three, which show a slightly different behavior (see [16, Chapter 8] for details).

Our second main result, Theorem 1.2, establishes the general result for ℓ-uniform hypergraphs. We believe that the matching 0-statement also holds for "most" hypergraphs F, but we will not study this here.

Theorem 1.2. *Let F be an ℓ-uniform hypergraph with maximum degree at least 2 and let $r \in \mathbb{N}$. There exists a constant $C > 0$ such that for any sequence of probabilities $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ satisfying $p_n \geq Cn^{-1/m_F}$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left(G^{(\ell)}(n, p_n) \to (F)_r\right) = 1\,.$$

Theorem 1.2 was conjectured by Rödl and Ruciński [30, Conjecture 1.23]. In [30] and in [31] such a result was already established for the special cases when $F = K_4^{(3)}$ and for ℓ-partite, ℓ-uniform hypergraphs F. Theorem 1.2 follows from the more general result presented in Section 2 and we present the reduction in Section 3.

2. Main technical result

In this section we introduce a general environment allowing to prove Theorem 1.1 and Theorem 1.2 along the same lines. We note that the earlier results of Rödl and Ruciński in [28, 30] were based on applications of the regularity lemma for graphs and 3-uniform hypergraphs [35, 6]. Due to the somewhat technical nature of the regularity lemma for hypergraphs, proving even special cases of Theorem 1.2 presented several technical difficulties. Although the approach taken here uses some ideas from [28], we will, similarly as in [31], avoid the use of the regularity lemma. The approach here can be viewed as a refinement of the work in [32], where related extremal and Turán-type problems for random subsets of the integers and random hypergraphs were studied.

2.1. **Statement of main result.** It will be convenient to consider sequences of k-uniform hypergraphs $\boldsymbol{H} = (H_n)_{n\in\mathbb{N}}$. In the context of Theorem 1.1 for a given irredundant, partition regular $\ell \times k$ matrix, one may think of the vertex set $V(H_n)$ to be $[n]$ and the edges being the solutions $(x_1, \ldots, x_k)$ of $\mathcal{L}(A)$ with $x_i \neq x_j$ for $1 \leq i < j \leq k$. In view of Theorem 1.2, for a given ℓ-uniform hypergraph F with k edges we may think of $V(H_n)$ being the edge set of $K_n^{(\ell)}$ and every edge of

$E(H_n)$ corresponds to a copy of F in $K_n^{(\ell)}$. The two main assumptions allowing to apply the main result, Theorem 2.5, are (r,ζ)-Ramseyness (cf. Definition 2.1) and $(K,\boldsymbol{p})$-boundedness (cf. Definition 2.3). Roughly speaking, $\boldsymbol{H}$ will be (r,ζ)-Ramsey, if a quantitative Ramsey-type result for the original structure holds. For Rado's theorem such a strengthening was deduced from Deuber's theorem in [4] and for Ramsey's theorem it follows directly from Ramsey's original argument. The $(K,\boldsymbol{p})$-boundedness will impose a lower bound on $\boldsymbol{p}$ and we will verify this condition for Theorem 1.1 and Theorem 1.2 in Section 3.

Definition 2.1. *Let $H=(V,E)$ be a k-uniform hypergraph and $r\in\mathbb{N}$. We say H is r-Ramsey, if for every partition $V^1\dot\cup\dots\dot\cup V^r$ of V there exists an $s\in[r]$ such that $e(V^s)\neq 0$.*

For a subset $U\subseteq V$ and $\zeta>0$, we say the induced subhypergraph $H[U]$ is (r,ζ)-Ramsey, if for every partition $U^1\dot\cup\dots\dot\cup U^r$ of U there exists an $s\in[r]$ such that $e(U^s)\geq\zeta|E|$.

Moreover, for a sequence $\boldsymbol{H}=(H_n)_{n\in\mathbb{N}}$ of k-uniform hypergraphs, we say $\boldsymbol{H}$ is (r,ζ)-Ramsey, if for all but finitely many n the hypergraph H_n is (r,ζ)-Ramsey.

We note that in the definition of (r,ζ)-Ramseyness the number of required monochromatic edges is given in terms of the global number $e(H)$ of the edges of H and not in terms of $e(U)$. The next observation follows directly from the Definition 2.1.

Fact 2.2. *Let $r_1\dots,r_\ell$ be positive integers, let $\zeta>0$, let $H=(V,E)$ be a k-uniform hypergraph, and let $U^1\dot\cup\dots\dot\cup U^\ell$ be a partition of $U\subseteq V$. If $H[U]$ is $(\sum_{j=1}^{\ell} r_j,\zeta)$-Ramsey, then there exists an $j\in[\ell]$ such that $H[U^j]$ is (r_j,ζ)-Ramsey.* □

For a k-uniform hypergraph $H=(V,E)$, $i\in[k-1]$, $v\in V$, and $U\subseteq V$ we denote by $\deg_i(v,U)$ the number of edges of H containing v and having at least i vertices in $U\setminus\{v\}$, i.e.,

$$\deg_i(v,U)=|\{e\in E\colon\ |e\cap(U\setminus\{v\})|\geq i \text{ and } v\in e\}|\,.$$

For $q\in(0,1]$ we let $\mu_i(H,q)$ denote the expected value of the sum over all such degrees squared with $U=V_q$ being the binomial random subset of V with probability q

$$\mu_i(H,q)=\mathbb{E}\left[\sum_{v\in V}\deg_i^2(v,V_q)\right]\,. \tag{7}$$

Definition 2.3. *Let $\boldsymbol{H}=(H_n)_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let $\boldsymbol{p}=(p_n)_{n\in\mathbb{N}}$ be a sequence of probabilities, and let $K\geq 1$. We say $\boldsymbol{H}$ is $(K,\boldsymbol{p})$-bounded if the following is true.*

For every $i\in[k-1]$, there exists n_0 such that for every $n\geq n_0$ and $q\geq p_n$ we have

$$\mu_i(H_n,q)\leq Kq^{2i}\frac{|E(H_n)|^2}{|V(H_n)|}\,. \tag{8}$$

We will use the following recursive function.

Definition 2.4. *We define the function $R\colon\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ recursively by setting*

$$R(1,r)=1\,,\quad R(i,1)=1\,,$$

and

$$R(i+1,r+1)=R(i,r+1)+(r+1)R(i+1,r)$$

for every i, $r \in \mathbb{N}$.

The results stated in the introduction are consequences of the following theorem.

Theorem 2.5. *Let* $\boldsymbol{H} = (H_n)_{n\in\mathbb{N}}$ *be a sequence of* k*-uniform hypergraphs, let* $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ *be a sequence of probabilities satisfying* $p_n \to 0$ *and* $p_n^k|E(H_n)| \to \infty$ *as* $n \to \infty$*, and let* $\zeta > 0$*,* $K \geq 1$*, and* $r \in \mathbb{N}$*.*

If $\boldsymbol{H}$ *is* $(R(k,r),\zeta)$*-Ramsey and* $(K,\boldsymbol{p})$*-bounded, then there exists a* $C \geq 1$ *such that for* $q_n \geq Cp_n$ *a.a.s. the binomial random subset* V_{n,q_n} *of* $V(H_n)$ *induces an* r*-Ramsey hypergraph.*

We remark that typically satisfying the $(K,\boldsymbol{p})$-boundedness will be the more restrictive assumption on $\boldsymbol{p}$ compared to $p_n^k|E(H_n)| \to \infty$. The proof of Theorem 2.5 is based on induction on $k+r$ and for the induction we will strengthen the statement (see Lemma 2.7 below).

For a k-uniform hypergraph $H = (V,E)$ subsets $W \subseteq U \subseteq V$, and any integer $i \in \{0,\dots,k\}$ we consider those edges of $H[U]$ which have at least i vertices in W and we denote this family by

$$E_U^i(W) = \{e \in E(U)\colon\ |e \cap W| \geq i\}\,.$$

Note that

$$E_U^0(W) = E(U) \quad \text{and} \quad E_U^k(W) = E(W) \tag{9}$$

for every $W \subseteq U$.

The next technical definition is crucial to our induction scheme.

Definition 2.6. *Let* $H = (V,E)$ *be a* k*-uniform hypergraph and* $W \subseteq U \subseteq V$*. Let* $i \in [k]$*,* $r \in \mathbb{N}$*,* $\xi > 0$ *and* $q \in (0,1]$*. We say* $H[W]$ *is* (i,r,ξ,q,U)*-Ramsey, if for every partition* $W^1\dot\cup\dots\dot\cup W^r$ *of* W *there exists an* $s \in [r]$ *such that*

$$\left|E_U^i(W^s)\right| \geq \xi q^i |E|\,.$$

The next lemma states that under some fairly general assumptions $(R(i,r),\zeta)$-Ramseyness of $H[U]$ implies (with probability close to 1) that $H[U_q]$ is (i,r,ξ,q,U)-Ramsey.

Lemma 2.7. *Let* $\boldsymbol{H} = (H_n = (V_n,E_n))_{n\in\mathbb{N}}$ *be a sequence of* k*-uniform hypergraphs, let* $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ *be a sequence of probabilities satisfying* $p_n^k|E_n| \to \infty$ *as* $n \to \infty$*, and let* $K \geq 1$*. Suppose* $\boldsymbol{H}$ *is* $(K,\boldsymbol{p})$*-bounded.*

For every $i \in [k]$*,* $r \in \mathbb{N}$*,* $\zeta > 0$*, and* $(\omega_n)_{n\in\mathbb{N}}$ *with* $\omega_n \to \infty$ *as* $n \to \infty$ *there exist* $\xi > 0$*,* $b > 0$*,* $C \geq 1$*, and* n_0 *such that for every* $n \geq n_0$*, every* q *with* $1/\omega_n \geq q \geq Cp_n$ *the following holds.*

If $U \subseteq V_n$ *and* $H_n[U]$ *is* $(R(i,r),\zeta)$*-Ramsey, then the binomial random subset* U_q *satisfies*

$$\mathbb{P}\left(H[U_q] \text{ is } (i,r,\xi,q,U)\text{-Ramsey}\right) \geq 1 - 2^{-bq|V_n|}\,.$$

Theorem 2.5 follows from Lemma 2.7 applied with $i = k$ and $U = V_n$. Note that the property of being r-Ramsey is monotone and, hence, it suffices to verify Theorem 2.5 for $q = o(1)$.

2.2. **Probabilistic tools.** We will use Chernoff's inequality [1] in the following form.

Theorem 2.8 (Chernoff's inequality (see, e.g., [16, Corollary 2.3]))**.** *Let $X \subseteq Y$ be finite sets and $p \in (0,1]$. For every $0 < \varrho \le 3/2$ we have*

$$\mathbb{P}\left(\big||X \cap Y_p| - p|X|\big| \ge \varrho p|X|\right) \le 2\exp(-\varrho^2 p|X|/3)\,. \qquad \square$$

We will also use Janson's inequality [15].

Theorem 2.9 (Janson's inequality (see, e.g. [16, Theorem 2.14]))**.** *Let $H = (V, E)$ be a k-uniform hypergraph, $U \subseteq V$, $i \in \{2, \dots, k\}$, and $q \in (0, 1]$. For every edge $e \in E(U)$ fix some i-element subset $I(e) \subseteq e$ (in an arbitrary way) and set*

$$\mathbb{1}(e) = \begin{cases} 1\,, & \text{if } I(e) \subseteq U_q \\ 0\,, & \text{otherwise.} \end{cases}$$

For every $\varrho > 0$ the binomial random subset U_q satisfies

$$\begin{aligned}\mathbb{P}\left(|E^i_U(U_q)| \le (1-\varrho)q^i|E(U)|\right) &\le \mathbb{P}\left(\sum_{e\in E(U)} \mathbb{1}(e) \le (1-\varrho)q^i|E(U)|\right)\\ &\le \exp\left(-\frac{\varrho^2 q^{2i}|E(U)|^2}{2\overline{\Delta}_i}\right)\,,\end{aligned}$$

where $\overline{\Delta}_i = \mathbb{E}\left[\sum\sum\{\mathbb{1}(e)\cdot\mathbb{1}(e')\colon\ e, e' \in E(U) \text{ and } I(e) \cap I(e') \neq \emptyset\}\right]$. $\square$

We note that $\overline{\Delta}_i$ can be bounded by $q\mu_{i-1}(H, q)$. In fact, it follows from the linearity of the expectation that

$$\begin{aligned}\overline{\Delta}_i &= \mathbb{E}\left[\sum\sum\{\mathbb{1}(e)\cdot\mathbb{1}(e')\colon\ e, e' \in E(U) \text{ and } I(e)\cap I(e') \neq \emptyset\}\right]\,,\\
&\le \mathbb{E}\left[\sum_{u\in U_q}\Big|\big\{(e,e')\colon\ u \in I(e)\cap I(e')\,,\ I(e)\subseteq U_q\,,\ \text{and } I(e')\subseteq U_q\big\}\Big|\right]\\
&= \sum_{u\in U} q\mathbb{E}\Bigg[\Big|\big\{(e,e')\colon\ u\in I(e)\cap I(e')\,,\ (I(e)\setminus\{u\})\subseteq U_q\,,\\
&\qquad\qquad\qquad\qquad \text{and } (I(e')\setminus\{u\})\subseteq U_q\big\}\Big|\Bigg]\\
&\le q\sum_{v\in V}\deg^2_{i-1}(v, V_q)\\
&= q\mu_{i-1}(H, q)\,. \end{aligned} \tag{10}$$

We also use an approximate concentration result for $(K, \boldsymbol{p})$-bounded hypergraphs. The boundedness of $\boldsymbol{H}$ only bounds the expected value of the quantity $\sum_v \deg_i^2(v, V_p)$. In the proof of Lemma 2.7 we need an exponential upper tail bound and, unfortunately, it is known that such bounds usually do not hold. However, it was shown by Rödl and Ruciński in [28] that on the prize of deleting a few elements such a bound can be obtained.

Proposition 2.10 (Upper tail [28, Lemma 4])**.** *Let $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ be a sequence of probabilities, and let $K \ge 1$. Suppose $\boldsymbol{H}$ is $(K, \boldsymbol{p})$-bounded.*

For every $i \in [k-1]$ and every $\eta > 0$ there exist $b > 0$ and n_0 such that for every $n \ge n_0$ and every $q \ge p_n$ the binomial random subset $V_{n,q}$ has the following *property with probability at least* $1-2^{-bq|V_n|}$. *There exists a set* $|X|\subseteq V_{n,q}$ *with* $X\le \eta q|V_n|$ *such that*

$$\sum_{v\in V_n}\deg_i^2(v,V_{n,q}\setminus X)\le 2Kq^{2i}\frac{|E_n|^2}{|V_n|}\,.$$

The proof follows the lines of [28, Lemma 4] (see also [32]).

Proof. Suppose $\boldsymbol{H}$ is $(K,\boldsymbol{p})$-bounded and $i\in[k-1]$ and $\eta>0$ are given. We set

$$b=\frac{\eta}{4(k-1)^2}$$

and let n_0 be sufficiently large, so that (8) holds for every $n\ge n_0$ and $q\ge p_n$.

For every $j=i,\dots,2(k-1)$ we consider the family $\mathscr{S}_j$ defined as follows

$$\mathscr{S}_j=\Big\{(S,v,e,e')\colon\ S\subseteq V_n,\ v\in V_n,\ e,e'\in E_n \text{ such that } |S|=j,\\ v\in e\cap e',\ S\subseteq(e\cup e')\setminus\{v\},\ |e\cap S|\ge i \text{ and } |e'\cap S|\ge i\Big\}.$$

Let $\mathcal{S}_j$ be the random variable denoting the number of elements (S,v,e,e') from $\mathscr{S}_j$ with $S\in\binom{V_{n,q}}{j}$. By definition we have $\sum_{j=i}^{2k-2}\mathbb{E}\left[\mathcal{S}_j\right]=\mu_i(H_n,q)$ and due to the $(K,\boldsymbol{p})$-boundedness of $\boldsymbol{H}$ we have

$$\max_{j=i,\dots,2(k-1)}\mathbb{E}\left[\mathcal{S}_j\right]\le\sum_{j=i}^{2k-2}\mathbb{E}\left[\mathcal{S}_j\right]=\mu_i(H_n,q)\le Kq^{2i}\frac{|E_n|^2}{|V_n|}\,.$$

Let Z_j be the random variable denoting the number of sequences

$$((S_r,v_r,e_r,e'_r))_{r\in[z]}\in\mathscr{S}_j^z$$

of length

$$z=\left\lceil\frac{\eta q|V_n|}{4(k-1)^2}\right\rceil\le\left\lceil\frac{\eta q|V_n|}{2(k-1)j}\right\rceil$$

which satisfy

(*i*) the sets S_r are contained in $V_{n,q}$ and

(*ii*) the sets S_r are mutually disjoint, i.e., $S_{r_1}\cap S_{r_2}=\emptyset$ for all $1\le r_1<r_2\le z$.

Clearly, we have

$$\mathbb{E}\left[Z_j\right]\le|\mathscr{S}_j|^zq^{jz}=\left(\mathbb{E}\left[\mathcal{S}_j\right]\right)^z\le\left(Kq^{2i}\frac{|E_n|^2}{|V_n|}\right)^z\,. \tag{11}$$

On the other hand, if

$$\sum_{v\in V_n}\deg_i^2(v,V_{n,q}\setminus X)\ge 4k^2Kq^{2i}\frac{|E_n|^2}{|V_n|}\ge\sum_{j=i}^{2k-2}j\cdot 2Kq^{2i}\frac{|E_n|^2}{|V_n|}$$

for any $X\subseteq V_{n,q}$ with $|X|\le\eta q|V_n|$, then there exists some $j_0\in\{i,\dots,2k-2\}$ such that

$$Z_{j_0}\ge\left(2Kq^{2i}\frac{|E_n|^2}{|V_n|}\right)^z\,.$$

Markov's inequality bounds the probability of this event by

$$\begin{aligned}\mathbb{P}\left(\exists j_0 \in \{i,\dots,2k-2\}\colon\ Z_{j_0} \geq 2^z\left(Kq^{2i}\frac{|E_n|^2}{|V_n|}\right)^z\right)\\ \leq \sum_{j=i}^{2k-2}\mathbb{P}\left(Z_j \geq 2^z\left(Kq^{2i}\frac{|E_n|^2}{|V_n|}\right)^z\right) \overset{(11)}{\leq} \sum_{j=i}^{2k-2}\mathbb{P}\left(Z_j \geq 2^z(\mathbb{E}\left[Z_j\right])^z\right)\\ \leq 2k\cdot 2^{-z} \leq 2^{-bq|V_n|+1+\log_2 k}\,,\end{aligned}$$

which concludes the proof of Proposition 2.10. □

The next lemma, also due to Rödl and Ruciński from [28], states the if a binomial random subset enjoys a monotone property with very high probability, then a slightly enlarged random subset will have a "robust" variant of this property with similar probability. Here we say an event holds with very high probability if the probability of failing is exponentially small in the expected size of V_q.

Proposition 2.11 ([28, Lemma 3])**.** *Let U be a set and let $\mathcal{P}$ be a family of subsets of U closed under supersets. For all $\delta \in (0,1)$ and $b > 0$ satisfying $\delta(3-\log_2\delta) \leq b$ and $q \in (0,1]$ the following holds. If*

$$\mathbb{P}\left(U_{(1-\delta)q} \in \mathcal{P}\right) \geq 1 - 2^{-bq|U|}$$

then

$$\mathbb{P}\left(\forall X \subseteq U_q \text{ with } |X| \leq \delta q|U|/2 \text{ we have } (U_q \setminus X) \in \mathcal{P}\right) \geq 1 - 2^{-\delta^2 q|U|/20}\,. \quad \square$$

2.3. **Proof of main result.** We start with a simple observation.

Fact 2.12. *Let $H = (V,E)$ be a k-uniform hypergraph, let $U \subseteq V$, and let $\zeta > 0$ and $K \geq 1$. If $\sum_{v\in V}\deg^2_{k-1}(v,V) \leq K|E|^2/|V|$ and $e(U) \geq \zeta|E|$, then the set*

$$Y = \left\{u \in U\colon\ \deg_{k-1}(u,U) \geq \frac{\zeta}{2}\frac{|E|}{|V|}\right\}$$

satisfies

$$|Y| \geq \frac{\zeta^2}{4K}|V|\,.$$

Proof. Due to the definition of Y we have

$$\sum_{y\in Y}\deg_{k-1}(y,U) \geq e(U) - \frac{\zeta}{2}\frac{|E|}{|V|}|U| \geq \frac{\zeta}{2}|E|\,.$$

Hence, it follows from the Cauchy-Schwarz inequality

$$\begin{aligned}\frac{\zeta^2}{4}|E|^2 \leq \left(\sum_{y\in Y}\deg_{k-1}(y,U)\right)^2 &\leq |Y|\sum_{y\in Y}\deg^2_{k-1}(y,U)\\ &\leq |Y|\sum_{v\in V}\deg^2_{k-1}(v,V) \leq |Y|\cdot K\frac{|E|^2}{|V|}\,,\end{aligned}$$

which yields the claim. □

Proof of Lemma 2.7. Let $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ be a sequence of probabilities such that $p_n^k|E_n| \to \infty$ and $\boldsymbol{H}$ is $(K, \boldsymbol{p})$-bounded for some $K \geq 1$. We prove Lemma 2.7 by induction on $i + r$.

Induction start for $i = 1$ and $r \in \mathbb{N}$. In this case we need to show that for given r, ζ and $(\omega_n)_{n\in\mathbb{N}}$ there exist ξ, b, C, and n_0 so that for $n \geq n_0$ the $(R(1, r), \zeta)$-Ramseyness of $H_n[U]$ implies $(1, r, \xi, q, U)$-Ramseyness of $H[U_q]$ with very high probability. This will follow from Chernoff's inequality and Fact 2.12. In fact, let $\zeta > 0$ (the sequence $(\omega_n)_{n\in\mathbb{N}}$ will play no role in the induction start and we only need the upper bound on q in the induction step). We set

$$\xi = \frac{\zeta^3}{16krK}\,, \qquad b = \frac{\zeta^2}{49K}\,, \qquad C = 1\,,$$

and let n be sufficiently large. Note that for sufficiently large n, the $(K, \boldsymbol{p})$-boundedness of $\boldsymbol{H}$ (applied for $q = 1$) yields

$$\mu_{k-1}(H_n, 1) = \sum_{v\in V} \deg^2_{k-1}(v, V) \leq K\frac{|E_n|^2}{|V_n|}\,.$$

For every $U \subseteq V_n$ satisfying $H_n[U]$ is $(R(1, r), \zeta)$-Ramsey we have $e(U) \geq \zeta|E_n|$. Consequently, we infer from Fact 2.12 that

$$|Y| \geq \frac{\zeta^2}{4K}|V_n|\,. \tag{12}$$

Furthermore, due to the definition of Y we have for every $q \in (0, 1]$

$$|E^1_U(U_q)| \geq \frac{1}{k}|Y \cap U_q| \cdot \frac{\zeta}{2}\frac{|E_n|}{|V_n|}$$

and for every partition $U^1_q\dot\cup \dots \dot\cup U^r_q$ of U_q there exists an $s \in [r]$ such that

$$|E^1_U(U^s_q)| \geq \frac{\zeta}{2kr}|Y \cap U_q|\frac{|E_n|}{|V_n|}$$

Finally, it follows from Chernoff's inequality that $|Y \cap U_q| \geq q|Y|/2$ with probability at least $1 - 2\exp(-q|Y|/12)$. Hence, the choice of ξ and b and (12) yields

$$\mathbb{P}\left(H_n[U_q] \text{ is } (1, r, \xi, q, U)\text{-Ramsey}\right) \geq 1 - 2\exp(-q|Y|/12) \geq 1 - 2^{-bq|V_n|}$$

for sufficiently large n and $q \geq p_n$, since $q|V_n| \geq p_n|V_n| \geq p_n|E_n|^{1/k} \to \infty$.

Induction start for $i \geq 2$ and $r = 1$. This case follows from Janson's inequality. For $\zeta > 0$ we set

$$\xi = \frac{\zeta}{2}\,, \qquad b = \frac{\zeta^2}{8K}\,, \qquad C = 1\,.$$

Let n be sufficiently large and $q \geq p_n$ so that

$$\mu_{i-1}(H_n, q) \leq Kq^{2i-2}|E_n|^2/|V_n|\,.$$

For every $U \subseteq V_n$, which is $(R(i, 1), \zeta)$-Ramsey we have $e(U) \geq \zeta|E_n|$. Consequently, (10) combined with Janson's inequality applied with $\varrho = 1/2$ yields

$$\mathbb{P}\left(|E^i_U(U_q)| \leq \frac{\zeta}{2}q^i|E_n|\right) \leq \exp\left(-\frac{\zeta^2q^{2i-1}|E_n|^2}{8\mu_{i-1}(H_n, q)}\right) \leq 2^{-bq|V_n|}\,,$$

which yields the lemma for $r = 1$.

Induction step. We will verify the lemma for $i+1 \geq 2$ and $r+1 \geq 2$ and suppose the lemma holds for i and $r+1$ and for $i+1$ and r. Let $\zeta > 0$ and $\boldsymbol{\omega} = (\omega_n)_{n\in\mathbb{N}}$ be given.

Outline. We will expose the random set U_q in L rounds. Suppose U_{q_1} is the outcome of the first round and let $U_{q_1}^1 \dot\cup \dots \dot\cup U_{q_1}^{r+1}$ be an arbitrary partition of U_{q_1}. Due to the induction assumption applied for i and $r+1$ we will infer that there must be some color $s_1 \in [r+1]$ such that $|E_U^i(U_{q_1}^{s_1})| = \Omega(q_1^i|E|)$. We consider the set $W_1 \subseteq U$ of vertices such that every vertex $w \in W_1$ is contained in at least $\Omega(q_1^i|E|/|V|)$ edges from $E_U^i(U_{q_1}^{s_1})$. Note that if for some later round for $\ell > 1$ some $w \in W_1$ appears in U_{q_ℓ} and w will be also colored with the same color s_1, i.e., $w \in U_{q_\ell}^{s_1}$, then this will create edges in $E_U^{i+1}(U_{q_1}^{s_1} \cup U_{q_\ell}^{s_1}) \subseteq E_U^{i+1}(U_q^{s_1})$. Moreover, we will infer from the $(K,\boldsymbol{p})$-boundedness of $\boldsymbol{H}$ that W_1 is of linear order, i.e., $|W_1| = \Omega(|V|)$.

In the second round we will repeat the same argument and obtain some $s_2 \in [r+1]$ and a set W_2. However, we will also ensure that $W_2 \setminus W_1$ is large. For that we will apply the induction assumption to $U \setminus W_1$. In fact, this is the reason for allowing an arbitrary subset $U \subseteq V$ in Lemma 2.7. As a result we will ensure that $|W_2 \setminus W_1| \geq \lambda|V|$ for some fixed $\lambda > 0$ (only depending on K, $i+1$, $r+1$, and ζ). Recall that we can only apply the induction assumption for i and $r+1$ to $U \setminus W_1$ only, if $H[U \setminus W_1]$ is still $(R(i,r+1),\zeta)$-Ramsey.

In general we will repeat the argument from the first round and obtain sets $W_1,\dots,W_{\ell-1}$ such that $|W_j \setminus \bigcup_{i=1}^{j-1} W_i| \geq \lambda|V|$ for every $j \in [\ell-1]$ and integers $s_1,\dots,s_{\ell-1} \in [R+1]$ such that $|E_U^i(U_{q_j}^{s_j})| = \Omega(q_j^i|E|)$ for every $j \in [\ell-1]$. We stop when $H[U \setminus W]$ fails to be $(R(i,r+1),\zeta)$-Ramsey for $W = \bigcup_{j=1}^{\ell-1} W_j$. Clearly, after at most $1/\lambda < L$ rounds we arrive at the situation, that $H[U \setminus W]$ is not $(R(i,r+1),\zeta)$-Ramsey and then we will argue as follows.

Since $H[U]$ was $(R(i+1,r+1),\zeta)$-Ramsey, by Fact 2.2 and the definition of $R(\cdot,\cdot)$ we then have that $H[W]$ must be $((r+1)\cdot R(i+1,r),\zeta)$-Ramsey. Consequently, there must be some $t \in [r+1]$ such that for

$$W^t = \bigcup_{j: s_j = t} W_j$$

we have that $H[W^t]$ is $(R(i+1,r),\zeta)$-Ramsey. In other words, we are ready to apply the induction assumption with $i+1$ and r to W^t. By definition of W^t every vertex of W^t is contained in $\Omega(q^i|E|/|V|)$ edges from $E_U^i(\bigcup_{j=1}^{\ell-1} U_{q_j}^t)$ and, therefore, if a substantial fraction of the vertices $U_{q_\ell} \cap W^t$ will be assigned the color t, then we have $|E_U^{i+1}(\bigcup_{j=1}^{\ell} U_{q_j}^t)| = \Omega(q^{i+1}|E|)$, which is what we have to show. If, on the other hand, the number of vertices of color t in $U_{q_\ell} \cap W^t$ is negligible, then the induction assumption applied for $i+1$ and r to W^t will yield that $E_U^{i+1}(U_{q_\ell}^s)$ is large for some $s \in [r+1] \setminus \{t\}$.

In the proof we will need that the error probabilities in the later rounds will have to beat the number of $(r+1)$-colorings of the earlier rounds. For that we will chose q_ℓ in such a way that $q_\ell \gg \sum_{j=1}^{\ell-1} q_j$ and $q_l = \Theta(q)$ for every $\ell \in [L]$. This will require that all statements in the proof have to hold with very high probability. We now give the details of this proof and first define all constants involved in the proof.

Constants. The number of rounds L will depend on the constant $\xi(i,r+1,\zeta,\boldsymbol{\omega})$, which is given by the induction assumption. More precisely, let

$$\xi' = \xi(i,r+1,\zeta,\boldsymbol{\omega})\,, \quad b' = b(i,r+1,\zeta,\boldsymbol{\omega})\,,$$
$$C' = C(i,r+1,\zeta,\boldsymbol{\omega})\,, \quad \text{and} \quad n' = n_0(i,r+1,\zeta,\boldsymbol{\omega})$$

be given by the induction assumption applied for i, $r+1$, ζ, and $\boldsymbol{\omega}$. We set

$$L = \left\lceil \frac{2^{2i+3}K}{(\xi')^2} + 1 \right\rceil . \tag{13}$$

Moreover, we will appeal to the induction assumption for $i+1$, r, ζ, and $\boldsymbol{\omega}$ and let

$$\xi^* = \xi(i+1,r,\zeta,\boldsymbol{\omega})\,, \quad b^* = b(i+1,r,\zeta,\boldsymbol{\omega})\,,$$
$$C^* = C(i+1,r,\zeta,\boldsymbol{\omega})\,, \quad \text{and} \quad n^* = n_0(i+1,r,\zeta,\boldsymbol{\omega})$$

be the corresponding constants. Let $1/2 \geq \delta > 0$ be sufficiently small so that

$$\delta(3 - \log_2 \delta) \leq \min\left\{\frac{b'}{2}\,, \frac{b^*}{2}\right\} \tag{14}$$

and set

$$\hat{b} = \frac{\delta^2\zeta^2}{80K}\,. \tag{15}$$

Furthermore, we appeal to Proposition 2.10 with K, i, and

$$\eta = \frac{\delta\zeta^2}{8K} \tag{16}$$

and obtain

$$\overline{b} = b(K,i,\eta), \qquad \overline{C} = C(K,i,\eta), \qquad \text{and} \qquad \overline{n} = n_0(K,i,\eta)\,. \tag{17}$$

Next we set

$$b_{\min} = \min\left\{\frac{\zeta^2}{25K}\,, \frac{\hat{b}}{2}\,, \frac{\overline{b}}{2}\right\} \qquad \text{and} \qquad B = \max\left\{\frac{4\log_2(r+1)}{b_{\min}}\,, \frac{32K}{\delta\zeta^2}\right\} + 1 \tag{18}$$

and finally let

$$\xi = \min\left\{\frac{\delta\zeta^2\xi'(1-\delta)^i}{32kK}\,, \xi^*(1-\delta)^{i+1}\right\} \cdot \left(\frac{B-1}{B^L-1}\right)^{i+1}\,, \tag{19}$$

$$b = \frac{b_{\min}}{3}\frac{B-1}{B^L-1}\,, \tag{20}$$

$$C = \frac{\max\left\{C'\,, C^*\,, \overline{C}\right\}}{1-\delta} \cdot \frac{B^L-1}{B-1}\,, \tag{21}$$

and let $n_0 \geq \max\{n', n^*, \overline{n}\}$ be sufficiently large. Let $n \geq n_0$, let $q \in (0,1]$ such that

$$Cp_n \leq q \leq \frac{1}{\omega_n}\,.$$

Finally, appealing to the assumptions of Lemma 2.7, let $U \subseteq V_n$ such that

$$H_n[U] \text{ is } (R(i+1,r+1),\zeta)\text{-Ramsey}\,. \tag{22}$$

From now on we drop the subscript n for a simpler notation. We have to show that $H[U_q]$ is $(i+1,r+1,\xi,q,U)$-Ramsey with very high probability.

As mentioned above we will expose U_q in L rounds, where the elements in the ℓ-th round will be included with probability q_ℓ. For that let q_1 be the solution of the equation

$$1 - q = \prod_{\ell=1}^{L} \left(1 - B^{\ell-1} q_1\right)$$

and set

$$q_\ell = B^{\ell-1} q_1$$

for every $\ell = 2, \dots, L$. For sufficiently large n we have

$$q_1 \geq \frac{q}{\sum_{\ell=1}^{L} B^{\ell-1}} = q \frac{B-1}{B^L - 1}, \tag{23}$$

since $\omega_n \to \infty$ and $q \leq 1/\omega_n$. Due to the choice of C in (21) and $q \geq C p_n$ we have

$$q_L > \dots > q_1 \geq \frac{\max\left\{C', C^*, \overline{C}\right\}}{1-\delta} \cdot p_n . \tag{24}$$

The choice of B in (18) yields for every $\ell = 2, \dots, L$

$$\begin{aligned}\sum_{j=1}^{\ell-1} q_j = q_1 \sum_{j=0}^{\ell-2} B^j = q_1 \frac{B^{\ell-1} - 1}{B - 1} \\ \overset{(18)}{\leq} \min\left\{ \frac{b_{\min}}{4\log_2(r+1)}, \frac{\delta\zeta^2}{32K} \right\} B^{\ell-1} q_1 \leq \frac{\delta\zeta^2}{32K} q_\ell \end{aligned} \tag{25}$$

holds.

For later reference we note that due to the choice of constants in (21) and (24) the following statements hold by induction assumption. For every subset $S \subseteq U$ and $\ell \in [L]$ we have

$$\begin{aligned} &H[S] \text{ is } ((R(i, r+1), \zeta)\text{-Ramsey} \\ &\quad \Rightarrow \mathbb{P}\left(H[S_{(1-\delta)q_\ell}] \text{ is } (i, r+1, \xi', (1-\delta)q_\ell, S)\text{-Ramsey}\right) \geq 1 - 2^{-b'(1-\delta)q_\ell|S|} \end{aligned} \tag{26}$$

and

$$\begin{aligned} &H[S] \text{ is } ((R(i+1, r), \zeta)\text{-Ramsey} \\ &\quad \Rightarrow \mathbb{P}\left(H[S_{(1-\delta)q_\ell}] \text{ is } (i+1, r, \xi^*, (1-\delta)q_\ell, S)\text{-Ramsey}\right) \geq 1 - 2^{-b^*(1-\delta)q_\ell|S|}. \end{aligned} \tag{27}$$

Details of the induction step. For our analysis we require some notation. Recall, the random subsets of the L rounds by $U_{q_1}, \dots, U_{q_L}$ and let $U_q = \bigcup_{\ell\in[L]} U_{q_\ell}$. Moreover, we let $\chi_\ell \colon U_{q_\ell} \to [r+1]$ be a partition of U_{q_ℓ} and we denote the partition classes by $U^1_{q_\ell} \dot\cup \dots \dot\cup U^{r+1}_{q_\ell}$, i.e., for every $s \in [r+1]$ and $\ell \in [L]$

$$U^s_{q_\ell} = \chi_\ell^{-1}(s) .$$

Since the sets U_{q_j} and $U_{q_{j'}}$ may not be disjoint we will require that the partitions χ_j and $\chi_{j'}$ are consistent, i.e., those functions agree on $U_{q_j} \cap U_{q_{j'}}$.

In the proof those vertices of U, which are contained in many edges in $E^i_U(U^s_{q_\ell})$ play a crucial role. For that we define for every $\ell \in [L]$ and $s \in [r+1]$ the set

$$W^s_\ell = \left\{ u \in U \colon \ \deg_{i,U}(u, U^s_{q_\ell}) \geq \frac{\xi'(1-\delta)^i}{2} q^i_\ell \frac{|E|}{|V|} \right\} , \tag{28}$$

where

$$\deg_{i,U}(u, U^s_{q_\ell}) = \left|\left\{e \in E(U)\colon\ u \in e \text{ and } |(e \setminus \{u\}) \cap U^s_{q_\ell}| \geq i\right\}\right|$$

is the degree of the vertex u in the edge set $E^i_U(U^s_{q_\ell})$. It follows directly from the definition that

$$\deg_{i,U}(u, U^s_{q_\ell}) \leq \deg_{i,V}(u, U^s_{q_\ell}) \leq \deg_{i,V}(u, U_{q_\ell}) = \deg_i(u, U_{q_\ell})\,. \tag{29}$$

Finally, for $\ell \in [L]$ we denote by W_ℓ the set of vertices with large degree in some partition class, i.e.,

$$W_\ell = \bigcup_{s\in[r+1]} W^s_\ell\,.$$

The following claim, roughly speaking, says that given subsets $U_{q_1}, \dots, U_{q_{\ell-1}}$ of U and consistent partitions $\chi_j\colon\ U_{q_\ell} \to [r+1]$ for $j \in [\ell-1]$ the random set U_{q_ℓ} satisfies the following with very high probability:

For any $(r+1)$-partition of U_{q_ℓ} either W_ℓ contains $\Omega(|V|)$ new elements disjoint from $W_1, \dots, W_{\ell-1}$ (see (*ii.a*) below) or there exists some $s \in [r+1]$ such that $E^{i+1}_U(\bigcup_{j=1}^{\ell} U^s_{q_j})$ will be large (see (*ii.b*) below).

Claim 1. *Let $\ell \in [L]$, let subsets $U_{q_1}, \dots, U_{q_{\ell-1}}$ of U satisfy*

$$\left|\bigcup_{j=1}^{\ell-1} U_{q_j}\right| \leq 2\sum_{j=1}^{\ell-1} q_j|U| \tag{30}$$

and let consistent $(r+1)$-partitions $\chi_j\colon\ U_{q_j} \to [r+1]$ for $j \in [\ell-1]$ be given.

With probability at least

$$1 - 2^{-b_{\min} q_\ell |V|}$$

the random set U_{q_ℓ} satisfies the following:

(*i*) $|U_{q_\ell}| \leq 2q_\ell|U|$

and for every partition $\chi_\ell\colon\ U_{q_\ell} \to [r+1]$

(*ii.a*) *either*

$$\left|W_\ell \setminus \bigcup_{j=1}^{\ell-1} W_j\right| \geq \frac{(\xi')^2}{2^{2i+3}K}|V|\,,$$

(*ii.b*) *or there exists an $s \in [r+1]$ such that*

$$\left|E^{i+1}_U\left(\bigcup_{j=1}^{\ell} U^s_{q_j}\right)\right| \geq \xi q^{i+1}|E|\,.$$

We first deduce Lemma 2.7 from Claim 1. Let $\mathscr{A}$ denote the event that $H[U_q]$ is $(i+1, r+1, \xi, q, U)$-Ramsey and for given $\boldsymbol{\mathcal{U}}_q(\ell-1) = (U_{q_1}, \dots, U_{q_{\ell-1}})$ and for given $\boldsymbol{\chi}(\ell-1) = (\chi_1, \dots, \chi_{\ell-1})$ with $\chi_j\colon\ U_{q_j} \to [r+1]$ being consistent for $j = 1, \dots, \ell-1$, let $\mathscr{B}_{\boldsymbol{\chi}(\ell-1)}$ be the event that the conclusion of Claim 1 holds. In other words, Claim 1 states that for the randomly chosen set U_{q_ℓ} we have

$$\mathbb{P}\left(\mathscr{B}_{\boldsymbol{\chi}(\ell-1)} \mid \boldsymbol{\mathcal{U}}_q(\ell-1)\right) \geq 1 - 2^{-b_{\min} q_\ell |V|}\,. \tag{31}$$

Note that $\boldsymbol{\mathcal{U}}_q(0)$ and $\boldsymbol{\chi}(0)$ are vectors of length 0. For $\ell = 1$ we set

$$\mathbb{P}\left(\mathscr{B}_{\boldsymbol{\chi}(0)} \mid \boldsymbol{\mathcal{U}}_q(0)\right) = \mathbb{P}\left(\mathscr{B}_{\boldsymbol{\chi}(0)}\right)\,,$$

where $\mathscr{B}_{\boldsymbol{\chi}(0)}$ denotes the event that

(*i*) $|U_{q_1}| \leq 2q_1|U|$

and for every partition $\chi_1\colon\ U_{q_1} \to [r+1]$

(*ii.a*) either $|W_1| \geq (\xi')^2|V|/(2^{2i+3}K)$,

(*ii.b*) or there exists an $s \in [r+1]$ such that $|E^{i+1}_U(U^s_{q_1})| \geq \xi q^{i+1}|E|$.

Again Claim 1 states that

$$\mathbb{P}\left(\mathscr{B}_{\boldsymbol{\chi}(0)}\right) \geq 1 - 2^{-b_{\min} q_1 |V|}\,.$$

Note that if $\mathscr{B}_{\boldsymbol{\chi}(\ell)}$ holds for every $\ell \in [L]$, then alternative $(ii.a)$ cannot alway occur since

$$\frac{2^{2i+3}K}{(\xi')^2} \overset{(13)}{<} L\,.$$

Hence, if $\mathscr{B}_{\boldsymbol{\chi}(\ell)}$ holds for every $\ell \in [L]$, then conclusion $(ii.b)$ in Claim 1 must hold for some $\ell \in [L]$. Consequently, for every partition of $\bigcup_{j=1}^{\ell} U_{q_j}$ into $r+1$ classes there exists some $s \in [r+1]$ such that $|E_U^{i+1}(\bigcup_{j=1}^{\ell} U_{q_j}^s)| \geq \xi q^{i+1}|E|$. In other words, since $\bigcup_{j=1}^{\ell} U_{q_j} \subseteq U_q$, the hypergraph $H[U_q]$ is $(i+1, r+1, \xi, q, U)$-Ramsey and event $\mathscr{A}$ occurs. Below we will verify that this happens with a sufficiently high probability

$$\mathbb{P}\left(\neg\mathscr{A}\right) \leq \sum_{\ell=1}^{L} \sum_{\boldsymbol{\mathcal{U}}_q(\ell-1)} \sum_{\boldsymbol{\chi}(\ell-1)} \mathbb{P}\left(\neg\mathscr{B}_{\boldsymbol{\chi}(\ell-1)} \mid \boldsymbol{\mathcal{U}}_q(\ell-1)\right) \mathbb{P}\left(\boldsymbol{\mathcal{U}}_q(\ell-1)\right)\,,$$

where the middle sum runs over all choices of $\boldsymbol{\mathcal{U}}_q(\ell-1) = (U_{q_1}, \dots, U_{q_{\ell-1}})$ satisfying (30) and the inner sum runs over all $(r+1)^{2|V|\sum_{j=1}^{\ell-1} q_j}$ partitions $\boldsymbol{\chi}(\ell-1)$ of $\boldsymbol{\mathcal{U}}_q(\ell-1)$. Therefore, (31) yields

$$\mathbb{P}\left(\neg\mathscr{A}\right) \leq \sum_{\ell=1}^{L} (r+1)^{2|V|\sum_{j=1}^{\ell-1} q_j} \cdot 2^{-b_{\min} q_\ell |V|}\,.$$

Since $\sum_{j=1}^{\ell-1} q_j \leq \frac{b_{\min}}{4\log_2(r+1)} q_\ell$ by (25) and $q_1 \leq q_\ell$ we have

$$\mathbb{P}\left(\neg\mathscr{A}\right) \leq L \cdot 2^{-b_{\min} q_1 |V|/2} \overset{(20),(23)}{\leq} 2^{-bq|V|}$$

where the last inequality holds for sufficiently large n. This concludes the proof of Lemma 2.7 and it is left to verify Claim 1. □

Proof of Claim 1. Let $\ell \in [L]$, $U_{q_1}, \dots, U_{q_{\ell-1}}$ and partitions $\chi_1, \dots, \chi_{\ell-1}$ be given. Note that this defines the sets W_j^s for $j \in [\ell-1]$ and $s \in [r+1]$ as well. We first observe that property (i) of Claim 1 holds with high probability. Since (22) holds, we have $e(U) \geq \zeta|E|$ and, therefore, the $(K, \boldsymbol{p})$-boundedness of $\boldsymbol{H}$ combined with Fact 2.12 yields

$$|U| \geq \frac{\zeta^2}{4K}|V|\,.$$

Hence, Chernoff's inequality yields

$$\mathbb{P}\left(|U_{q_\ell}| \geq 2q_\ell|U|\right) \leq 2\exp(-q_\ell|U|/3) \overset{(18)}{\leq} 2^{-2b_{\min} q_\ell |V|}\,. \tag{32}$$

for sufficiently large n.

For the rest of the proof we distinguish two cases depending on the structure of the complement of $\bigcup_{j=1}^{\ell-1} W_\ell$ in U. For that we set

$$\hat{U} = U \setminus \bigcup_{j=1}^{\ell-1} W_j\,.$$

Case 1 ($H[\hat{U}]$ is $(R(i,r{+}1),\zeta)$-Ramsey)**.** Note that it follows from the the Ramsey-ness of $H[\hat{U}]$ that $e(\hat{U})\geq\zeta|E|$. Therefore, the $(K,\boldsymbol{p})$-boundedness of $\boldsymbol{H}$ combined with Fact 2.12 yields

$$|\hat{U}|\geq\frac{\zeta^2}{4K}|V|\,. \tag{33}$$

In this case we appeal to induction assumption for i and $r+1$ and focus to the restriction on $H[\hat{U}]$ (cf. (26)). In fact, the induction assumption for i and $r+1$ yields

$$\begin{aligned}\mathbb{P}\Big(H\big[\hat{U}_{(1-\delta)q_\ell}\big]\text{ is }(i,r+1,\xi',(1-\delta)q_\ell,\hat{U})\text{-Ramsey}\Big)&\geq 1-2^{-b'(1-\delta)q_\ell|\hat{U}|}\\&\geq 1-2^{-(b'/2)q_\ell|\hat{U}|}\end{aligned}$$

Since being $(i,r+1,\xi',(1-\delta)q_\ell,\hat{U})$-Ramsey is closed under supersets, in view of (14) we infer from Proposition 2.11 that with probability at least

$$1-2^{-\delta^2 q_\ell|\hat{U}|/20}\overset{(15),(33)}{\geq}1-2^{-\hat{b}q_\ell|V|} \tag{34}$$

the random set $\hat{U}_{q_\ell}$ has the property that $H[\hat{U}_{q_\ell}\setminus X]$ is $(i,r+1,\xi',(1-\delta)q_\ell,\hat{U})$-Ramsey for every $X\subseteq\hat{U}_{q_\ell}$ with $|X|\leq\delta q_\ell|\hat{U}|/2$. In other words, for every such set X and every partition $\hat{U}^1_{q_\ell}\dot\cup\ldots\dot\cup\hat{U}^{r+1}_{q_\ell}$ of $\hat{U}_{q_\ell}$ there exists an $s\in[r{+}1]$ such that

$$\Big|E^i_{\hat{U}}(\hat{U}^s_{q_\ell}\setminus X)\Big|\geq\xi'(1-\delta)^i q_\ell^i|E|\,. \tag{35}$$

Recalling the definition

$$E^i_{\hat{U}}(\hat{U}^s_{q_\ell}\setminus X)=\{e\in E(\hat{U})\colon\ |e\cap(\hat{U}^s_{q_\ell}\setminus X)|\geq i\}\,.$$

and recalling that $i+1\leq k$ we note that

$$\begin{aligned}\Big|E^i_{\hat{U}}(\hat{U}^s_{q_\ell}\setminus X)\Big|&\leq\sum_{u\in\hat{U}}\big\{e\in E(\hat{U})\colon\ u\in e\text{ and }(e\setminus\{u\})\cap(\hat{U}^s_{q_\ell}\setminus X)|\geq i\big\}\\&=\sum_{u\in\hat{U}}\deg_{i,\hat{U}}(u,\hat{U}^s_{q_\ell}\setminus X)\,.\end{aligned}$$

Consequently, (35) implies

$$\sum_{u\in\hat{U}}\deg_{i,\hat{U}}(u,\hat{U}^s_{q_\ell}\setminus X)\geq\xi'(1-\delta)^i q_\ell^i|E|\,. \tag{36}$$

Moreover, due Proposition 2.10 and the choice of constants in (17) with probability at least

$$1-2^{-\bar{b}q_\ell|V|} \tag{37}$$

there exists a set $X\subseteq V_{q_\ell}$ of size at most

$$|X|\leq\eta q_\ell|V|\overset{(16)}{\leq}\delta q_\ell|\hat{U}|/2$$

such that

$$\sum_{u\in\hat{U}}\deg^2_{i,\hat{U}}(u,\hat{U}^s_{q_\ell}\setminus X)\overset{(29)}{\leq}\sum_{v\in V}\deg^2_i(v,V_{q_\ell}\setminus X)\leq 2Kq_\ell^{2i}\frac{|E|^2}{|V|}\,. \tag{38}$$

Let

$$\hat{W}^s_\ell=\left\{u\in\hat{U}\colon\ \deg_{i,\hat{U}}(u,\hat{U}^s_{q_\ell})\geq\frac{\xi'(1-\delta)^i}{2}q_\ell^i\frac{|E|}{|V|}\right\}\,. \tag{39}$$

Note that by definition $\hat{W}_\ell^s \subseteq W_\ell$ and W_ℓ^s is disjoint from $\bigcup_{j=1}^{\ell-1} W_j$.

Summarizing, due to (34) and (37) with probability at least $1-2^{-\hat{b}q_\ell|V|}-2^{-\bar{b}q_\ell|V|}$ the random set $\hat{U}_{q_\ell}$ satisfies properties (36) and (38) for every partition of $\hat{U}_{q_\ell}$ and we infer by the Cauchy-Schwarz inequality that

$$2Kq_\ell^{2i}\frac{|E|^2}{|V|} \overset{(38)}{\geq} \sum_{u\in\hat{U}} \deg^2_{i,\hat{U}}(u,\hat{U}^s_{q_\ell}\setminus X) \geq \sum_{u\in\hat{W}^s_\ell} \deg^2_{i,\hat{U}}(u,\hat{U}^s_{q_\ell}\setminus X)$$
$$\geq \frac{1}{|\hat{W}^s_\ell|}\left(\sum_{u\in\hat{W}^s_\ell} \deg_{i,\hat{U}}(u,\hat{U}^s_{q_\ell}\setminus X)\right)^2.$$

Moreover

$$\sum_{u\in\hat{W}^s_\ell} \deg_{i,\hat{U}}(u,\hat{U}^s_{q_\ell}\setminus X)$$
$$\geq \sum_{u\in\hat{U}} \deg_{i,\hat{U}}(u,\hat{U}^s_{q_\ell}\setminus X) - \sum_{u\in\hat{U}\setminus\hat{W}^s_\ell} \deg_{i,\hat{U}}(u,\hat{U}^s_{q_\ell}\setminus X)$$
$$\overset{(36),(39)}{\geq} \frac{\xi'}{2}(1-\delta)^i q_\ell^i|E|.$$

Combining the last two estimates and $\delta\leq 1/2$ we obtain

$$|\hat{W}^s_\ell| \geq \frac{(\xi')^2}{2^{2i+3}K}|V|.$$

In other words, in this case property $(ii.a)$ holds with probability at least

$$1-2^{-\hat{b}q_\ell|V|}-2^{-\bar{b}q_\ell|V|}$$

and in view of (32) and the choice of $b_{\min}$ in (18) for sufficiently large n this yields the proof of Claim 1 in this case.

Case 2 ($H[\hat{U}]$ is not $(R(i,r+1),\zeta)$-Ramsey)**.** Due to (22), the assumption of this case combined with Fact 2.2, and the definition of the function $R(\cdot,\cdot)$ in Definition 2.4 we have

$$H[U\setminus\hat{U}] \text{ is } ((r+1)\cdot R(i+1,r),\zeta)\text{-Ramsey.} \tag{40}$$

Recall that

$$U\setminus\hat{U} = \bigcup_{j=1}^{\ell-1} W_j = \bigcup_{j=1}^{\ell-1}\bigcup_{s=1}^{r+1} W_j^s.$$

For $s\in[r+1]$ let

$$W^s = \{w\in U\setminus\hat{U}\colon\ w\in W_j^s \text{ for some } j\in[\ell-1]\} = \bigcup_{j=1}^{\ell-1} W_j^s.$$

Clearly,

$$W^1\cup\dots\cup W^{r+1} = U\setminus\hat{U}.$$

We remark that this $W^1\cup\dots\cup W^{r+1}$ is not necessarily a partition of $U\setminus\hat{U}$. However, it follows from Fact 2.2 and (40), that there exists a $t\in[r+1]$ such that

$$H[W^t] \text{ is } (R(i+1,r),\zeta)\text{-Ramsey.} \tag{41}$$

Next combine (41) with Fact 2.12 and the $(K, \boldsymbol{p})$-boundedness of $\boldsymbol{H}$ to obtain

$$|W^t| \geq \frac{\zeta^2}{4K}|V|\,. \tag{42}$$

Moreover, due to (41) we can apply the induction assumption for $i+1$ and r to $H[W^t]$ (cf. (27)). This yields

$$\begin{aligned}\mathbb{P}\left(H\left[W^t_{(1-\delta)q_\ell}\right] \text{ is } (i+1, r, \xi^*, (1-\delta)q_\ell, W^t)\text{-Ramsey}\right) &\geq 1 - 2^{-b^*(1-\delta)q_\ell|W^t|}\\ &\geq 1 - 2^{-(b^*/2)q_\ell|W^t|}\,.\end{aligned}$$

Similarly as in the former case, we infer from Proposition 2.11 that with probability at least

$$1 - 2^{-\delta^2 q_\ell |W^t|/20} \overset{(15),(42)}{\geq} 1 - 2^{-\hat{b} q_\ell |V|} \tag{43}$$

the random set $W^t_{q_\ell}$ has the property that

$$H[W^t_{q_\ell} \setminus X] \text{ is } (i+1, r, \xi^*, (1-\delta)q_\ell, W^t)\text{-Ramsey} \tag{44}$$

for every $X \subseteq W^t_{q_\ell}$ with $|X| \leq \delta q_\ell |W^t|/2$.

Note that in the statement above only partitions into r classes are considered, while we have to deal with $(r+1)$-partitions here. Let $\chi_\ell\colon U_{q_\ell} \to [r+1]$ be an arbitrary partition. Depending on the cardinality of $\chi_\ell^{-1}(t) \cap W^t_{q_\ell}$ we will argue in two different ways. In fact, if

$$\left|\chi_\ell^{-1}(t) \cap W^t_{q_\ell}\right| \geq \frac{\delta}{4} q_\ell |W^t|\,, \tag{45}$$

then we infer from the fact that $W^t = \bigcup_{j=1}^{\ell-1} W^t_j$ and (28)

$$\begin{aligned}\left|E_U^{i+1}\left(\bigcup_{j=1}^{\ell} U^t_{q_j}\right)\right| &= \left|\left\{e \in E(U)\colon \left|e \cap \bigcup_{j=1}^{\ell} U^t_{q_j}\right| \geq i+1\right\}\right|\\ &\geq \frac{1}{k} \sum_{u \in \chi_\ell^{-1}(t) \cap W^t_{q_\ell}} \deg_{i,U}\left(u, \bigcup_{j=1}^{\ell} U^t_{q_j}\right)\\ &\geq \frac{1}{k} \sum_{u \in \chi_\ell^{-1}(t) \cap W^t_{q_\ell}} \max_{j \in [\ell-1]} \deg_{i,U}(u, U^t_{q_j})\\ &\geq \frac{1}{k} \cdot \frac{\delta}{4} q_\ell |W^t| \cdot \frac{\xi'(1-\delta)^i}{2} q_1^i \frac{|E|}{|V|}\,.\end{aligned}$$

Hence, if χ_ℓ satisfies (45), then since $q_1 \leq q_\ell$

$$\begin{aligned}\left|E_U^{i+1}\left(\bigcup_{j=1}^{\ell} U^t_{q_j}\right)\right| &\overset{(42)}{\geq} \frac{\delta\zeta^2\xi'(1-\delta)^i}{32kK} q_1^{i+1}|E|\\ &\overset{(23)}{\geq} \frac{\delta\zeta^2\xi'(1-\delta)^i}{32kK}\left(\frac{B-1}{B^{L-1}-1}\right)^{i+1} q^{i+1}|E| \overset{(19)}{\geq} \xi q^{i+1}|E|\,.\end{aligned}$$

In other words, if χ_ℓ satisfies (45), then the resulting partition satisfies conclusion $(ii.b)$ of Claim 1.

If, on the other hand, (45) does not hold, then setting

$$X = \left(\chi_\ell^{-1}(t) \cap W^t_{q_\ell}\right) \cup \bigcup_{j=1}^{\ell-1} U_{q_j}$$

we have

$$|X| \overset{(30)}{\leq} \frac{\delta}{4}q_\ell|W^t| + 2|U|\sum_{j=1}^{\ell-1} q_j \leq \frac{\delta}{4}q_\ell|W^t| + 2|V|\sum_{j=1}^{\ell-1} q_j \overset{(42)}{\leq} \frac{\delta}{4}q_\ell|W^t| + \frac{8K}{\zeta^2}|W^t|\sum_{j=1}^{\ell-1} q_j \overset{(25)}{\leq} \frac{\delta}{2}q_\ell|W^t|\,.$$

Since $W^t \subseteq U$ and $\chi_\ell^{-1}(s) \cap (W^t_{q_\ell} \setminus X) \subseteq \bigcup_{j=1}^{\ell} U^s_{q_j}$, it follows from (44) that there exists some $s \in [r+1] \setminus \{t\}$ such that

$$\Big|E_U^{i+1}\Big(\bigcup_{j=1}^{\ell} U^s_{q_j}\Big)\Big| \geq \big|E_{W^t}^{i+1}\big(\chi_\ell^{-1}(s) \cap (W^t_{q_\ell} \setminus X)\big)\big| \geq \xi^*(1-\delta)^{i+1}q_\ell^{i+1}|E| \overset{(23),(19)}{\geq} \xi q^{i+1}|E|\,,$$

which again implies conclusion (*ii.b*) of Claim 1.

Summarizing, it follows from (43), that in this case conclusion (*ii.b*) of Claim 1 holds for any $\chi_\ell \colon U_{q_\ell} \to [r+1]$ with probability at least $1-2^{-\hat{b}q_\ell|V|}$. This combined with (32) concludes the proof of Claim 1 , since

$$1 - 2^{-\hat{b}q_\ell|V|} - 2^{-2b_{\min}q_\ell|V|} \overset{(18)}{\geq} 1 - 2^{-b_{\min}q_\ell|V|}$$

for sufficiently large n. □

3. Proof of the new results

In this section we deduce Theorem 1.1 and Theorem 1.2 from Theorem 2.5.

Proof of Theorem 1.1. Note that due to (3) it suffices to verify the 1-statement of Theorem 1.1 and we will show that this follows from Theorem 2.5. Let A be an irredundant, partition regular $\ell \times k$ matrix. Without loss of generality we may assume that A has full rank, i.e., $\text{rank}(A) = \ell$. As mentioned in the introduction, it follows from Rado's characterization of partition regular matrices that $k \geq \ell + 2$.

For every $n \in \mathbb{N}$ we consider the k-uniform hypergraph $H_n = ([n], E_n)$ where the edges of H_n are the k-sets $\{x_1, \dots, x_k\} \subseteq [n]$ such that (for some ordering) the vector $(x_1, \dots, x_k)$ is a solution of $\mathcal{L}(A)$. (Note that we disregard solutions of $\mathcal{L}(A)$ which consist of less than k distinct integers). Let $p_n = n^{-1/m_A}$ (cf. (2)).

The conclusion of Theorem 2.5 yields Theorem 1.1, since (by definition) $H[V_{n,q_n}]$ is r-Ramsey if and only if $[n]_{q_n} \to (A)_r$ and we have to show that $\boldsymbol{H}$ and $\boldsymbol{p}$ satisfy the assumptions of Theorem 2.5. This means, we have to verify the the following

(*a*) $p_n = n^{-1/m_A} \to 0$ as $n \to \infty$,
(*b*) $p_n^k|E_n| \to \infty$ as $n \to \infty$,
(*c*) for every $R \in \mathbb{N}$ exists some $\zeta > 0$ such that $\boldsymbol{H}$ is (R, ζ)-Ramsey, and
(*d*) $\boldsymbol{H}$ is $(K, \boldsymbol{p})$-bounded for some $K \geq 1$.

Clearly, $p_n = n^{-1/m_A} \to 0$ due to the definition of m_A. It was shown in [29, Proposition 2.2 ii)] that $m_A \geq k-1$ and due to Rado's characterization we have $k - \ell \geq 2$, which yields $|E_n| = \Omega(n^2)$. Therefore, we have

$$p^k|E_n| = \Omega(n^{-k/(k-1)} \cdot n^2) = \Omega\big(n^{\frac{k-2}{k-1}}\big)\,.$$

Moreover, it follows from [4, Theorem 1], that for every $R \in \mathbb{N}$ there exists some $\zeta > 0$ for which $\boldsymbol{H}$ is (R, ζ)-Ramsey. Consequently, it suffices to verify that $\boldsymbol{H}$ is

$(K, \boldsymbol{p})$-bounded for some $K \geq 1$. For $i \in [k-1]$ and $q \geq n^{-1/m_A}$ we have to show that

$$\mu_i(H_n, q) = O\left(q^{2i}\frac{|E_n|^2}{n}\right).$$

Recalling the definition of $\mu_i(H_n, q)$ in (7) and $H_n = ([n], E_n)$ we have

$$\mu_i(H_n, q) = \mathbb{E}\left[\sum_{x\in[n]} \deg_i^2(x, V_{n,q})\right] = \sum_{x\in[n]} \mathbb{E}\left[\deg_i^2(x, V_{n,q})\right]. \tag{46}$$

Note that $\mathbb{E}\left[\deg_i^2(x, V_{n,q})\right]$ is the expected number of pairs $(X, Y) \in [n]^k \times [n]^k$ such that

(*i*) $x \in X \cap Y$,

(*ii*) $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ are solutions of $\mathcal{L}(A)$, where

$$A\boldsymbol{x} = A\boldsymbol{y} = \boldsymbol{0}$$

for $\boldsymbol{x} = (x_1, \dots, x_k)^t$ and $\boldsymbol{y} = (y_1, \dots, y_k)^t$, and

(*iii*) $|X \cap ([n]_q \setminus \{x\})| \geq i$ and $|Y \cap ([n]_q \setminus \{x\})| \geq i$.

For fixed x and (X, Y) let $w \geq 1$ be the largest integer such that there exist indices $i_1, \dots, i_w$ and $j_1, \dots, j_w$ for which

$$x_{i_1} = y_{j_1}, \dots, x_{i_w} = y_{j_w}. \tag{47}$$

Consequently,

$$x \in \{x_{i_1}, \dots, x_{i_w}\} = \{y_{j_1}, \dots, y_{j_w}\} \tag{48}$$

Set $W_1 = \{i_1, \dots, i_w\}$ and $W_2 = \{j_1, \dots, j_w\}$.

For fixed sets $W_1, W_2 \subseteq [k]$ we are going to describe all $(2k-w)$-tuples $X \cup Y$ satisfying (*ii*) and (47). To this end consider the $2\ell \times (2k-w)$ matrix B, which arises from two copies A_1 and A_2 of A with permuted columns. We set $A_1 = (A_{\overline{W}_1} \mid A_{W_1})$ and $A_2 = (A_{W_2} \mid A_{\overline{W}_2})$ where for every $\alpha = 1, \dots, w$ the column of A_{W_1} which is indexed by i_α aligns with that column of A_{W_2} which is indexed by j_α. Then let

$$B = \left(\begin{array}{c|c|c} A_{\overline{W}_1} & A_{W_1} & \boldsymbol{0} \\ \hline \boldsymbol{0} & A_{W_2} & A_{\overline{W}_2} \end{array}\right).$$

Without loss of generality we may assume that $\operatorname{rank}(A_{\overline{W}_1}) \geq \operatorname{rank}(A_{\overline{W}_2})$ and, therefore,

$$\operatorname{rank}(B) \geq \operatorname{rank}(A) + \operatorname{rank}(A_{\overline{W}_1}).$$

Clearly, the number of $(2k-w)$-tuples $X \cup Y$ satisfying (*ii*) and (47) equals the number of solutions of the homogeneous system given by B, which is $O(n^{2k-w-\operatorname{rank}(B)})$. Since A is an irredundant, partition regular matrix, it follows from [29, Proposition 2.2 i)] that $\operatorname{rank}(A') = \operatorname{rank}(A)$ for every matrix A' obtained from A by removing one column. Consequently, any matrix B' obtained from B by removing one of the middle columns (i.e., one of the w columns of B which consist of a column of A_{W_1} and a columns of A_{W_2}) satisfies

$$\operatorname{rank}(B') \geq \operatorname{rank}(A) + \operatorname{rank}(A_{\overline{W}_1}) = \ell + \operatorname{rank}(A_{\overline{W}_1}).$$

Therefore, it follows from (48) that the number of such $(2k-w)$-tuples that also satisfy condition (*i*) for some fixed $x \in [n]$ is at most

$$O(n^{2k-w-1-\ell-\operatorname{rank}(A_{\overline{W}_1})}). \tag{49}$$

Finally, we estimate the probability that a $(2k-w)$-tuple $X \cup Y$ satisfying (i), (ii), and (47) also satisfies (iii). Since $|X \cap Y \cap ([n]_q \setminus \{x\})| = j \leq w-1$ and $q \leq 1$ this probability is bounded by

$$\sum_{j=0}^{w-1} q^{2i-j} = O(q^{2i-w+1}).$$

In view of (49) we obtain

$$\sum_{x\in[n]} \mathbb{E}\left[\deg_i^2(x, V_{n,q})\right]$$
$$= \sum_{x\in[n]} \sum_{w=1}^{k} \sum_{\substack{W_1,W_2\subseteq[k]\\|W_1|=|W_2|=w}} O(n^{2k-w-1-\ell-\mathrm{rank}(A_{\overline{W}_1})} q^{2i-w+1}). \quad (50)$$

Note that if $w = 1$, then again due to [29, Proposition 2.2 i)] we have $\mathrm{rank}(A_{\overline{W}_1}) = \ell$ and, therefore, the contribution of those terms satisfies

$$\sum_{x\in[n]} \sum_{\substack{W_1,W_2\subseteq[k]\\|W_1|=|W_2|=1}} O(n^{2k-2\ell-2} q^{2i}) = O(n^{2k-2\ell-1} q^{2i}) = O\left(q^{2i}\frac{|E_n|^2}{n}\right). \quad (51)$$

For $w \geq 2$ and $W_1 \subseteq [k]$ with $|W_1| = w$ we obtain from the definition of m_A and $q \geq n^{-1/m_A}$ that

$$q^{w-1} \geq n^{-w+1-\mathrm{rank}(A_{\overline{W}_1})+\ell}.$$

Consequently,

$$\sum_{x\in[n]} \sum_{w=2}^{k} \sum_{\substack{W_1,W_2\subseteq[k]\\|W_1|=|W_2|=w}} O(n^{2k-w-1-\ell-\mathrm{rank}(A_{\overline{W}_1})} q^{2i-w+1})$$
$$= \sum_{x\in[n]} \sum_{w=2}^{k} \sum_{\substack{W_1,W_2\subseteq[k]\\|W_1|=|W_2|=w}} O(n^{2k-2-2\ell} q^{2i})$$
$$= O(n^{2k-2\ell-1} q^{2i}) = O\left(q^{2i}\frac{|E_n|^2}{n}\right). \quad (52)$$

Finally, combining (46), (50), (51), and (52) we obtain

$$\mu_i(H_n, q) = O\left(q^{2i}\frac{|E_n|^2}{n}\right),$$

which concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let F be an ℓ-uniform hypergraph with $k = e(F) \geq \Delta(F) \geq 2$ edges. For every $n \in \mathbb{N}$ we consider the k-uniform hypergraph $H_n = (V_n, E_n)$ with $V_n = E(K_n^{(\ell)})$ and edges of H_n correspond to (unlabeled) copies of F in $K_n^{(\ell)}$. Furthermore, let $p_n = n^{-1/m_F}$ (cf. (6)). It is easy to see that the conclusion of Theorem 2.5 yields Theorem 1.2, i.e., $H[V_{n,q_n}]$ is r-Ramsey if and only if $G^{(\ell)}(n, q_n) \to (F)_r$. Therefore, it is left to show that $\boldsymbol{H}$ and $\boldsymbol{p}$ satisfy the assumptions of Theorem 2.5 (see properties (a)-(d) in the proof of Theorem 1.1).

Note that it follows directly from the definition of $\boldsymbol{H} = (H_n)_{n\in\mathbb{N}}$ and $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ that $p_n \to 0$ and $p_n^k|E_n| = \Omega(n)$, since $\Delta(F) \geq 2$. Moreover, the original proof of Ramsey's theorem implies that for every $R \in \mathbb{N}$ there exists some $\zeta > 0$ such that the sequence $\boldsymbol{H} = (H_n)_{n\in\mathbb{N}}$ is (R,ζ)-Ramsey. Consequently, it is left to verify that $\boldsymbol{H}$ is $(K,\boldsymbol{p})$-bounded for some constant $K \geq 1$.

To this end observe that H_n is a regular hypergraph with $\binom{n}{\ell}$ vertices and every vertex is contained in $\Theta(n^{v(F)-\ell})$ edges and that $|E_n| = \Theta(n^{v(F)})$. We will show that for $q \geq n^{-1/m_F}$ and $i \in [k-1]$ we have

$$\mu_i(H_n,q) = \mathbb{E}\left[\sum_{v\in V_n} \deg_i^2(v,V_{n,q})\right] = \sum_{v\in V} \mathbb{E}\left[\deg_i^2(v,V_{n,q})\right] = O\left(q^{2i}\frac{|E_n|^2}{|V_n|}\right).$$

Due to the definition of $\boldsymbol{H}$ every $v \in V_n$ corresponds to an edge $e(v)$ in $K_n^{(\ell)}$. Therefore, the number $\mathbb{E}\left[\deg_i^2(v,V_{n,q})\right]$ is the expected number of pairs (F_1,F_2) of copies F_1 and F_2 of F in $K_n^{(\ell)}$ satisfying $e(v) \in E(F_1)\cap E(F_2)$ and both copies F_1 and F_2 have at least i edges in $E(G^{(\ell)}(n,q)) \setminus \{e(v)\}$. Summing over all such pairs F_1 and F_2 we obtain

$$\begin{aligned}\mathbb{E}\left[\deg_i^2(v,V_{n,q})\right] &\leq \sum_{F_1,F_2\colon\, e(v)\in E(F_1)\cap E(F_2)} \sum_{j=0}^{|E(F_1)\cap E(F_2)|-1} q^{2i-j} \\ &= O\left(\sum_{F_1,F_2\colon\, e(v)\in E(F_1)\cap E(F_2)} q^{2i-(|E(F_1)\cap E(F_2)|-1)}\right)\end{aligned} \tag{53}$$

since $q \leq 1$. Furthermore,

$$\begin{aligned}&\sum_{F_1,F_2\colon\, e(v)\in E(F_1)\cap E(F_2)} q^{2i-(|E(F_1)\cap E(F_2)|-1)} \\ &\qquad\qquad = O\left(\sum_{J\colon\, e(v)\in E(J} n^{2v(F)-2v(J)}q^{2i-(e(J)-1)}\right),\end{aligned} \tag{54}$$

where the sum on the right-hand side is indexed all hypergraphs $J \subseteq K_n^{(\ell)}$ which contain $e(v)$ and which are isomorphic to a subhypergraph of F. It follows from the definition of m_F and $q \geq n^{-1/m_F}$ that $n^{v(J)}q^{e(J)} = \Omega(qn^\ell)$. Combining this with (53) and (54) we obtain

$$\begin{aligned}\mathbb{E}\left[\deg_i^2(v,V_{n,q})\right] &= O\left(\sum_{J\colon\, e(v)\in E(J)} n^{2v(F)-2v(J)}q^{2i-(e(J)-1)}\right) \\ &= O\left(\sum_{J\colon\, e(v)\in E(J)} n^{2v(F)-v(J)-\ell}q^{2i}\right).\end{aligned}$$

Moreover, since $v(J) \geq \ell$ we have

$$\mathbb{E}\left[\deg_i^2(v,V_{n,q})\right] = O\left(\sum_{J\colon\, e(v)\in E(J)} n^{2v(F)-2\ell}q^{2i}\right),$$

and, consequently,

$$\mu_i(H_n, q) = \sum_{v \in V_n} O(n^{2v(F)-2\ell} q^{2i}) = O(n^{2v(F)-\ell} q^{2i}) = O\left(q^{2i} \frac{|E_n|^2}{|V_n|}\right),$$

which concludes the proof of Theorem 1.2. □

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