Definable sets over valued fields

Ehud Hrushovski

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Plan

- Review of basics on definable sets.
- Imaginaries. Joint work with Deirdre Haskell, Dugald Macpherson (monograph), Ben Martin (ArXiv)
- Topology. Joint work with François Loeser. (ArXiv, F.L. web page.)
- Definable types and generically stable types.
- Geometric imaginaries: sketch of proof.
- Topological finiteness: rough structure of proof.

Setting

K denotes a valued field.

- Algebraic varieties V. $V(K)$ = points of V in a field K. For most of this talk, can think of V as affine, $V(K) = \{x \in K^n : f_1(x) = \cdots = f_k(x) = 0\}$.
- A *semi-algebraic* or *constructible* $Z \subset V$ is defined by valuation inequalities such as $\mathrm{val} f \geq \mathrm{val} g$; again $Z(K) = \{x \in V(K) : \mathrm{val} f \geq \mathrm{val} g\}$, etc.
- $\mathcal{O}$ is defined by: $\mathrm{val} x \geq 0$.
- $(\Gamma, +, <)$ denotes the value group, val the valuation map. $\Gamma_\infty = \Gamma \cup \{\infty\}$.
- k is the residue field; $\mathrm{res} : \mathcal{O} \to k$ the residue map.
- For $a \in K$ and $\gamma \in \Gamma$ denote $B_{\geq\gamma}(a)$ (resp. $B_{>\gamma}(a)$) the **closed** (resp. **open**) **ball** of valuative radius γ around a.

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Geometric imaginaries

- $S_n := GL_n/GL_n(\mathcal{O}) \cong B_n/B_n(\mathcal{O})$.
- $T_n := GL_n/GL_n(\mathcal{O})^o$, where:
  $1 \to GL_n(\mathcal{O})^o \to GL_n(\mathcal{O}) \to GL_n(k) \to 1$ exact.
- A *definable subset* of S_n or T_n is the image of a definable subset of GL_n. A *definable map* $U \to V$ is a definable subset f of $U \times V$, that always defines a function.

$n = 1$: Γ and k

- $\Gamma := S_1 = GL_1/GL_1(\mathcal{O})$.
- A linearly ordered group: $+, <$ are definable (their pullbacks are $\cdot, x \in \mathcal{O}y$.)
- pure / QE: Any definable subset of Γ^n is a Boolean combination of $\mathbb{Q}$-linear inequalities.
- A natural topology, determined by the ordering. $\Gamma_\infty := \Gamma \cup \{\infty\}$.
- $\mathrm{k} = \mathcal{O}/\mathcal{M}$; $\mathrm{k}^* = GL_1(\mathcal{O})/GL_1(\mathcal{O})^o$; a pure field.
- $RV := T_1 = GL_1/GL_1(\mathcal{O})^o$ also has a definable set structure that can be explicitly described;

$$1 \to k^* \to GL_1/GL_1(\mathcal{O})^o \to \Gamma$$

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$\mathbb{Q}_p$

We will occasionally consider $Th(\mathbb{Q}_p)$, where quantifers range over $\mathbb{Q}_p$ and not over the algebraic closure. The principal difference is that Γ is now discrete; QE still holds if *arithmetic sequences* are added to the basic structure.

Elimination of imaginaries

Theorem (H., Haskell, Macpherson)

Let $X \subset U \times V$ be semi-algebraic. Let $X_u = \{v : (u,v) \in X\}$. Then there exists a definable map $f : U \to S_n \times T_n \times \mathbb{A}^n$ such that

$$X_u = X_v \iff f(u) = f(v)$$

- Equivalent statement: Let $E \subset U^2$ be a semi-algebraic equivalence relation. Then there exists n, a definable subgroup $H \leq GL_n(\mathcal{O})$ as above, and a definable embedding $U/E \to GL_n/H$.
- The same result holds for definabity in $\mathbb{Q}_p$. In this case, only the S_n are needed. (H.-Martin)
- Probably also for ultraproducts of the $\mathbb{Q}_p$. (Certain cases, conjectured by Cluckers-Denef, proved.)
- All proofs use same strategy: study germs for definable types; geometry of definable types in terms of generically stable types. To be explained.

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Elimination of imaginaries

Corollary (Rationality)

Let $X \subset \Gamma \times U$, $E \subset \Gamma \times U \times U$ be $Th(\mathbb{Q}_p)$-definable, such that E_n is an equivalence relation on X_n, with a finite number of classes $\alpha(n)$.
Then piecewise, $\alpha(n)$ is an exponential polynomial $\sum b_{kl} n^k p^{ln}$.

Piecewise: divide $\mathbb{N}$ according to residue mod some M, with a finite exceptional set. Combinatorial formulation: $\sum \alpha(n) t^n$ is rational.

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Proof of corollary: counting classes of definable equivalence relations

- Denef (1984) showed the same statement for p-adic integrals $\beta(n) = \int_{\mathbb{Q}_p^m} f(x, n)dx$ varying definably with $n \in \Gamma$.
- Denef's theorem is now understood as part of motivic integration; cf. Scanlon's talk. It can be shown via iterated integration, reduction to dimension one.
- Let μ be the right invariant volume form on GL_n. If X is a finite set of right $GL_n(\mathcal{O})$-cosets, then $|X/GL_n(\mathcal{O})| = (\int 1_X d\mu)/(\int 1_{GL_n(\mathcal{O})} d\mu)$.
- By elimination of imaginaries, every equivalence relation reduces to the one above ($GL_n(\mathcal{O})$-cosets).
- Hence counting reduces to volumes.
- In fancy language: the Grothendieck ring of definable sets, even of imaginary sorts, maps into the Grothendieck ring of normalized volumes.

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Two examples of imaginaries arising geometrically

- Cluckers-Denef 2007: Orbital integrals. X a homogeneous space for an algebraic group G. Study $X(\mathbb{Q}_p)/G(\mathbb{Q}_p)$ uniformly in p.
- H. - Martin. Irreducible representations of finitely generated nilpotent groups, up to 1-dimensional twists.
  - $G \leq U_n(\mathbb{Z}_p)$. U_n=upper triangular matrices. G has infinitely may 1-dimensional representations, but up to tensoring with them, only finitely many (a_n) irreducible continuous representations of dimension p^n. Then again $\sum a_n x^n$ is rational. Here X=1-dimensional representations of subgroups; E= same induced representation to G, up to a twist.

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  - $G \leq U_n(\mathbb{Z}_p)$. U_n=upper triangular matrices. G has infinitely may 1-dimensional representations, but up to tensoring with them, only finitely many (α_n) irreducible continuous representations of dimension p^n. Then again $\sum \alpha_n t^n$ is rational. Here X=1-dimensional representations of subgroups; E= same induced representation to G, up to a twist.

From now on we will restrict attention to the theory $ACVF_F$ of algebraically closed valued fields, containing a given valued field F. Thus for subsets of algebraic varieties, semi-algebraic $=$ constructible $=$ definable (Robinson.) For subsets of the imaginary sorts, we prefer the term "definable".

Topology

We consider the Berkovich topology of *algebraic varieties*. We are given a valued field F, an ordered group A and a valuation $v : F \to A \cup \{\infty\}$. Mostly (with Berkovich) we will consider only the $A = \mathbb{R}$.

- V an algebraic variety over F. A *Berkovich point* is a Grothendieck point, i.e. a K-irreducible subvariety U of V, along with an extension to $F(U)$ of the valuation on F *into the same group A.*
- $B_F(V)$ denotes the set of Berkovich points. If X is cut out of V by some valuation inequalities, let $B_F(X)$ be the subset where these inequalities hold.
- Let f be a regular function on V. For any $p = (U_p, v_p) \in B_F(V)$, have $\mathrm{val} f(p) := v_p(f|U) \in \mathbb{R}$. Thus while f does not extend to $B_F(V)$, $\mathrm{val} \circ f$.
- For affine V, topologize $B_F(V)$ minimally so that the functions $\mathrm{val} \circ f : B_F(V) \to A_\infty$ are continuous, for any regular f on V. (in general, patch.)

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Topological finiteness for Berkovich spaces

Let X be a definable subset of a quasi-projective variety V.

Theorem (H.-Loeser)

1. *There exists a deformation retraction from $B_F(X)$ to a subspace S homeomorphic to a finite simplicial complex.*
2. *Let $f : X \to Y$ be a morphism, $X_b = f^{-1}(b)$. Then there are finitely many possibilities for the homotopy type of $B_F(X_b)$, as b runs through $Y(F)$.*

(1) was proved by Berkovich assuming the base field F is nontrivially valued, and certain weak smoothnes assumptions on the ambient varieties.

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$\widehat{X}$

In the model-theoretic treatment, Berkovich points are replaced by generically stable types. The set of generically stable types on X is denoted $\widehat{X}$.
They are defined for any valued field, not necessarily with value group $\subset \mathbb{R}$. This is related to the finiteness theorem (2).
We will define the points from several viewpoints; show that they form a pro-definable set; define a topology on this set; and discuss the relation of $\widehat{X}(F)$ to $B_F(X)$, when the latter is defined.
But first we must consider a more general notion, of a definable type. Asides from serving as a natural setting for picking out the generically stable types, we will use them to define and prove most of the significant properties of $\widehat{X}$,

from Martin Hils' Segovia tutorial: The notion of a definable type

- $T = ACVF_F$, $L = +, \cdot, \mathrm{val}$

Definition

Let $\mathcal{M} \models T$ and $A \subseteq M$. A type $p(x) \in S_n(M)$ p is *A-definable* if for every L formula $\phi(x, y)$ there is an L_A-formula $d_p\phi(y)$ s.t.

$$\phi(x, b) \in p \ \Leftrightarrow \ \mathcal{M} \models d_p\phi(b) \quad \text{(for every } b \in M)$$

We say p is *definable* if it is definable over some $A \subseteq M$.

The collection $(d_p\phi)_\phi$ is called a defining scheme for p.

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Definable types

I prefer to take the defining scheme itself to be the definable type.

Definition

A *definable type* $p(x)$ is a Boolean retraction $L_{x,y_1,y_2,\ldots}$ to $L_{y_1,y_2,\ldots}$,

$$\phi \mapsto (d_p x)\phi$$

Analogy: a *finite measure* on a compact space X can be defined as a retraction from continuous functions on $X \times Y$, to continous functions on Y.

Example, $Th(\mathbb{C})$: let V be an irreducible variety. $(d_p x)\phi =$ "for generic $x \in V$, ϕ" = for some proper Zariski closed $Z \subset V$, $(\forall x \in V \setminus Z)\phi$.

Example, $Th(\mathbb{R})$ Let V be a variety and let $g : (a, b] \to V$ be a parameterized curve. $(d_p x)\phi =$ "for all t sufficiently close to b, $\phi(g(t))$. Definition of definable compactness in o-minimality.

In ACVF, both kinds of example occur; in fact we will see that every definable type decomposes into a composition of the two.

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Operations on definable types

(from M.H. tutorial)

- **(Realised types are definable)**
  Let $a \in M^n$. Then $\mathrm{tp}(a/M)$ is definable.
  (Take $d_p\phi(y) = \phi(a, y)$.) constant definable types

- **(Preservation under definable functions)**
  Let $b \in \mathrm{dcl}(M \cup \{a\})$, i.e. $f(a) = b$ for some M-definable function f. Then, if $\mathrm{tp}(a/M)$ is definable, so is $\mathrm{tp}(b/M)$.
  Pushforward, f_*p:

$$(d_{f_*p}y)\theta(y, u) := (d_px)\theta(f(x), u)$$

- **(Transitivity)** Let $a \in N$ for some $\mathcal{N} \succ \mathcal{M}$, $A \subseteq M$. Assume
  - $r = \mathrm{tp}(a/M)$ is A-definable;
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Definable types: germs and limits

- Let f, g be definable functions. f, g have the same p-germ if $(d_p x)(f(x) = g(x))$ (iff whenever $c \models p|M$, where f, g are defined oer M, we have $f(c) = g(c)$.)
- Assume $f : D \to X$, p a definable type on X, and X carries a (definable) topology. Write $\lim_p f = a$ if for any definable open U of a, $a \in U \implies (d_p x)(f(x) \in U)$

$\widehat{V}$: generically stable types on V

1. Definable types, orthogonal to the value group: $f_* p$ for any $f : V \to \Gamma$.
2. Stably dominated types: p dominated by $g_* p$ for some definable $g : V \to E$, E a finite dimensional space over k.
3. The center of the monoid of definable types: for any q, $p(x) \otimes q(y) = q(y) \otimes p(x)$
4. (When $V \leq \mathbb{A}^n$ is an affine variety). Γ-seminorms $\nu : K[X_1, \ldots, X_n] \to \Gamma_\infty$
   $\nu(fg) = \nu(g) + \nu(g), \nu(f+g) \geq \min(\nu(f), \nu(g)), \nu(c) = \mathrm{val}(c)$
   such that $\nu(f) = \infty$ if $f|V = 0$,
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5. (When $\Gamma(F) \leq \mathbb{R}$). An element of $B_F(V)$, functorially extendible to $B_{F'}(V)$ for $F' \geq F$ As Antoine Ducros pointed out, for this statement we must consider arbitrary F'; for those with value group $\mathbb{R}$, a theorem of Poineau extends any Berkovich point functorially.

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$\widehat{V}$: generically stable types on V

1. Definable types, orthogonal to the value group: $f_* p$ for any $f : V \to \Gamma$.
2. Stably dominated types: p dominated by $g_* p$ for some definable $g : V \to E$, E a finite dimensional space over k.
3. The center of the monoid of definable types: for any q, $p(x) \otimes q(y) = q(y) \otimes p(x)$
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- An F-definable, generically stable type need not be dominated by an F- definable map into k^n. The vector space E may have the form $\Lambda/\mathcal{M}\Lambda$, Λ an F-definable lattice.
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- $\{\Lambda_d(p) : p \in \widehat{V}\}$ is in fact *definable*. (It is easily seen to be a countable intersection of definable sets. Using stable domination, it is also a countable union of definable sets. By compactness it must be definable.)

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Proof of equivalence

- $2 \Rightarrow 3$ Since p is determined by $g_* p$, and $g_* p \otimes q = q \otimes g_* p$.
- $3 \Rightarrow 1$: Symmetry implies symmetry of pushfoward. A type on Γ commuting with itself is constant.
- $1 \Rightarrow 4$ $\nu(f) = (\mathrm{val} f)_* p$.
- $1 \Rightarrow 2$ follows from the decomposition theorem over maximally complete fields below, and a (still quite technical) descent theorem for stably dominated types.
- $4 \Rightarrow 1$: $(d_p x)(\mathrm{val} f \geq \mathrm{val} g) \iff \nu(f) \geq \nu(g)$.
- $1 \Rightarrow 5$ as definable types give types over any larger base.
- $5 \Rightarrow 1$: example of type 4 point.

Connection with Berkovich space

- F be a valued field, with value group $\leq \mathbb{R}$.
- F^{max} a spherically complete algebraically closed field, containing F, with value group $\mathbb{R}$, and residue field equal to the algebraic closure of the residue field of F. (unique up to isomorphism, by Kaplansky's theorem.)
- $\pi = \pi_X : \widehat{X}(F^{max}) \to B_F(X)$ (realization and restriction.)
- π_X is surjective.
- π is functorial in X. $\pi(\Gamma) = \mathbb{R}$.
- in particular a homotopy $h : \widehat{X} \times I \to \widehat{X}$ gives a homotopy $B_F(X) \times I(\mathbb{R}) \to B_F(X)$.
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Proposition

Let M be a spherically complete valued field, $N = M(a)$ a valued field extension. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a basis for $\Gamma(N)/\Gamma(M)$. Then there exists a unique $M(\gamma)$-definable type extending $tp(a/M(\gamma))$. This type is stably dominated.

Call a lattice Λ *diagonal* for a basis $(b_1, \ldots, b_n)$ if there exist $c_1, \ldots, c_n \in K$ with $\Lambda = \sum \mathcal{O} c_i b_i$. In other words, $\Lambda = \oplus_i \Lambda \cap K b_i$

Proposition

let D be a Γ-internal set of lattices, i.e. there exists a surjective map $\Gamma^m \to D$. Then there exist a finite partition $D = \cup_{i=1}^r D_i$ and bases $b^1, \ldots, b^r$ such that each $\Lambda \in D_i$ is diagonal in b^i.

Decomposition theorem

Theorem

Let p be an A-definable type on a variety V. Then there exist an A-definable type r on Γ^n and an A-definable r-germ of pro-definable maps into $\widehat{V}$, with $p = \int_r f$.

Example

Definable types on a curve C correspond to germs of definable paths on $\alpha : [a, b] \subset \Gamma \to \widehat{C}$. Generically stable types correspond to constant paths.

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- $n \leq \dim(V)$.
- The theorem holds also for *invariant types*, meaning a functorial Huber-Knebush point; r is then an invariant type on Γ^n.
- r and the r-germ of f are unique up to reparameterization; a canonical additional constraint on the parameterization of f exists.
- f itself may not exist over A, but only over a bigger base field. E.g., when $p = p_B =$ generic type of an open ball.

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Imaginaries

- Let A be a set of abstract imaginaries. Let $D \subset K^n$ be a nonempty A-definable set. Then there exists a definable type p on D (over $\mathbb{U}$) such that p has a finite orbit under $Aut(\mathbb{U}/A)$. reduces to dimension 1.
- Any definable type has a canonical base $B \subset S_n \times T_n \times K^n$, some n. (A unique minimal base of definition.) uses decomposition theorem.
- Let E be a definable equivalence relation on $\mathbb{A}^n$, let D be a class, a an (abstract) code for the class D. Let p be a definable type on D. Let b be the canonical base. Then D is b-definable, and b has finitely many a-conjugates $b_1, \ldots, b_m$. Hence a is equivalent to a *finite set* of geometric imaginaries.
- Explicitly code finite sets of lattices by a higher-dimensional lattice.

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Topological finiteness for $\widehat{V}$

Let X be a definable subset of a quasi-projective variety V, over F.

Theorem

1. *There exists a definable deformation retraction from $\widehat{X}$ to a definable subspace Υ, and a definable homoeomorphism $\Upsilon \to S \subset \Gamma_\infty^w$; w a finite set.*
2. *The image in S of any constructible $Y \subset X$ is definable using $<, +$ alone. (A hint of tropicality.)*
3. *Let $f : X \to Y$ be a morphism, $X_b = f^{-1}(b)$. Then the retractions $X_b \to \Upsilon_b$ and definable homeomorphisms $\Upsilon_b \to S_b \subset \Gamma_\infty^w$ are uniformly definable; and as b runs through $Y(F)$, there are finitely many possibilities for the homeomorphism type of $S_b(\mathbb{R}_\infty)$.*

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Remarks

- w is the set of roots of a polynomial over F. Γ_∞^w is homeomorphic to $\Gamma_\infty^{|w|}$; we use w in order to have an F-definable homeomorphism; in particular, Galois invariant.
- Semi-linearity of the image is automatic: any (ACVF) definable subset of Γ_∞^n is $<,+$-definable.
- Finite number of definable homotopy types: likewise automatic from the same statement in o-minimal case, once one notes that the family of skeleta S_b of the sets X_b, is uniformly definable. Any $ACFA_F$-definable subset of Γ_∞^n is $<,+$-definable with parameters from $\Gamma(F)$.

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Definable homotopies

- A definable homotopy is a continuous, pro-definable $H : \widehat{X} \times I \to \widehat{X}$, I a Γ-interval; with $h_{minI} = Id, h_{maxI} = h_1$. We seek a definable homotopy H to h_1 with $h_1(\widehat{X}) \cong S \subset \Gamma_\infty^w$.
- We construct a deformation of V, respecting finitely many definable subsets, and functions into Γ.
- Canonical extension: Any definable $h : V \to \widehat{U}$ extends to $H : \widehat{V} \to \widehat{U}$; similarly for $h : V \times I \to \widehat{U}$. $H(p) = \int_p h$.
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ACV^2F and continuity criteria

- ACV^2F is the theory ACV^2F of triples (K_2, K_1, K_0) of fields with surjective, non-injective places $K_2 \to_{r_{21}} K_1 \to_{r_{10}} K_0$.
- $0 \to \Gamma_{10} \to \Gamma_{20} \to \Gamma_{21} \to 0$
- $\widehat{V}_{210} = \widehat{V}_{20}$. Hence $\widehat{V}_{20} \to \widehat{V}_{21}$.
- An $ACVF$-definable map $f : W \to \widehat{V}$ extends to a continuous map $\widehat{W} \to \widehat{V}$ if and only if it is compatible with the natural maps among $\widehat{V}_{ij}$.

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Construction of a definable deformation

We obtain the deformation by a composition of four kinds of homotopies:

1. Deformations of (relative) curves.

   Arrange (after a blowup with finite center) that V is fibered by curves over a variety U. Apply (1) to each curve V_u.

   Away from a divisor D_{vert} on U, and after a fiber product with a finite Galois cover of U, obtain a deformation H on $\widehat{V}$ with final image definably homeomorphic to a subset Ω of $U \times \Gamma_{\infty}^{r}$.
2. Extend deformation H_U of $\widehat{U}$ to Ω.
3. Pre-compose with *inflation homotopy* in order to get away from D_{vert}. This homotopy does not move singular points, and slightly inflates smooth points to generics of small polydisks around them.
4. These steps already yield H as stated; but one also wants a *strong* deformation, i.e. that H fixes $h_1(\widehat{X})$.. This can be arranged by post-composing with a homotopy of $h_1(\widehat{X})$. This fourth homotopy lives entirely in the tropical world.

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