# Nonarchimedean globally valued fields
# Géométrie et théorie des modèles
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## Abstract

In a joint research project with Itaï Ben Yaacov, we study a class of fields enriched with a global structure tying together their various valuations by a product formula. This is an elementary class in the sense of continuous logic; the field of algebraic functions $\mathbb{C}(t)^{alg}$ is a prime example, and the Weil height function on projective space is a key example of a definable map into $\mathbb{R}$. I will describe some of the connections to algebraic geometry on the one hand, and stability theory on the other. This will also be the subject of a more extended class at IHP next semester.

# 1 Fields with a height function

Let $F$ be a field. A *height-type function* is a function $h : F \setminus (0) \to \mathbb{R}^{\geq 0}$ satisfying:

$$ht(xy) \leq ht(x)+ht(y),\ ht(x^{-1}) = ht(x),\ ht(x+y) \leq ht(x)+ht(y)+ht(2)$$

The field may be enriched by additional functions $\phi$ into $\mathbb{R}$, satisfying $|f(x)| \leq O(1)ht(x)$.

Standard examples of height:

- On $\mathbb{Q}$, $ht(\frac{a}{b}) = \max(\log|a|, \log|b|)$ for reduced fractions.
- For a number field $L$, Weil: $ht(a) = \sum_v v(a)^+$ where $v(a) = -\log|a|_v$, ranges over appropriately normalized absolute values of $L$, $x^+ = \max(x, 0)$.
- Let $F = k(t)^{alg}$, $\alpha > 0$. Any element $f \in F$ can be viewed as a morphism $f : C \to \mathbb{P}^1$ for some curve $C$ over $k$; we also have $t : C \to \mathbb{P}^1$.

$$ht(f) = \alpha \deg(f)/\deg(t)$$

Let $(F_i, h_i, \phi_i : i \in I)$ be fields with height-type functions (and one additional $\phi$ as above), and let $u$ be an ultrafilter on $I$. Assume $h_i(2)$ remains bounded. The *ultraproduct* is defined as follows. Let $(F_u, h_u, \phi_u : F_u \to \mathbb{R}_u)$ be the usual ultraproduct of fields; let $\bar{\mathbb{R}}$ be the convex hull of $\mathbb{R}$ in $\mathbb{R}_u$, and $st : \bar{\mathbb{R}} \to \mathbb{R}$ the standard part map; define

$$F = \{0\} \cup \{a \in F_u^* : h_u(a) \in \bar{\mathbb{R}}\}$$

$$h(a) = st(h_u(a)), \quad \phi(a) = st(\phi_u(a))$$

## 2 The language

The terms are polynomials over $\mathbb{Z}$; equality is a $\{0,1\}$-valued relation as usual.
Basic relations $R_t$: A symbol $R_t$ for each *tropical term* $t$ = term in the language $+, \min, 0, \alpha \cdot x$ of divisible ordered Abelian groups. to be interpreted as functions $(F^*)^n \to \mathbb{R}$.
Local interpretation of $R_t$ Let $(K, v)$ be a valued field, or a subfield of $\mathbb{C}$ with $v(x) = -\alpha \log |x|$. For $x$ with $x_i \neq 0$, interpret ${R_t}^v(x)$ as $t(vx_1, \ldots, vx_n)$.
Global intended intepretation: We think of $R_t(x)$ as the *expected value* of ${R_t}^v(x)$ with respect to an implied measure on valuations. Write a basic formula

$$R_t(f_1(x), \ldots, f_n(x)) =: \int t(vf_1x, \ldots, vf_nx)dv$$

.

Among them, the height: $x^+ = -\min(-x, 0)$.
$ht(x) = R_t(x) = \int v(x)^+ dv$

Connectives $\min, \max, 0, +, \alpha \cdot x$.

Quantifiers The analogue of quantifiers in real-valued logic is inf and sup operators. Let $\psi_{n,\epsilon}(t)$ be 1 on $[-n, n]$, 0 on $|t| > n+\epsilon$, and a linear interpolation on $[n, n+\epsilon]$. Let $\phi(x, y)$ be a formula. Then so is $\sup_x \psi_{n,\epsilon}(ht(x))\phi(x, y)$.

We view this as a quantifier over $x$ of height up to about $n$.

All formulas are preserved by ultrapowers.

It will turn out that the height function suffices to generate the language, at least in the purely non-archimedean case.

# 3 Universal axioms

Let $LVF$ be the set of pairs $(\phi, t)$ of formulas $\phi(x_1, \ldots, x_n)$ in the language of rings implying $\Pi_i x_i \neq 0$, $t$ a tropical term, such that the theory of valued fields implies $t$ is positive on the amoeba of $\phi$:

$$VF \models (\forall x)(\phi(x) \implies t(v(x_1), \ldots, v(x_n)) \geq 0)$$

Axioms GV *for purely non-archimedean globally valued domains:*

1. $(F, +, \cdot)$ is an integral domain.
2. The $R_t$ are compatible with permutations of variables and dummy variables.
3. (Linearity:) $R_{t_1+t_2} = R_{t_1} + R_{t_2}$. $R_{\alpha t} = \alpha R_t$.
4. (Local-global positivity for amoebas) If $(\phi, t) \in LVF$ and $\phi(a_1, \ldots, a_n)$ then $\int t(v(a_1), \ldots, v(a_n))dv \geq 0$.
5. (Product formula) $\int v(x)dv = 0$

Remarks:

A model of GV can be shown to admit the structure of an M-field in the sense of Gubler, meaning essentially that there is a measure $\mu$ on a set of representatives for the valuations of F, validating

$$R_t(f_1(x), \ldots, f_n(x)) =: \int t(vf_1x, \ldots, vf_nx)d\mu(v)$$

.

One can relax the *valued field* axiom to $v(x + y) \geq \min v(x), v(y) - e$, so as to allows archimedean valuations; obtaining $LVF_e$ and correspondingly $VF_e$. In this talk we will concentrate on the non-archimedean case.

# 4 Classical structures

**Function fields:** **over a constant field $k$:**

$L_{fn} = k(t)^{alg} = \cup_C k(C)$

For $f_1, \ldots, f_n \in k(C)$,

$$R_t(f_1, \ldots, f_n) = [k(C) : k(t)]^{-1} \sum_{c \in C(k)} t(v_c(f_1), \ldots, v_c(f_n))$$

Product axiom: $\sum_{c \in C(k)} v_c(f) = 0$.
We have $ht(t) = 1$.

For $K = k(C)$, and $X$ a variety over $k$, $X(K)$ can be identified with the morphisms $m : C \to X$, informally with their image $\bar{C} \leq X$. Let $Y$ be a subvariety; cut out by homogeneous $g_1, \ldots, g_m$ say. The field language only permits asking when $\bar{C} \subset Y$ for a subvariety $Y$: the GVF language allows discussing whether or not $\bar{C}$ intersects $Y$, or rather gives sthe intersection number: $\sum_{c \in C} \min_i v_c g_i - \min_i v_c x_i$. Indirectly, can discuss the homology class of $\bar{C}$.

**Lemma** (Artin-Whaples). *Let $C$ be a curve of genus $g$ over $k$. Let $x \in k(C) \smallsetminus k$, and let $r > 0$. Then $k(C)$ admits a unique GVF structure over $k$ with $ht(x) = r$.*

*Proof.* The nontrivial valuations of $k(C)$ over $k$ can be identified with the points of $C(k)$, a discrete space. We have to show that if $\mu$ is a measure on $C(k)$ satisfying the product formula, then $\mu$ gives equal weight to any two points $a, b \in C$. By Riemann-Roch $(n+g)a - nb$ is effective, $(n+g)a - nb + (f) = \sum m_i d_i$ with $m_i \geq 0$; so

$$(n+g)\mu(a) \geq n\mu(b)$$

Thus $(1+g/n)\mu(a) \geq \mu(b)$. Letting $n \to \infty$, $\mu(a) \geq \mu(b)$. □

# Number fields, asymptotically in height

$(b_r)$ The field $\mathbb{Q}^{alg}[r]$. For $f_1, \ldots, f_n \in L$, $[L : \mathbb{Q}] = d$,

$$R_t(f_1, \ldots, f_n) = \alpha \sum_v t(-\log|f_1|_v, \ldots, -\log|f_n|_v)$$

where the $v$ range over all absolute values of $L$, normalized so that the product formula holds, and $ht(2) = \log 2/r$.

Then $\mathbb{Q}^{alg}[r] \models GV_e$ for $e = \log 2/r$. However, let $u$ be an ultrafilter on $\mathbb{R}^{>0}$, concentrating on $r \to \infty$.

Let $L_{\#,u}$ be an ultraproduct of $\mathbb{Q}^{alg}[r]$. Then $L_{\#,u} \models GV$.

The GVF language allows sampling $\mathbb{Q}^a$ at one or two height scales, near 1 and near $r$. If $r$ is kept bounded we need the theory with absolute values; if $r \to \infty$ the non-archimedean theory applies.

# Example of global algebraic closure

In diophantine approximations, e.g. Roth's theorem, one considers rational approximations $b$ to $\alpha \in \mathbb{Q}_{v_0} \cap L$ at a place $v_0$ of $\mathbb{Q}$:

$$(1) \qquad v_0(b-\alpha) \geq \kappa h(b), \quad \kappa > 2$$

Here $L$ is a number field, Galois over $\mathbb{Q}$. Letting $S$ be the lifts of $v_0$ to $L$ and $\alpha_v$ the corresponding conjugates of $\alpha$, we obtain a glueing problem viewpoint:

$$(2) \qquad v(b-\alpha_v) \geq \kappa_v h(b), \ \sum_{v \in S} \kappa_v = \kappa > 2$$

But *we will let $b$ range over $\mathbb{Q}^{alg}$.* We are interested in $b$ of height $h$ above a certain height threshhold, $h_0$ so that $(\kappa - 2)h > ht(2)$.

Assume the *height* $h$ of $b$ is fixed. Then there are at most finitely many solutions $b$ of (2), even in a GVF extension. All are in fact definable over a base $A$ of definition for the data. [1] For suppose $b, b'$ are two solutions with the same type over $A$. Then and $v(b - \alpha_v), v(b' - \alpha_v) \geq \kappa_v h$ for $v \in S$, so

$$ht(b - b') \geq \sum_v v(b - b')^+ \geq \sum_{v \in S} \kappa_v h = \kappa h > 2h + ht(2)$$

a contradiction.

This exemplifies global algebricity; GV qf algebraic closure is not just ACF algebraic closure. .

[1] additional parameters of height $O(h)$ is required to capture the $\alpha_v$, $v_i \in S$, $\kappa_i h'$.

## A baby curve selection theorem

Let $X$ be a smooth projective variety over $\mathbb{Q}$; let $Y$, $Y_1, \ldots, Y_m$ be subschemes.

If $L$ is a number field and $x \in X(L)$, let $\delta(x,Y)^L = \int \delta_v(x,Y)dv$ be the weighted sum of the local distances from $x$ to $Y$, over all valuations (and $-\log|\cdot|$) $v$ of $L$.

Note that $\delta(x,Y)^L$ is the $L$-value of a certain quantifier-free formula $\phi_Y(x)$ in the language of GVF's.

**Proposition.** *Assume $a_n \in X(\mathbb{Q}^a)$, $ht(a_n) \to \infty$, with $\lim_{n\to\infty} \delta_{Y_k}(a_n)/ht(a_n) = e_k$; let $\epsilon > 0$. Then there exists a curve $C$ on $X$ such that for any sequence $a'_n \in C(\mathbb{Q}^a)$, $ht(a'_n) \to \infty$, we have $\lim_{n\to\infty} |\delta_{Y_k}(a'_n)/ht(a'_n) - e_k| < \epsilon$.*

*Proof.* Choose $r_i = ht(2)/ht(a_i)$ so that $\mathbb{Q}^a[r_i]$ gives $a_i$ height 1. Consider any non-principal ultrafilter $u$ on the index set $\mathbb{N}$, and let $(L,a)$ be the GVF ultraproduct of $(\mathbb{Q}^a[r_i], a_i)$. Then $(L,a)$ is a purely non-archimedean GVF, and $\delta_{Y_k}(a) = \phi(a)^L = e$. There exists $a' \in K = k(t)^{alg}$ with $e' = \phi(a')^K$

satisfying $|e' - e| < \epsilon$. In fact $a' \in k(C)$ for some curve $C$, so $a'$ corresponds to a morphism $g : C \to X$. We may choose $a'$ so that $g(C)$ avoids any given proper subvariety of $X$. By computing the meaning of $\phi$ in $k(t)^{alg}$ we see that $\bar{i}(C, Y_k) = e'$.

Conversely, if $C$ is a curve on $X$ defined over $\mathbb{Q}^a$, then for any sequence of distinct $a_i \in C(\mathbb{Q}^a)$ of bounded degree over $\mathbb{Q}$, $\delta_Y(a) \to i_Y(C)$. This follows upon taking normalized ultraproducts as above, from the uniqueness of the GVF structure on $k(C)$. □

In particular, there exists such a sequence $a'_i$ of bounded degree over $\mathbb{Q}$.

# Value distribution theory

Let $\mathcal{M}$ be the field of meromorphic functions. (Or a countably generated algebraically closed subfield.) Fix a function $\eta(r)$ (say $\log(r)$ or $r^d$), and also an ultrafilter $u$ on $\mathbb{R}^{>0}$, avoiding finite measure sets.

Let $\mu_r$ be the measure space on $\{a : 0 < |a| \le r\}$ giving mass $\log(r/a)/\eta(r)$ to each point $0 < |a| < r$, and the uniform measure of mass $1/\eta(r)$ to the circle $|t| = r$. Define

$$v_a(f) = ord_a f \text{ for } |a| < r, \quad v_t(f) = -\log|f(t)|$$

$$ht_{\eta,u}(f) = \lim_{r \to u} \max(v_a f, 0) d\mu_r a$$

$$\mathcal{M}[\eta, u] = \{f \in \mathcal{M} : ht_{\eta,u}(f) < \infty\}$$

$$R_t(f_1, \ldots, f_n) := \lim_{r \to u} \int t(v_a f_1, \ldots, v_a f_n) d\mu_r a$$

.

.

The product axiom is Jensen's formula:

$$\sum_{0<|a|<r} \log \frac{r}{a} ord_a(f) + \frac{1}{2\pi} \int_0^{2\pi} - \log |f(re^{i\theta})| \, d\theta = O(1)$$

- Not a GVF for fixed $r$ (O(1) term in Jensen's formula.)
- $\mathcal{M}[\eta, u]$ is a purely non-archimedean GVF.
- Vojta's dictionary: number theory $\leftrightarrow$ Nevanlinna theory.
- In GVF language, $\mathcal{M}[\eta]$, has the same *universal* theory as the ultraproduct of the $\mathbb{Q}^a[r]$, and also as $\mathbb{C}(t)^{alg}[1]$.
- Conjecturally the same *theory*: a GVF *isomorphism of ultrapowers*.
- We will show this for the universal theory. This explains a *part* of Vojta's dictionary. But the GVF language will not permit e.g. *truncations* , *support, discriminants* .

## 5 Geometry: Divisors and curves

Let $X$ be a smooth projective variety over $k = k^{alg}$.

$H$ will denote an irreducible hypersurface.

$NS(X) = Pic(X)/Pic^0(X)$ is the group of formal combinations $\sum n_i H_i$ up to algebraic equivalence; by Lang-Néron 1951, it is finitely generated. Let $N^1(X) = NS(X) \otimes \mathbb{R}$. A finite dimensional vector space over $\mathbb{R}$. The *effective cone of divisors* $N^1_{eff}(X)$ consists of elements $\sum \alpha_i [H_i]$ with $\alpha_i \in \mathbb{R}^{\geq 0}$.

Duality of divisors and curves: given a (Cartier) divisor $D$ and a curve $C$ not lying on $D$, define $D \cdot C$ to be the number of intersection points, with multiplicity. If $D = (f)$ is principal, then $D \cdot C = 0$.

Let $N_1(X) = N^1(X)^*$ be the dual space, = space of 1-cycles. Let

$$N_1^+(X) = \{c \in N_1(X) : (\forall H)(c, H) \geq 0\}$$

# 6 Curves on $X$ and qf types

Fix a constant field $k = k^{alg}$. $K = k(X)$, $X$ a smooth projective variety over k.

Basic objects: pairs $\mathbf{X} = (X, e)$, $e \in N_1^+(X)$.

For each irreducible hypersurfaces $H$ of $X$ we have a valuation on $K$: $v_H(f)$ is the order of vanishing of $f$ at $H$.

A *GVF structure on* $X$ is a GVF structure on $K$, given by a measure concentraing on the valuations $v_H$.

There is a 1-1 correspondence $e \mapsto \mathrm{k}(X, e)$ between $N_1^+(X)$ and GVF structures on $X$.

Given $e \in N_1^+(X)$, define a measure concentrating on $\{v_H\}$, and giving $v_H$ mass $e \cdot H$.

Positivity: $e \in N_1^+$.

Product formula: $e \cdot (f) = 0$ for a principal divisor $(f)$.

The GVF structures on varieties over $k$ are dense in the space of quantifier-free types over $k$.

If $\pi : Y \to X$, $\pi_* e_Y = e_X$, write $\pi : \mathbf{Y} \to \mathbf{X}$. *Do not expect* $K_{\mathbf{X}} \leq K_{\mathbf{Y}}$*! But they agree on Cartier divisors of* $X$*.*

For $L$ a GVF over k, define $\mathbf{X}(L)$ to be the set of points $b$ of $X(L)$ such that for $f \in L$, $(e, (f)^+_X) = ht_L(f)$. I.e. $L$ is the limit of $K_{e_i}$ over an inverse system $(X_i, e_i)$ of blowups converging to the field $K$, with $(\pi_{ij})e_i = e_j$ for $i > j$.

The field $\mathrm{k}(\mathbf{X})$ can be viewed as a "generic point" of $\mathbf{X}$.

## Geometric description of quantifier-free types over k

Let $K$ be a finitely generated field over $k = k^{alg}$. Let $N_1^+(K)$ be the inverse limit of the cones $N_1(X)^+$, $K = k(X)$. It sits in the dual space to the direct limit $N^1(K)$ of the $N^1(X)$. Let $S_K$ be the set of all GVF structures on $K$ over $k$.

We define a map $\alpha : S_K \to N_1^+(K)$ via the pairing

$$S_K \times N^1(K) \to \mathbb{R}$$

$$(p, D) \mapsto \int v(D) d_p(v)$$

**Theorem.** $\alpha : S_K \to N_1^+(K)$ *is a homeomorphism.*

Comparing the topologies:

If $D$ is a Cartier divisor, can find a very ample. $A$ such that $D + A$ is also very ample Then $D + A = (f)_\infty$ and $A = (g)_\infty$. Thus $ht(f) = e \cdot (D + A)$ and $ht(g) = e \cdot A$; so $e \cdot D = ht(f) - ht(g)$.

A similar description of quantifier-free types holds over $k' = k(C)$, $C$ a curve.

But over a general base field, the description using 1-cycles is difficult to work with. How to amalgamate 1-cycles $e, e'$ of $X, X'$ over $U$?

# 7 Relative dimension one.

## Representation of qf types

Let $\pi : X \to U$, $n = \dim(X) = \dim(U) + 1$.

**Theorem.** *For all* $e \in N_1^+(U)$ *away from a lower-dimensional semi-algebraic set, any GVF structure on* $X$ *above* $\mathbf{U} = (U, e)$ *is given by some* $Q \in N^1(X)$*, by:*

$$D \mapsto e \cdot \pi_*(Q \cdot D)$$

*Proof.* Awkward proof using Hodge index on surfaces in $X$, hard Lefschetz. There should be a soft one. □

But once we have this representation, the quantifier-free type over $k(U)$ more precisely a functorial map base change map.

## Extendible qf types

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Let $\mathbf{X}/\mathbf{U}$ be given by the divisor $Q$. Given $\mathbf{U}' \to \mathbf{U}$, pull back $Q$ to a resolution $X'$ of $X \times_U U'$ and use the same formula over $\mathbf{U}'$ to obtain a candidate GVF structure $\mathbf{X}'_Q$. Is it positive?

At the limit, given any GVF $L$ and $b \in \mathbf{U}(L)$, we obtain a candidate GVF structure on $L(X_b)$.

Duality criterion Let $M \models GV$, and let $b \in \mathbf{U}(M)$. Let $Q \in N^1(X)$, giving a GVF structure $\mathbf{X}/\mathbf{U}$.
*Either*
the canonical lift $\mathbf{X}'_Q$ to any $\mathbf{U}'$ realized in $M$ is is a GVF,
*or*
there exists a GVF structure $\mathbf{X}''$ on $X$ over $\mathbf{U}$ realized in $M$, *with* $\mathbf{X}'(Q)$ *negative.*

Why not both? Diagonal on $\mathbf{X} \times \mathbf{X}''$.

Note that the duality gives an axiom of the form: *if* $p|M$ is consistent, then $p|A$ is realized.

Why is this enough? *qf stability*: every qf type $p$ over a large $M$ is the canonical extension of $p|A$ for some $A$.

# 8 G and TM

**Theorem** (TM). *$K = k(t)^{alg}[1]$ is existentially closed. In other words if $K \leq L$ then for any qf formula $\phi(x)$ over $K$, for any $b \in L$ and $\epsilon > 0$ there exists $a \in K$ with $|\phi(b) - \phi(a)| < \epsilon$. In particular, all algebraically closed, nontrivial purely non-archimedean globally valued fields share the same universal theory.*

A weighted curve class in $N_1(X)$ is one of the form $\alpha[C] = (D \mapsto \alpha D \cdot C)$ $C$ an irreducible curve on $X$, $\alpha > 0$.

**Theorem** (G). *Let $X$ be a smooth projective variety over $k$. Then the weighted curve classes in $N_1^+(X)$ (not contained in a prescribed hypersurface) are dense in $N_1^+(X)$.*

In fact the density in (G) is already true for rational push-forwards of $n-1$-fold products of very ample divisors. This is a strong version of a theorem of Boucksom, Demailly, Paun, Peternell (BDPP) to the effect that the moveable curves generate a *dense convex subcone* of $N_1^+(X)$.

G implies TM for $\phi$ over k: Assume $G$. The topologies are the same; so an arbitrary GVF structure on $K \supset \mathbb{Q}^a$ can be approximated by one of the form $\mathbb{Q}^a(X,e)$; and moreover by (G), one can take $e$ to be $\alpha\cdot$ the class of an irreducible curve $c : C \to X$, avoiding some given subvarieties. We can also assume that $\phi$ involves only $v(f(x))$ with $f : X \to \mathbb{P}^1$ a morphism. Now view $c$ as a k$(C)$-point of $X$, and give k$(C)$ the GVF structure with weight $\alpha$ points. Check that $\phi$ has the same value in k$(C)$ and in k$(X,e)$.

TM implies G An element $0 \neq c \in N_1^+(X)$ endows $k(X)$ with a GVF structure; extend it to $L = k(X)^{alg}$. So $X(k(X)) \leq X(L)$ has a tautological point $b$ realizing the corresponding qf type. Choose $t \in L$ which is not constant (i.e. $c \cdot t^{-1}(\infty) \neq 0$.) Using (A), find $a \in K = k(t)^{alg}$. We can approximate the GVF $L$ by an element $X(c')$ with $c' \in k(C)$, $C$ a curve, $t : C \to \mathbb{P}^1$. . It corresponds to a morphism $f : C \to \widetilde{X}$, $\pi : \widetilde{X} \to X$ finite; composing with $\pi$ we can assume $f : C \to X$. Then $\frac{\deg(f)}{\deg t|C}[fC]$ is as close as we wish to $c$.

# 9 Proof of theorem G

The proof follows Boucksom, Favre, Jonsson, with an additional convexity ingredient gleaned from Gromov.

Boucksom, Favre, Jonsson, Differentiability of volumes of divisors and a problem of Teissier. J. Algebraic Geom. 18 (2009), no. 2, 279-308.

Gromov, M. Convex sets and Kähler manifolds. Advances in differential geometry and topology, 1-38, World Sci. Publ., Teaneck, NJ, 1990.

## Okounkov's picture

$K = k(X)$, $d = \dim(X)$. Fix an auxiliary valuation $w : K \twoheadrightarrow \mathbb{Z}^d \leq \mathbb{R}^d$.

Let $D$ be a divisor in the interior of the effective cone. Let

$$K(nD) = \{f \in K^* : (f)_\infty \leq nD\}$$

$W_n = \frac{1}{n} w(K(nD))$ - a finite set of size $\dim K(nD)$. Let $\omega_n$ be the measure giving each point mass $n^{-d}$.

$Ok(D)$ is the closed convex hull of $\cup_n W_n$. The $\omega_n$ converge to Lebesgue measure;

$$\mathrm{vol}(D) := d!\mathrm{vol}(Ok(D))$$

So the volume measures the growth of the dimension of the space of sections of $nD$.

From Brunn-Minkowski we obtain a fundamental hyperbolicity:

$\mathrm{vol}^{1/d}$ is concave on the interior of the effective cone.

Restricted to the ample cone when $d = 2$, this is equivalent to the Hodge index theorem; the intersection pairing has signature $(1, -1, \ldots, -1)$. (Weil, Hodge, ... Khovanskii, Tessier,... Lazarsfeld, Mustata...)

There is also Okounkov's finiteness principle:

A compact $C \subset Ok(D)^o$ is contained in image of a finitely generated subalgebra.

From this it is easy to deduce Fujita's approximation theorem.

*Let $D$ be an effective divisor on $X$, and let $\epsilon > 0$. Then there exists a birational morphism $\phi : X' \to X$, $D' = \phi^* D$, with $D' \geq A/m$ for some divisor $A$ generated by global sections, with* $\mathrm{vol}(A/m) > (1 - \epsilon)\mathrm{vol}(D)$. *Also $A$ is $\epsilon$-close to an ample divisor.*

$$Fuj(X, D) = \{(f, A) : f : X' \to X \text{ birational }, A \text{ ample on } X', A \leq f^*(D)\}$$

Fujita's theorem expresses the volume as a *positive intersection product (BFJ)*:

$$\mathrm{vol}(D) = \sup\{A^d : (f, A) \in Fuj(X, D)\}$$

Let $U$ be the interior of the effective cone of divisors.

**Theorem** (BFJ)**.** vol *is differentiable on $U$, continuous on $cl(U)$ and vanishes on the boundary. On $U$ we have*

$$d\mathrm{vol}(x)/d = \psi(x) := \sup\{f_* y^{d-1} : (f, y) \in Fuj(X, x)\}$$

**Proposition.** *The interior of the moveable cone of curves lies in the image of* $\psi$.

Let $\phi = \text{vol}^{1/d}$, so $\phi$ is concave and $d\phi$ is proportional to $\psi$. The differential $l$ of $\phi =$ at a point $u$ defines a linear function $l$ on $U$; by concavity of $\phi$, $l$ lies above the graph of $\phi$ on $U$. Conversely if a linear function $l$ lies above $\phi$ on $U$, we can tilt to $\gamma l$ $(0 < \gamma \leq 1)$ it until it first meets the graph of $\phi$; then $\gamma l$ is in the image of $d\phi$. It follows that $Im(\psi) = \mathbb{R}^{>0} Im(d\phi) \supset (U^*)^o = N_1^+(X)^o$. □

Theorem (G) follows at once from this Yau-type result: approximate the amples in $Fuj(X, D)$ by rational multiples of very ample Cartier divisors; Bertini.

# 10 Beyond the universal theory

13/01/2016, at 10am, IHP room 314, and subsequent Wednesdays.

## Working goals

GV has a model companion $\widetilde{GVF}$.

$\widetilde{GVF}$ admits quantifier-elimination to the level of algebraically closed sets (model theoretically).

Stability: basic geometric types; qf types (blowing ups); bounded quantifiers.

## global relative modularity

**Proposition.** *Assume* $(a_i)$ *forms an indiscernible sequence over* $K$*, in* $L$*. Assume it is a Morley sequence locally almost everywhere. Also assume it is pairwise independent. Then it is independent.*

A Morley sequence locally almost everywhere: for almost all choices of nontrivial valuations $v$ of $K$ (if any) and $v_i$ of $K(a_i)$ over $v$, the measure $\mu_L$ relative to $(v_i)$ concentrates on the independent amalgam

## Abelian varieties

Let $A$ be an Abelian variety, and let $M$ be the group of points of height 0. There is a natural Hilbert space structure on $A/M$. On the other hand the structure on $M$ can be conjectured to be stable of finite U-rank, when the variety has no isotrivial components over the constants. (related to Bogomolov conjecture.)