A simple proof of Bazzi’s theorem

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Abstract

In 1990, Linial and Nisan asked if any polylog-wise independent distribution fools any function in $\text{AC}^0$. In a recent remarkable development, Bazzi solved this problem for the case of DNF formulas. The aim of this note is to present a simplified version of his proof.

In the 1990s, it was shown in a series of papers [LMN93, BRS91, ABFR94] that Boolean functions computable by constant depth polynomial size circuits can be well approximated (in various contexts) by low degree polynomials. Around the same time, Linial and Nisan [LN90] conjectured that any such function can be fooled by a polylog-wise\(^1\) independent probability distribution. By linear duality, this conjecture is an approximation problem of precisely the kind considered in [LMN93, BRS91, ABFR94]. Therefore, it is quite remarkable that the only noticeable progress in this direction was achieved only last year by Bazzi [Baz07]. Namely, he showed that any DNF formula of polynomial size is fooled by (any) $O(\log n)^2$-independent distribution. We refer the reader to [Baz07] for motivations and applications of this result; the purpose of this note is to give a simplified version of Bazzi’s proof.

For a probability distribution $\mu$ on $\{0, 1\}^n$ and a function $f : \{0, 1\}^n \rightarrow \mathbb{R}$, $E_\mu(f)$ is the expected value of $f$ w.r.t. this distribution (in particular, if $f :\)

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\(^1\)As literally stated in [LN90] the conjecture is false [LV96], so we relax the parameters appropriately.
\{0,1\}^n \longrightarrow \{0,1\} is a Boolean function then \( E_\mu(f) = \Pr_{x \sim \mu}[f(x) = 1] \) is the probability that \( f(x) = 1 \). If \( \mu \) is uniform on \( \{0,1\}^n \), \( E_\mu(f) \) is abbreviated to \( E(f) \). The bias of \( f \) w.r.t. \( \mu \) is defined as \( |E_\mu(f) - E(f)| \), and for an integer \( k \geq 0 \), \( \text{bias}(f;k) \overset{\text{def}}{=} \max_{\mu} |E_\mu(f) - E(f)| \), where the maximum is taken over all \( k \)-independent probability distributions on \( \{0,1\}^n \).

In this note we give a simplified proof of the following theorem:

**Theorem 1 (Bazzi [Baz07])** If the Boolean function \( f : \{0,1\}^n \longrightarrow \{0,1\} \) is computable by an \( m \)-term DNF formula then \( \text{bias}(f;k) \leq m O(1) \exp(-\Omega(\sqrt{k})) \).

From now on we will identify a DNF formula \( F = A_1 \lor \ldots \lor A_m \) and the Boolean function it represents. The first step in the proof of Theorem 1 is to reduce the problem to the case when every conjunctive term \( A_i \) has only a few variables, that is \( F \) is an \( s \)-DNF for a sufficiently small \( s \). This simple step is borrowed from [Baz07] without any changes:

**Lemma 2 ([Baz07])** Let \( k \geq s \geq 1 \) be integers, and \( F \) be an \( m \)-term DNF. Then 
\[
\text{bias}(F;k) \leq \max_G \text{bias}(G;k) + m 2^{-s},
\]
where the maximum is taken over all \( m \)-terms \( s \)-DNF \( G \).

The next relatively simple step in Bazzi’s proof that we also reproduce here without alterations is to estimate the bias of an \( s \)-DNF \( F \) in terms of a constrained version of \( \ell_2 \)-approximation by low degree polynomials called in [Baz07] zero-energy. Let us first recall the unconstrained version.

**Definition 3** For a function \( f : \{0,1\}^n \longrightarrow \mathbb{R} \) and an integer \( t \geq 0 \), let 
\[
\text{energy}(f;t) \overset{\text{def}}{=} \min_{\deg(g) \leq t} E((f - g)^2).
\]

This quantity is equal to the sum of squares \( \sum_{|S| > t} \hat{f}(S)^2 \) of high order Fourier coefficients of \( f \). But we do not need this interpretation in our proof, besides making connection to the following celebrated result by Linial, Mansour and Nisan [LMN93]:

**Lemma 4 ([LMN93])** If \( f \) is a Boolean function computable by an \( \{\neg,\land,\lor\} \)-circuit of size \( m \) and depth \( d \) then for any \( t > 0 \), 
\[
\text{energy}(f;t) \leq 2m \cdot 2^{-t^{1/d}/20}.
\]
Definition 5 ([Baz07])

\[
\text{zeroEnergy}(f; t) \overset{\text{def}}{=} \min_{\deg(g) \leq t} E((f - g)^2),
\]

where this time the minimum is taken over all degree \( \leq d \) polynomials \( g \) that satisfy one additional zero-constraint: \( g(x) = 0 \) whenever \( f(x) = 0 \) \( (x \in \{0, 1\}^n) \).

Clearly, \( \text{energy}(f; t) \leq \text{zeroEnergy}(f; t) \). Also, bias is related to zero-energy with the following lemma:

Lemma 6 ([Baz07]) Let \( F \) be an \( m \)-term \( s \)-DNF formula and let \( k \geq s \) be an integer. Then

\[
\text{bias}(F; k) \leq m \cdot \text{zeroEnergy}(F; \lfloor (k - s)/2 \rfloor).
\]

In the opposite direction, bounding zero-energy in terms of energy of certain auxiliary functions is where the bulk of work is done in Bazzi’s proof. And this is where our simplification comes in:

Theorem 7 Let \( F \) be an \( m \)-term \( s \)-DNF and \( t \) be an integer. Then

\[
\text{zeroEnergy}(F; t) \leq m^2 \cdot \max_G \text{energy}(G; t - s),
\]

where the maximum is again taken over all \( m \)-term \( s \)-DNF formulas \( G \).

Proof. Let \( F = A_1 \lor \ldots \lor A_m \), where \( A_i \) are conjunctive terms of size \( \leq s \) each. We claim that \( F \) can be expressed in the form

\[
F = \sum_{i=1}^{m} A_i(1 - E[G_i]),
\]

where \( G_i \) are specially constructed random sub-DNFs of \( F \) and the expectation sign is understood pointwise: \( E[G_i](x) \overset{\text{def}}{=} E[G_i(x)] \ (x \in \{0, 1\}^n) \). But before exhibiting the distributions of \( G_i \) with this property, let us see why their mere existence already implies the statement of Theorem 7.

Indeed, denoting the maximum \( \max_G \text{energy}(G; t - s) \) in (1) by \( \epsilon \), we have (random) polynomials \( g_i \) of degree \( \leq t - s \) such that with probability one we have the bound \( E((G_i - g_i)^2) \leq \epsilon \). And now we simply let

\[
g \overset{\text{def}}{=} \sum_{i=1}^{m} A_i(1 - E[g_i]).
\]
Since every term $A_i$ has at most $s$ variables, $\deg(g) \leq t$. $F(x) = 0$ implies $\forall i \in [m](A_i(x) = 0)$ which in turn implies $g(x) = 0$. Therefore, $g$ satisfies the zero-constraint. And we bound the $\ell_2$-distance between $F$ and $g$ as follows:

$$E((F - g)^2) = E\left(\left(\sum_{i=1}^{m} A_i \cdot E[G_i - g_i]\right)^2\right)$$

$$\leq \text{Cauchy-Schwartz} \quad m \cdot \sum_{i=1}^{m} E\left((A_i \cdot E[G_i - g_i])^2\right)$$

$$\leq \text{since } |A_i| \leq 1 \quad m \cdot \sum_{i=1}^{m} E\left(E[G_i - g_i]^2\right)$$

$$\leq \text{Cauchy-Schwartz} \quad m \cdot \sum_{i=1}^{m} E\left(E[(G_i - g_i)^2]\right)$$

$$= m \cdot \sum_{i=1}^{m} E\left(E[(G_i - g_i)^2]\right) \leq \epsilon m^2.$$

It remains to exhibit $G_1, \ldots, G_m$ such that the identity (2) holds. For that purpose, we first pick $p \in [0, 1]$ uniformly at random. And then we let $G_i$ be the sub-DNF of $(A_1 \lor \ldots \lor A_{i-1} \lor A_{i+1} \lor \ldots \lor A_m)$ in which every term is removed, independently of others, with probability $p$ and kept alive with probability $1 - p$.

Fix an input $x \in \{0, 1\}^n$, and let $w \defeq |\{i \in [m] | A_i(x) = 1\}|$. If $w = 0$ then both sides of (2) are equal to 0.

If, on the other hand, $w > 0$ then there are precisely $w$ non-zero terms in the expression $\sum_{i=1}^{m} A_i(x)(1 - E[G_i](x))$. And every one of them contributes to the sum precisely

$$\int_0^1 (1 - E[G_i(x)|p = p])dp = \int_0^1 P[G_i(x) = 0|p = p]dp = \int_0^1 p^{w-1}dp = \frac{1}{w}.$$

Thus, $\sum_{i=1}^{m} A_i(x)(1 - E[G_i](x)) = 1$ ($w > 0$), and this completes the proof of (2) and of Theorem 7. ■

Like in Bazzi’s proof, Theorem 1 immediately follows from Lemma 2, Lemma 6, Theorem 7 and Lemma 4.

**Remark.** After the preliminary version of this note was disseminated, Avi Wigderson observed that the proof can be further simplified by (deterministically!) letting $G_i$ in (2) be equal $A_1 \lor \ldots \lor A_{i-1}$. This is definitely
simpler, but our version has the potential advantage of being more symmetric.

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References


