Voyage to the land of Erdös
Macintyre meeting

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1. $\aleph_1$-categoricity. Structures, fields, fields with extra structure. How big are finite sets?

2. Pseudo finite structures / fields with extra structure. Erdös geometry.

3. Fine dimension δ, associated measures μ_α; coarse dimension $\boldsymbol{\delta}$. Stability phenomena for δ-generics.

4. Lachlan-Zilber, Zilber's conjecture.

5. Probability logic. Hoover (Gromov, Binyamini-Schram.) Probability logic on varieties.

6. Tao's induced structure theorem for $\boldsymbol{\delta}$-generics.

7. Extensions towards probability logic on varieties (joint with Bukh-H.-Zimmerman). Limiting examples, Archimedean moonshine.

8. Metrically approximately subgroups. (Beyond local compactness; Set theoretic phenomena?)

2 $\delta, \boldsymbol{\delta}, \mu_\alpha$.

For subsets of Y^n, we define *the coarse dimension*

$$\boldsymbol{\delta}(X) = st(\log|X|/\log|Y|)$$

The *fine dimension*: $\delta(X) = \log|X|$ modulo the convex hull of $\mathbb{R}$ in $\mathbb{R}^*$.

And the measure at fine dimension α : If $\delta(X) = \delta(X') = \alpha$, $\mu_\alpha(X)/\mu_\alpha(X') = st(|X/X'|)$.

So $\boldsymbol{\delta}(X)$ is a real number, and given $\alpha(X)$, $\mu_a(X)$ is a real number.

3 Lachlan-Zilber, Zilber's conjecture

$X \subset F$, $\boldsymbol{\delta}(X) = 1$. Let $\mathfrak{R}$ be the set of all subvarieties V of F^n such that $\dim(V) = \boldsymbol{\delta}(V \cap X^n)$.

By a $(2, 3, 2)$ pseudo-plane (for short, in this talk, pseudo-plane) we mean : interpretable ∞-definable sets P, L lying on algebraic varieties $\underline{P}, \underline{L}$, and a constructible set $\underline{I}$ such that for any two points of $\underline{P}$, $\underline{I}(a) \cap \underline{I}(a')$ is finite, and dually; and $\boldsymbol{\delta}(P) = \dim(\underline{P}) = 2$, $\boldsymbol{\delta}(L) = \dim(\underline{L}) = 2, \boldsymbol{\delta}(I) = \dim(I) = 3$.

Proposition 3.1. *If some $V \in \mathfrak{R}$ is not modular (1-based in every cut), then (X, F) interprets a pseudo-plane on W/E, with $W, E \in \mathfrak{R}$.*

Conjecture 3.2. *Assume $\mathfrak{R}$ is not modular. Then it interprets a field k with $\boldsymbol{\delta}(k) > 0$. In fact $\boldsymbol{\delta}(k) = 1$.*

cf. Rabinovich.

Proposition 3.3 (Trotter-Szemeredi, Elekes-Szabo, Solymosi-Tao

). *The conjecture is true in internal characteristic zero; in fact* $\mathfrak{R}$ *interprets no* $(2,3,2)$*-pseudoplane.*

4 Probability logic

Probability logic is best viewed in the framework of real-valued logic. It is presented as an operator, taking a formula $\phi(x, y)$ to a formula $Ex\phi(x, y)$ representing the x-expectation of ϕ.

Hoover quantifier-elimination.

The usual quantifiers of real-valued logic can be defined (using essential inf for inf) via: (for f taking values in a bounded region of $\mathbb{R}$)

$$inf_x f := \lim_n E_{x_1,,x_n} f_n$$

where $f_n(x_1, \ldots, x_n, y) = \min f(x_1, y), f(x_2, y),, f(x_n, y)$ Thus we have a full-fledged real-valued theory, so various statements apply; in particular the complete theory determines the isomorphism type of a compact model. In the special case of a language consisting of a metric alone, this is the Gromov (or Gromov-Vershik) theorem that the statistics of a compact measured metric space determines it up to isomorphism.

Theorem 4.1. *Let G_n be a sequence of locally finite metric spaces, convergent in probability logic (=Binyamini-Schramm) and increasingly 1-homogeneous. Assume a 2-ball is a union of k 1-balls. Then the limit (Γ, X) admits a homomorphism to a vertex transtive graph B of bounded degree, such that each fiber is commensurable to a Riemannian homogeneous space.*

4.2 Probability logic on varieties

Add bounded quantifiers. Equivalently, a system of measures on varieties, compatible with products and pushforwards.

5 Tao's 'algebraic Szemeredi lemma'

Strong definability of measure: $\boldsymbol{\delta}$ determines δ; $\boldsymbol{\delta}$ and μ_a definable. (cf. Macpherson-Steinhorn asymptotic classes.)

Explicitly: write $X \approx X'$ if $||X| - |X'|| = o(|X|^{1-\epsilon})$ for some rational $\epsilon > 0$. Let X, Y be sorts and $U \subset X \times Y$ be a definable set. Y can be partitioned into finitely many definable sets Q_ν, such that $U^b / \approx$ is constant in each class Q_ν.

Example: pseudo-finite fields (Lang-Weil.)

Theorem 5.1. *Let A be a pseudo-finite structure, with strong definability of measure. Let B be a subset, $\boldsymbol{\delta}(B) = \boldsymbol{\delta}(A)$, and assume B meets every 0-definable* unary *subset of A of positive measure. Then $Th_{prob}(B) = Th_{prob}(A)$ (and they have a common elementary submodel.)*

This is obvious when $\mu(B) = 1$. However, in nonstandard terms, the assumption is only that $|B| = |A|^{1-\epsilon}$; so $\mu(B) = 0$ and it seems like an amazing coincidence that $\mu_B = \mu_A(\phi)$.

One can delete the assumption that B meets every 0-definable subset of A of positive measure, and describe $Th_{prob}(B)$ as the induced from $Th_{prob}(A)$ in a natural sense; the usual notion of 'induced structure to a definable set' generalizes to a probability distribution on 1-types. Explicitly, for any definable $U \subset X^n$ one can find a definable partition $X = \cup_{i=1}^r X_i$, such that the theorem holds for each $(X_i, X_j, U \cap (X_i \times X_j)$.

6 Extensions; towards probability logic on varieties

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Partly in Tao, partly joint work with Bukh, Zimmerman.

Definition 6.1. *Let $\phi, B \subset Y$. Say B meets ϕ properly (in Y) if $(\phi \cap B) \times Y \approx B \times \phi$.*

Say B is 1-dense if for any definable $D \subset F$ with $\boldsymbol{\delta}(D) = 1$, we have $\boldsymbol{\delta}(D \cap B) = k$.

Theorem 6.2. *Let F be a pseudo-finite field Let $V \subset \mathbb{A}^n$ be a subvariety of codimension 1, projecting onto each $\mathbb{A}^{n-1}$. Then V meets properly every 1-dense $D \subset F$,* unless *V is conjugate (by multi-valued correspondences) to the graph of $\sum_{i=1}^n x_i = 0$ in some algebraic group.*

In fact: there exists a definable partition of F, such that for any pieces $P_1, \ldots, P_n$, and any $B_i \subset P_i$ with $\boldsymbol{\delta}(B_i) = \boldsymbol{\delta}(P_i)$, Then $\Pi_i B_i$ meets V properly;

Theorem 6.3. *Let F be a pseudo-finite field, $B \subset F$, $\boldsymbol{\delta}(B) = \boldsymbol{\delta}(F)$. Let $V \subset \mathbb{A}^n$ be a subvariety of codimension 2, not contained in a subvariety of codimension 1 of the above type. Then B^n meets V properly,* unless *there exist a simple Abelian variety A and curves C_i on A, such that (F^n, V) is isogenous to $(C_1 \times \cdots \times C_n, + \cap (C_1 \times \cdots \times C_n)$.*

Example 6.4 (Too few relations). G a 2-dimensional simple Abelian variety, $A = F(\mathbb{F}_p)$. Find subsets $Y \subset A$ of size about $(1/10)|A|$ such that the equation $y_1 + + y_{10} = 0$ has no solutions in Y^{10}. E.g. find a homomorphism $f : A \to \mathbb{Z}/m$ with $m > 10$, and let Y be the inverse image of $1, \ldots, [m/10] - 1$.

Find a hyperplane section C of Y such that $Y \cap C$ has $> 1/11|C(F)|$ points.

Let $V = \{(x_1, , x_{10}) \in C^{10} : f(x_1) + . + f(x_{10}) = 0\}$. This is a codimension 2 subvariety of C^{10}, which is not group-like in the sense of one-dimensional groups, but it has empty intersection with X^{10}.

On the other hand if one takes p such that $|G(\mathbb{F}_p)|$ has no small prime factors, so that over the ultraproduct $G(\mathbb{F})$ has no subgroups of finite index, it can be shown that no definable partition can be responsible for this; the predicted intersection number with any product of 10 pieces will be nonzero.

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7 Caucy-Schwartz

Let M be a pseudo-finite structure. Let X, Y be sorts and $U \subset X \times Y$ be a definable set. When $U \subset X \times Y$, $a \in X, b \in Y$ we will write $U_a = \{y : (a, y) \in U$ and $U^b = \{x : (x, b) \in U\}$

Proposition 7.1. *Assume* $|U_a|/\approx$ *is constant for* $a \in X(M)$*; and* $Y \times Y$ *can be partitioned into finitely many definable sets* Q_ν*, such that* $(U^b \cap U^{b'})/\approx$ *is constant in each class* Q_ν*. Similarly,* $Y = \dot{\cup}_{\nu'} Q'_\nu$ *and* $U^b/\approx$ *is constant in each class* $Q'_{\nu'}$*.*

Let B *be a subset of* Y *meeting properly any (generic)* Q_ν *or* Q'_ν*.*

Then B *meets* $U(a, y)$ *properly for almost all* a *(all but* $|X|^{1-\epsilon}$ *for some* $\epsilon > 0$*).*

The following was noticed also by Starchenko-Pillay:

Remark 7.2. *In codimension* 0*, the* Q_ν *(intersection measure) are stable relations on* $Y \times Y$*.*

Hence a large enough B meets the Q_ν properly; this serves as an induction base.

Proof of 7.1. Consider:

$$\frac{1}{X}\Sigma_{a\in X}||U_a(B)||Y|-|U_a||B|| \tag{1}$$

By Cauchy-Scwhartz, or just convexity of the parabola $y = x^2$, $(1)^2$ is bounded above by:

$$\frac{1}{|X|}\Sigma_{a\in X}(|U_a(B)||Y|-|U_a||B|)^2 =$$

$$\frac{1}{|X|}\Sigma_{a\in X}\Sigma_{b,b'\in B}(1_U(a,b)|Y|-|U_a|)(1_U(a,b')|Y|-|U_a|)$$

$=$

$$\frac{1}{|X|}\Sigma_{b,b'\in B}(U^b\cap U^{b'})+\cdots$$

□

8 Metrically approximate subgroups

Let G be a group with a metric d invariant under left and right translations. A (K, r)-*approximate subgroup* is a subset X of G containing 1, such that the product set XX is covered by at most K translates of XB_r, where B_r is the ball of radius r around 1. (Tao, de Saxce, $\cdots$.) A closely related condition is that $N_r(XX) \leq KN_r(X)$, where $N_r(X)$ is the number of r-balls needed to cover X.

Proposition 8.1. *Fix $K_1, K_2, \ldots$, and let $k > 0$. Then for some $m \in \mathbb{N}$, for all triples (G, X, d), and $r_m, \ldots, r_1$ with $2r_{i+1} < r_i$, if for X is (K_i, r_i)-approximate, then there exists $Y \subset X^4$ satisfying, for at least k values of i,*

$$1 \in Y = Y^{-1}, Y^k \subset X^4 B_{r_k}, \; N_{r_i}(Y) \geq \frac{1}{m} N_{r_i}(X) \qquad (2)$$

Stabilizer theorem for hyperimaginaries.