

1 From groups to graphs

Setting: G, H, X . G a group, H a **subgroup**; X a set of generators for H . Often we assume X is *symmetric*: $1 \in X, X = X^{-1}$.

We have an equivalence relation \mathbf{H} , the (left) coset equivalence relation; and a graph \mathbf{X} , the Cayley graph. \mathbf{H} is generated by \mathbf{X} .

Transpose to: G a set; H an **equivalence relation**. $X \subset G^2$ a generating set. *Symmetric:* the diagonal on G is contained in X ; and $X = X^t$. So $H = \cup X \circ \dots \circ X$.

In both cases, an associated **metric** $d = d_X$, generated by: $d(x, y) \leq 1$ if $(x, y) \in X$. If we are given a family X_i , let $d(x, y) \leq n$ iff there exist $x = x_1 \dots x_n = y$ with $(x_k, x_{k+1}) \in \cup X_i$.

Definition 1.1. A symmetric $X \subset G^2$ is a ***k-approximate equivalence relation*** if the valency is of a fixed order of magnitude $|X(a)| \leq k|X(b)|$ for all $a, b \in G$, and *every 2-ball is a union of k 1-balls*.

- Say two metrics d, d' are ***k-commensurable at scale α*** if an α -ball of d' is contained in $\leq k$ balls of d -radius α , and vice versa.
- A metric space is ***k-doubling at scale α*** if $d, (1/2)d$ are k -commensurable at scale α .
- Thus: for a k - ***approximate equivalence relation*** X , d_X is k -doubling at scale 1.
- X is an ***approximate subgroup*** of G iff \mathbf{X} is an *approximate equivalence relation* on G .

Theorem 1.2 (strong approximation: groups). Let $F = \mathbb{F}_p$, p nonstandard. Let $G = GL_n(F)$, and let X_i be a family of definable subsets. Then there exists a definable H such that:

1. H is a subgroup of G
2. $H \subset \langle \cup X_i \rangle$.
3. X_i/H is finite (bounded.)

Moreover, 'definable' here can be made explicit as follows: there exists a homomorphism of algebraic groups (with bounded data), with finite kernel

$$h : \tilde{H} \rightarrow G$$

such that $H = h(\tilde{H}(F))$.

Applied to the family of one-dimensional unipotent subgroups X_i of an arbitrary subgroup Γ of G , this shows that Γ contains a *definable* normal subgroup H with Γ/H Abelian-by-bounded. Thus the image of a Zariski dense subgroup of $SL_n(\mathbb{F}_p)$ has bounded index, and is as above. (Weisfeiler 1984, Nori 1987, Gabber, precedents by Eichler 1938, Kneser 1966,...).

Theorem 1.3 (strong approximation: graphs). *Let $F = \mathbb{F}_p$, p nonstandard, G definable over F . Let $X \subset G^2$ be definable. Then there exists $m, m' \in \mathbb{N}$ and a 0-definable $H \leq G^2$ such that:*

1. H is an equivalence relation on G
2. If $(a, b) \in H$ then $d_X(a, b) \leq m$.
3. X/H has valency (degree) $\leq m'$.

Moreover, H is algebraic: $(a, b) \in H$ iff $\phi(h^{-1}(a)) = \phi(h^{-1}(b))$ for some morphism of varieties $h : \tilde{G} \rightarrow G$ with finite fibers, and regular functions ϕ on \tilde{G} , with $\phi \circ h^{-1}$ well-defined.

Generate as long as dimension increases; then show there are no definable approximate equivalence relations.

If stated for standard primes: the bounds m, m' on valency and on diameter are independent of p ; H varies through only finitely many possible definitions (given G, X); and the complexity of h, \tilde{G}, ϕ is bounded independently of p .

Example 1.4. Fix an algebraic group G_0 , e.g. $G_0 = SL_k$. let $F = GF(p)$, let Ω_1, Ω_2 be unipotent orbits in $G_0(F)$, and make $\Gamma = \Omega_1 \times \Omega_2$ into a graph by letting (a, b) be adjacent to $(a, a^{-1}ba)$ and to $(b^{-1}ab, b)$.

Invariants of the connected components:
the group $\langle a, b \rangle$ generated by (a, b) . The fact that this is a definable invariant is the strong approximation lemma for groups!

The trace $tr(ab)$.

Further algebraic invariants.

For pairs (a, b) that generate SL_k , I do not know if further invariants are needed; Gamburd and Sarnak have results on related graphs.

2 Stabilizers

Theorem 2.1 (H. Sanders 2009). *Let X be a k -approximate group. Then there exists Y with $Y^8 \subset X^4$, X contained in boundedly many cosets of Y .*

Theorem 2.2. *Let X be a k - approximate equivalence relation on G . Then there exists $S \subset G^2$ such that $S^{\circ 8} \subset X^{\circ 4}$, and for all $a \in \Omega$ outside an ϵ -slice U , $|S(a)| \geq O_k(1)|X(a)|$.*

Moreover S is 0-definable, uniformly in (Ω, X) , in an appropriate logic; in particular $\text{Aut}(\Omega, X)$ leaves U, S invariant.

The invariance implies the group-theoretic version.

The 0-definability of S will be essential, in moving from the approximate symmetry of a graph to automorphisms of an associated locally compact space.

Closely related to Lovász-Szegedy *graphons* in cases of bounded diameter at least. However, to know that the definition of $W \circ W$ agrees with the one we give on elements, we need to know essentially the independence theorem; so it does not appear to give a new proof of the stabilizer theorem.

3 Approximate symmetry

A distance between finite graphs: (Keisler-Hoover, Gromov (measured metric spaces), Benjamini-Schramm)

Definition 3.1.

$$\rho(\Omega, \Omega') = \sup\left\{\frac{1}{m} : (\exists \Gamma) |\Gamma| = m, |Pr(\Gamma, \Omega) - Pr(\Gamma, \Omega')| \geq \frac{1}{m}\right\}$$

Where $Pr(\Gamma, \Omega) = |Hom(\Gamma, \Omega)|/\Omega^m$.

A similar definition applies to *pointed graphs*.

Definition 3.2. (Ω, X) is ϵ -homogeneous if $\rho_{ptd}((\Omega, a), (\Omega, b)) \leq \epsilon$ for all $a, b \in \Omega$.

Definition 3.3. $\Omega_n \rightarrow \Omega$ if $\rho_{ptd}(\Omega_n, a_n), (\Omega, a) \rightarrow 0$ for all $a_n \in \Omega_n, a \in \Omega$.

Of course, this only makes sense if the Ω_n are increasingly ϵ -homogeneous.

We will really have a stronger notion of convergence: there will be a metric d on Ω_n , $d(x, y) = 1$ iff $(x, y) \in X$, such that Gromov-Hausdorff convergence holds with respect to the metrics.

A **Riemannian homogeneous space** is a Riemannian manifold, with transitive isometry group (Classified by Wolf when the stabilizer acts irreducibly on tangent space.)

A **Riemannian model** is a Riemannian homogeneous space, with compact point stabilizer, and with the approximate equivalence relation: $d(x, y) \leq 1$.

Riemannian models have ϵ -homogeneous approximations for any ϵ . Let G/K be a Riemannian homogeneous space; G a Lie group, K compact. Let Λ be a lattice of large covolume. Let n be large, and choose n points at random on $\Lambda \backslash G/K$.

Theorem 3.4. *Let (G_n, X_n) an approximately homogeneous sequence of approximate equivalence relations. Then some subsequence approaches a limit (Γ, X) , admitting a homomorphism to a vertex transitive graph B of bounded degree, such that each fiber is commensurable to a Riemannian model.*

4 Partial Bourgain systems

Theorem 4.1. *Fix $k \in \mathbb{N}$. Then there exists $e^* \in \mathbb{N}$ such that the following holds: Let G be any group, X a finite subset, and assume $|XX^{-1}X| \leq k|X|$.*

Then there are $2 \leq e, c \leq e^$, and $N > 2^{2^{2^{ec}}}$ subsets $X_N \subseteq X_{N-1} \subseteq \dots \subseteq X_1 \subseteq X^{-1}XX^{-1}X$ such that X, X_1 are e -commensurable, and for $1 \leq m, n < N$ we have:*

1. $X_n = X_n^{-1}$
2. $X_{n+1}X_{n+1} \subseteq X_n$
3. X_n is contained in the union of c translates of X_{n+1} .
4. $[X_n, X_m] \subseteq X_k$ whenever $k \leq N$ and $k < n + m$.

Theorem 4.2. *Fix $k \in \mathbb{N}$. Then there exists $e^* \in \mathbb{N}$ and $\epsilon > 0$ such that the following holds: Let (G, X) be an ϵ -homogeneous approximate equivalence relation. Then there are $e, c \leq e^*$ and $N > 2^{2^{2^{eck}}}$, and a metric d_N on X such that:*

X -balls are covered by $\leq e$ d_N -balls of radius 1, while d_N -balls of radius 1 are contained in X -balls of radius 4. (and so in k^4 X -balls of radius 1); and for $1 \leq m, n < N$, d_N is c -doubling at scale 2^{-n} , i.e. d_N -balls of radius 2^n are contained in c balls of radius 2^{n+1} .

Problem 4.3. *Complete this with an analogue of (4).*

Problem 4.4. *Exploit homogeneity on types to obtain a statement without the approximate homogeneity assumption. (embedding of sections into a Riemannian homogeneous space.)*

5 Proof of stabilizer lemma

- $xS_n y$ iff $\mu\{z : |\mu(R(x) \cap R(z)) - \mu(R(y) \cap R(z))| \geq 2^{-n}\} \leq 2^{-n}$
- At limit, $\cap_n S_n$: for almost all z , $\mu(R(x) \cap R(z)) = \mu(R(y) \cap R(z))$. It is cobounded.

- $S_{n+1} \circ S_{n+1} \subset S_n$. (Away from measure 0).
- $S_n \subset R^{\circ 4}$, for large n .
- S_n is definable in terms of R using *probability logic*. This definability will be essential, showing that (approximate) symmetries of the graph, are (approximate) symmetries of the associated refining metric.
- The proof uses stability: $\mu(R(x) \cap R(z))$ is a stable real-valued formula. New proofs of this by Tao.

6 Approximately homogeneous approximate equivalence relations (proof)

- Ultraproduct. Obtain two equivalence relations: \tilde{E} = finite distance. Γ = infinitesimal distance.
- Let Ω be a class of \tilde{E} ; then Ω/Γ is locally compact.
- $G := \text{Aut}(\Omega/\Gamma)$ acts transitively on Ω , by isometries of the fine metric. Keisler, Gromov-Vershik,
- A locally compact structure on G (compact-open topology.) The stabilizer of a point is compact.
- By Gleason-Yamabe, an open subgroup H , a small normal compact subgroup N , with H/N a Lie group.
- From Ω to an H -orbit: locally bounded distortion. (R induces a graph of bounded degree on Ω/H .)
- Factor out N . Obtain a coarser equivalence relation than the original distance-zero, but still contained in $d_R \leq 4$.
- Now the Lie group H/N acts transitively on Ω/Γ , compact point stabilizer. Find an invariant Riemannian metric. This metric is doubling up to distance 1, and the distance 1 relation is commensurable with d_R .
- For partial Bourgain systems: return information to finite factors, up to scale $\Psi(c)$.

7 Comparison

Theorem 7.1 (Benjamini- Finucane-Tessera 2012). *1. Let (X_n) be an unbounded sequence of finite, connected, vertex transitive graphs with bounded degree such that $|X_n| = o(\text{diam}(X_n)^q)$ for some $q > 0$. After rescaling by the diameter, some subsequence converges in the Gromov Hausdorff distance to a torus of dimension $< q$, with an invariant metric.*

2. If q is close to 1, then the scaling limit of (X_n) is S^1 , even if X_n is only roughly transitive