LECTURES ON REPRESENTATION THEORY AND L FUNCTIONS FOR $U(2,1)$

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General references on smooth representations of p-adic groups

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1. Group theory of the unitary group in three variables

1.1. The unitary group.

1.1.1. *Linear algebra.* Let E/F be a quadratic field extension, $char.F \neq 2$.[1]

Let $z \mapsto \bar{z}$ be the non-trivial automorphism of E.

Let $E^1 = \ker N_{E/F} : E^\times \to F^\times$.

[1]This assumption is unfortunate. Bilinear forms exhibit pathologies in characteristic 2. However, even when we consider fields of characteristic 0, as number fields or p-adic fields, integral structures inevitably lead to the study of the group modulo p, and when $p = 2$ one needs to confront issues ignored here.

Let $\iota \in E$, $\bar{\iota} = -\iota$, $\mathrm{Im}(z) = (z - \bar{z})/2\iota$, $\mathrm{Re}(z) = (z + \bar{z})/2$, so that $z = \mathrm{Re}(z) + \iota\,\mathrm{Im}(z)$.

Let V be a 3-dimensional vector space over E, and

$$(\,,\,) : V \times V \to E \tag{1.1}$$

a hermitian non-degenerate bilinear form, conjugate-linear in the first variable.

Hypothesis: V has an isotropic vector, i.e. $0 \neq v$ satisfying $(v, v) = 0$.

Lemma 1.1. *After multiplying* $(\,,\,)$ *by a non-zero scalar from* F *we can find a basis* e_1, e_2, e_3 *w.r.t. which the matrix* $((e_i, e_j))$ *is*

$$S = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}. \tag{1.2}$$

Proof. Easy exercise. ∎

Note that for the purpose of studying the unitary group of $(V, (\,,\,))$ we may rescale the hermitian form as in the lemma. We do it, and fix a basis as in the lemma, identifying V with E^3. Then

$$(u, v) = {}^t\bar{u}Sv. \tag{1.3}$$

1.1.2. *The unitary group.* We set

$$U = \{g \in GL(V) | (gu, gv) = (u, v)\, \forall u, v\}. \tag{1.4}$$

This is actually an algebraic group over F. We write $\mathbf{U}$ for the algebraic group, so that $U = \mathbf{U}(F)$. This notational convention will apply to any algebraic group without further ado. In coordinates, $g \in GL_3(E)$ belongs to U if and only if

$$g^* g = 1 \tag{1.5}$$

where $g^* = S^{-1}\,{}^t\bar{g}S$. The group U is called the *quasi-split unitary group in three variables.* If V is anisotopic (does not have an isotropic vector), one gets a different group. When $F = \mathbb{R}$, for example, the group that we have just defined is commonly denoted $U(2, 1)$, and the symbol $U(3)$ is reserved to the compact unitary group preserving the standard inner product on $\mathbb{C}^3$. On the other hand, when F is a non-archimedean local field, by the [Serre, Ch. IV, Theorem 6] every 3-dimensional hermitian space has an isotropic vector, so this is the only unitary group in three variables.

Lemma 1.2. *The special unitary group* $SU = U \cap SL(V)$ *is also the commutator subgroup of* U.

Proof. Let $\mathfrak{u}$ and $\mathfrak{su}$ be the Lie algebras of $\mathbf{U}$ and $\mathbf{SU}$. Since the commutator group of $\mathbf{U}$ is clearly contained in $\mathbf{SU}$, and $\mathbf{SU}$ is connected, it is enough to check that $[\mathfrak{u}, \mathfrak{u}] = \mathfrak{su}$, which is easy, to get that $\mathbf{SU} = \mathbf{U}'$ as algebraic groups. But $\mathbf{U}'(F) = U'$. ∎

If L is any field *containing* E then $L \otimes_F V \simeq (L \otimes_E V) \oplus (L \otimes_{\tau,E} V)$ where $\tau : E \hookrightarrow L$ is the conjugate of the given embedding, so

$$\mathbf{U}(L) \simeq GL_3(L) \tag{1.6}$$

(under projection to the first factor).

1.1.3. *The standard Borel subgroup.* The vector e_1 is isotropic, $\langle e_1\rangle^\perp = \langle e_1, e_2\rangle$, so the stabilizer P of the line $\langle e_1\rangle$ consists of the upper-triangular matrices in U. We have

$$P = MN \tag{1.7}$$

where

$$M = \left\{ m(t,s) = \begin{pmatrix} t & & \\ & s & \\ & & \bar{t}^{-1} \end{pmatrix} | t \in E^\times, s \in E^1 \right\} \tag{1.8}$$

and

$$N = \left\{ n(b,z) = \begin{pmatrix} 1 & b & z \\ & 1 & -\bar{b} \\ & & 1 \end{pmatrix} | z + \bar{z} = -b\bar{b} \right\} \tag{1.9}$$

(check!). As algebraic groups, M is a 3-dimensional torus with split rank 1, and N is a 3-dimensional unipotent group. The center of U is $C = E^1 \subset M$. The group N sits in a short exact sequence

$$0 \to Z \to N \to E \to 0 \tag{1.10}$$

where $Z = \{n(0,z) | z \in \iota F\} \simeq F$ is the center of N and the map $N \to E$ is $n(b,z) \mapsto b$. Note that $Z = N'$, so N is nilpotent of length 2.

The character group and cocharacter group of M are of rank 1. A generator of $X^*(\mathbf{M})$ is the map $\mathbf{M} \to \mathbb{G}_m$ given by $m(t,s) \mapsto t\bar{t}$. A generator of $X_*(\mathbf{M})$ is the map $\mathbb{G}_m \to \mathbf{M}$, $t \mapsto m(t,1)$.

1.1.4. *The Bruhat decomposition.* Let

$$w = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}. \tag{1.11}$$

Then $w^2 = 1$. (The sole advantage of w over S is that $w \in SU$. Rogawski [Rog] defines U using w instead of S.)

Proposition 1.3. *We have*

$$U = P \cup PwP \tag{1.12}$$

and $PwP = NwP = PwN$. The expression of an element from the "big cell" PwP as nwp (or pwn) with $p \in P$ and $n \in N$ is unique. We have

$$w^{-1} m(t,s) w = m(\bar{t}^{-1}, s). \tag{1.13}$$

A similar decomposition holds in SU.

We emphasize that the Bruhat decomposition is not a decomposition as algebraic groups. There are 6 double cosets of the Borel needed to cover $GL_3(E)$.

Proof. Suppose $g \notin P$, i.e. $ge_1 \notin \langle e_1\rangle$. If $ge_1 = ae_1 + be_2$ then $0 = (ge_1, ge_1) = b\bar{b}$, contradiction. Therefore ge_1 must involve e_3. There exists then a $p \in P$ satisfying $p^{-1}(e_3) = ge_1$ (check!) so that $pge_1 = e_3$. We get $pge_2 \in \langle e_3\rangle^\perp = \langle e_2, e_3\rangle$ so

$$pg = \begin{pmatrix} & & * \\ & * & * \\ * & * & * \end{pmatrix} \in wP \tag{1.14}$$

and $g \in PwP$. The rest is clear. ∎

Corollary 1.4. *The group* $\mathbf{P}$ *is, up to conjugation, the only* F*-rational proper parabolic subgroup of* $\mathbf{U}$.

Proof. Since $\mathbf{P}$ is a minimal parabolic subgroup, if $\mathbf{Q}$ is any F-rational parabolic subgroup, after conjugation we may assume $\mathbf{P} \subset \mathbf{Q}$. But then the F-rational points satisfy $P = Q$ and since both are affine geometrically connected F-groups, this is enough to guarantee that $\mathbf{P} = \mathbf{Q}$. ∎

1.1.5. *The root system.* Let $A \subset M$ be the group

$$A = \left\{ m(t,1) | t \in F^\times \right\} \tag{1.15}$$

and $\mathbf{A} \simeq \mathbb{G}_m$ the corresponding algebraic group. Then $\mathbf{A}$ is a maximal split torus of $\mathbf{M}$. We identify $X^*(\mathbf{A})$ with $\mathbb{Z}$, the character

$$\lambda : m(t,1) \mapsto t \tag{1.16}$$

sent to $1 \in \mathbb{Z}$. Note that $X^*(\mathbf{M}) \subset X^*(\mathbf{A})$ is then identified with $2\mathbb{Z}$.

The Lie algebra $\mathfrak{u}$ of $\mathbf{U}$ is a vector space over F. The root algebras $\mathfrak{u}_\lambda$ and $\mathfrak{u}_{2\lambda}$ are non-zero. We have

$$\mathfrak{u}_\lambda = \left\{ \begin{pmatrix} & b & \\ & & -\bar{b} \\ & & \end{pmatrix} | b \in E \right\}, \quad \mathfrak{u}_{2\lambda} = \left\{ \begin{pmatrix} & & z \\ & & \\ & & \end{pmatrix} | z \in \iota F \right\} \tag{1.17}$$

$\dim_F \mathfrak{u}_\lambda = 2$, $\dim_F \mathfrak{u}_{2\lambda} = 1$ and $[\mathfrak{u}_\lambda, \mathfrak{u}_\lambda] = \mathfrak{u}_{2\lambda}$. The root system $\Sigma = \{\pm\lambda, \pm 2\lambda\}$ is non-reduced. The Cartan subalgebra is

$$\mathfrak{h} = \left\{ \begin{pmatrix} t & & \\ & s & \\ & & -\bar{t} \end{pmatrix} | s \in \iota F, t \in E \right\} \tag{1.18}$$

and $\mathfrak{u} = \mathfrak{h} \oplus \bigoplus_{\mu \in \Sigma} \mathfrak{u}_\mu$. The splitting of these groups over E, where the group $\mathbf{U}$ becomes GL_3, is clear.

1.1.6. *The embedding of* $U(2)$ *in* $U(3)$. The group $\mathbf{H} = \mathbf{U(2)}$ is the stabilizer of e_2 (it is the *quasi-split* unitary group in two variables). In concrete terms

$$H = \left\{ \begin{pmatrix} * & & * \\ & 1 & \\ * & & * \end{pmatrix} \in U \right\}. \tag{1.19}$$

Its intersection with $\mathbf{U}'$ is $\mathbf{H}' = \mathbf{SU(2)}$. It is isomorphic, as an algebraic group over F, to $\mathbf{SL}_2$. Map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2 \mapsto \begin{pmatrix} a & \iota b \\ \iota^{-1} c & d \end{pmatrix} \in \mathbf{SU(2)}. \tag{1.20}$$

This gives the isomorphism. However, $\mathbf{U(2)}/\mathbf{SU(2)} \simeq \mathbf{E}^1$ while $\mathbf{GL}_2/\mathbf{SL}_2 \simeq \mathbf{F}^\times$, so $\mathbf{U(2)}$ and $\mathbf{GL}_2$ are not isomorphic.

1.2. **Now let F be p-adic.**

1.2.1. *The unimodular character of P.* Let F be a locally compact non-archimedean field of residue characteristic p. Let $\mathbb{C}$ be a fixed algebraically closed field of characteristic 0. Then there is a $\mathbb{C}$-valued finitely additive Haar distribution on U, which is both left and right invariant. Let π_E and q_E be a uniformizer and the residue field cardinality for E. Let

$$\omega = \omega_E : E^\times \to \mathbb{C}^\times \tag{1.21}$$

be the unramified character defined by $\omega_E(\pi_E) = q_E^{-1}$. If we similarly defined ω_F then $\omega_E = \omega_F \circ N_{E/F}$. The unimodular character of P is then

$$\delta(p) = \omega_E^2(t) \tag{1.22}$$

if $p = m(t,s)n(b,z) \in P$. Indeed

$$m(t_1,1)n(b,z)m(t_1,1)^{-1} = n(t_1 b, t_1\bar{t}_1 z) \tag{1.23}$$

so if $m = m(t_1,s_1)$, $U \subset N$ is open compact and dn is the Haar measure on N, $[mUm^{-1} : U] = \omega_E(t_1)\omega_F(t_1\bar{t}_1) = \omega_E(t_1)^2$ and

$$\int_U d(mnm^{-1}) = \int_{mUm^{-1}} dn = \omega_E^2(t_1)\int_U dn. \tag{1.24}$$

The left invariant Haar measure on P is

$$d_L p = d^\times t d^\times s dn \tag{1.25}$$

if $p = m(t,s)n$. The right invariant measure is

$$d_R p = \omega_E^2(t) d_L p. \tag{1.26}$$

1.2.2. *Characters of M.* A (smooth) character $\sigma : M \to \mathbb{C}^\times$ is called *regular* if $\sigma^* = w(\sigma) \neq \sigma$. Here

$$\sigma^*(m) = w(\sigma)(m) = \sigma(w^{-1}mw) = \sigma(\bar{m}^{-1}). \tag{1.27}$$

Clearly σ is *irregular* if and only if $\sigma(m\bar{m}) = 1$.

An *unramified character* (a character trivial on $E^1 \times E^1$) is completely determined by $\sigma(m(\pi_E,1))$. It is regular iff this is not ± 1.

The contragredient of σ is $\tilde{\sigma} = \sigma^{-1}$.

If σ is a smooth character of M we let $\mathbb{C}_\sigma$ be the one-dimensional space on which M acts via σ.

1.2.3. *The Iwasawa decomposition.* Let K be the compact subgroup of U stabilizing the lattice $\mathcal{O}_E^3$.

Proposition 1.5. *We have*

$$U = PK \tag{1.28}$$

Proof. Given $g \in U$ we have to show $g \in PK$. Write $l_1 = g^{-1}e_1$. Multiplying g on the left by an element of M we may assume that $l_1 \in \mathcal{O}_E^3$ is primitive. Write

$$l_1 = ae_1 + be_2 + ce_3, \tag{1.29}$$

then

$$0 = (l_1,l_1) = a\bar{c} + b\bar{b} + c\bar{a} = b\bar{b} + Tr_{E/F}(c\bar{a}). \tag{1.30}$$

If both a and c are non-units, b can not be a unit and l_1 is not primitive. Replacing g by gw if necessary, l_1 will be replaced by $w^{-1}l_1$, so we may assume that $c = 1$, and

$$-b\bar{b} = a + \bar{a}. \tag{1.31}$$

But this means that

$$n(-\bar{b}, a)w(e_1) = n(-\bar{b}, a)e_3 = l_1. \tag{1.32}$$

Letting $g_1 = n(-\bar{b}, a)w \in K$, $g_1(e_1) = l_1$, so $gg_1 \in P$ and $g \in PK$ as desired. ∎

1.2.4. *Iwahori factorization.*

Proposition 1.6. *Let K_n be the principal congruence group in K modulo π_E^n ($n \geq 1$). Let $N_n = K_n \cap N$, $N_n^w = K_n \cap N^w$, $M_n = K_n \cap M$. Then (i)*

$$K_n = N_n^w M_n N_n \tag{1.33}$$

and the representation of an element of K_n as a product of elements from the three groups is unique.

(ii) If $a = m(t, 1) \in A$ and $|t| \leq 1$ (we denote this semigroup by A^-) then $aN_na^{-1} \subset N_n$, and $a^{-1}N_n^w a \subset N_n^w$.

Proof. (i) If $g \in K_n$ and $g^{-1}e_3 = l_3$ then $l_3 \equiv e_3 \bmod \pi_E^n$ and there is a $p \in P_n = M_nN_n$ such that $p^{-1}e_3 = l_3$. Then $gp^{-1} \in K_n$ but also stabilizes e_3, so belongs to $P_n^w = N_n^w M_n$ and we get $g = (gp^{-1})p$. The rest is clear. (ii) is also clear. ∎

1.2.5. *Cartan decomposition.* This is the decomposition

$$U = K\tilde{A}K \tag{1.34}$$

where $\tilde{A} = \{m(t, 1)| t \in E^\times\}$. If E/F is unramified we may take A instead of $\tilde{A}$. More economically, we have

$$U = \bigcup_{n \geq 0} K \begin{pmatrix} \pi_E^n & & \\ & 1 & \\ & & \bar{\pi}_E^{-n} \end{pmatrix} K. \tag{1.35}$$

2. Smooth representations

In this section we write G for the unitary group U.

2.1. Generalities.

2.1.1. *Admissible representations and the contragredient.* From now on, let F be a local nonarchimedean field. All representations will be over $\mathbb{C}$. A representation of G is *smooth* if the stabilizer of every vector is open. If (ρ, V) is a smooth representation of G, we denote by $(\tilde{\rho}, \tilde{V})$ its contragredient (smooth dual). The space $\tilde{V}$ is the space of smooth vectors (vectors with open stabilizer) in the algebraic dual V' of V.

Recall that a smooth representation is *admissible* if and only if for every open subgroup U, $\dim V^U < \infty$. Let U be a compact open subgroup. Then the smooth G-module V decomposes as

$$V = V^U \oplus V(U) \tag{2.1}$$

where $V(U) = \ker \mathcal{P}_U$,

$$\mathcal{P}_U(v) = \int_U \rho(g)v \cdot dg \tag{2.2}$$

and the Haar measure is normalized so that $\int_U dg = 1$. If we let U vary over the principal congruence groups K_n we see that

$$V = \bigoplus_{\tau \in \widehat{K}} V(\tau) \tag{2.3}$$

decomposes according to K-types ($\widehat{K}$ is the smooth dual, which is of course also the unitary dual since K is compact) and

$$\tilde{V} = \bigoplus_{\tau \in \widehat{K}} V(\tau)' \tag{2.4}$$

where $V(\tau)'$ is the algebraic dual. The following lemma is then obvious.

Lemma 2.1. *The following are equivalent: (i) V is admissible (ii) for each τ, $\dim V(\tau) < \infty$ (iii) the canonical embedding of ρ in the contragredient of $\tilde{\rho}$ is an isomorphism (iv) $\tilde{V}$ is admissible.*

Corollary 2.2. *If ρ is admissible and irreducible, so is $\tilde{\rho}$.*

Smooth irreducible representations are in fact admissible, but this is harder. XXX

Lemma 2.3. *Finitely generated admissible representations are of finite length [Cass, 6.3.10].*

2.1.2. *The category of smooth representations.* The category of smooth representations is abelian. The category of admissible representations is a full abelian subcategory. *Schur's lemma* holds: if ρ is an (admissible) irreducible representation, then $End_G(\rho) = \mathbb{C}$. The converse is not true.

An admissible representation ρ is *unitarizable* if it carries an invariant inner product. In such a case it is completely decomposable: If $W \subset V$ is a sub-representation, then $V = W \oplus W^{\perp}$. This is because $W = \oplus_{\tau \in \widehat{K}} W(\tau)$ is an orthogonal direct sum and each $V(\tau) = W(\tau) \oplus W(\tau)^{\perp}$ by finite dimensionality.

2.2. **Parabolic induction and supercuspidals.**

2.2.1. *The Jacquet module.* Let V be a smooth N-module. Let $V(N)$ be the submodule generated by $\rho(n)v - v$ for $v \in V$ and $n \in N$. The *Jacquet module* is

$$V_N = V/V(N). \tag{2.5}$$

V_N is the largest quotient of V on which N acts trivially:

$$Hom_N(V, \mathbb{C}) = Hom(V_N, \mathbb{C}). \tag{2.6}$$

Here are three well-known easy properties of the Jacquet module.

1. $v \in V(N)$ if and only if there exists a compact open $N_0 \subset N$ such that

$$\mathcal{P}_{N_0}(v) = \int_{N_0} \rho(n)v dn = 0. \tag{2.7}$$

2. The functor $V \rightsquigarrow V_N$, from the category of smooth N-modules to vector spaces, is exact. This follows from the first property.

3. If V is a smooth P-module, then V_N is a smooth M-module.

Proposition 2.4. *If (ρ, V) is finitely generated (resp. admissible) as a G-module, then V_N is finitely generated (resp. admissible) as an M-module.*

Proof. Suppose X is a finite set generating V over G. Let U be open compact such that $X \subset V^U$. Let Γ be a finite set of G such that $G = P\Gamma U$. Then $\rho(\Gamma)X$ generates V_N as an M-module.

To prove that if V is G-admissible, V_N is M-admissible, let $n \geq 1$ and let

$$K_n = N_n M_n N_n^w \tag{2.8}$$

be the Iwahori decomposition of level π_E^n. We shall prove a more precise result, attributed to Borel, that the canonical projection maps V^{K_n} onto $V_N^{M_n}$. First note that the image of V^{K_n} in V_N is the same as the (a-priori larger) image of $V^{M_n N_n^w}$. Indeed, if $v \in V^{M_n N_n^w}$ and we average it over K_n, then $\mathcal{P}_{K_n}(v)$ is a finite linear combination of N_n-translates of v, but as they all have the same image in V_N, $\mathcal{P}_{K_n}(v)$ and v have the same image in V_N.

Next, let $W \subset V_N^{M_n}$ be a finite dimensional subspace. Let $\bar{v} \in W$. We may assume that the vector v representing $\bar{v}$ comes from V^{M_n}. It is then in $V^{M_n N_m^w}$ for some m (possible very large). But then for some $a = m(t, 1) \in M$ we have

$$\rho(a)v \in V^{M_n N_n^w}. \tag{2.9}$$

All that we have to make sure is that $aN_m^w a^{-1} \supset N_n^w$. Since W is finite dimensional, there is an a such that $\rho(a)W$ is fixed by $M_n N_n^w$. By the first part of the proof, the image of $\rho(a)W$ in V_N is contained in the image of V^{K_n}. Since V is admissible, the dimension of W is bounded, hence $\dim V_N^{M_n} < \infty$ and V_N is admissible. We may furthermore take now $W = V_N^{M_n}$ and get that $W = \rho(a)W$ is contained in the image of V^{K_n}. ∎

Corollary 2.5. *Let (ρ, V) be admissible and irreducible. Assume $V_N \neq 0$. Then there is a character σ of M and a non-zero homomorphism $V_N \to \mathbb{C}_\sigma$ of M-modules.*

Proof. V_N is finitely generated and admissible as an M-module. Zorn's lemma implies, since V_N is finitely generated, that it has a maximal proper submodule, hence an irreducible quotient W. Let $v \in V_N$ map to $w \in W$. There exists an open compact M_0 in M so that $v \in V_N^{M_0}$, and since V_N is admissible this is a finite dimensional space. Since M is abelian, $V_N^{M_0}$ is M-stable. Thus v, and also w, belong to a finite dimensional M-module. Since W is irreducible, W is finite dimensional, and since M is commutative it must be 1-dimensional. ∎

2.2.2. *Parabolic induction.* If σ is a smooth character of M then $Ind_P^G \sigma$ is the smooth representation of all locally constant functions $f : G \to \mathbb{C}$ satisfying

$$f(pg) = \sigma(p) f(g) \tag{2.10}$$

($p \in P$) with G acting by right translation. Recall that the unimodular character of P is $\delta(p) = \omega(t)^2$ (if $p = m(t, s)n \in MN$) and we write $\delta^{1/2}(p) = \omega(t)$. We let

$$i(\sigma) = Ind_P^G(\delta^{1/2}\sigma). \tag{2.11}$$

Here are again some well-known facts.

1. Let $\tilde{\sigma} = \sigma^{-1}$ (the contragredient of σ) then the contragredient of $i(\sigma)$ is $i(\tilde{\sigma})$. This follows from the fact that for $f_1 \in i(\sigma), f_2 \in i(\tilde{\sigma})$, the integral

$$\langle f_1, f_2 \rangle = \int_K f_1(g) f_2(g) dg \tag{2.12}$$

is G-invariant. See [Cass, 3.1.3],[Bump, 2.6.1].

2. If σ is unitary, $\sigma^{-1} = \bar{\sigma}$, then $i(\sigma)$ is unitarizable. *Proof:* by the first fact, the contragredient of $i(\sigma)$ is $i(\bar{\sigma}) = \overline{i(\sigma)}$, hence the pairing $\langle , \rangle$ defined above becomes an invariant inner product on $i(\sigma)$.

3. (Frobenius reciprocity [Cass, 3.2.4]) For any smooth (ρ, V)

$$\begin{aligned} Hom_G(V, i(\sigma)) &= Hom_P(V, \delta^{1/2}\sigma) \\ &= Hom_M(V_N, \delta^{1/2}\sigma). \end{aligned} \tag{2.13}$$

The maps are $T \leftrightarrow t$ where $Tv(g) = t(g.v)$ and $t(v) = Tv(1)$.

2.2.3. *Supercuspidals.*

Definition 2.1. *An irreducible admissible representation (ρ, V) of G is called* supercuspidal *if $V_N = 0$.*

Proposition 2.6. *An irreducible admissible representation (ρ, V) of G is* not supercuspidal *if and only if it* embeds in some $i(\sigma)$.

Proof. If V_N is not 0, we have seen that there is a homomorphism $V_N \to \mathbb{C}_\sigma$ hence, by Frobenius reciprocity, an embedding $V \hookrightarrow i(\delta^{-1/2}\sigma)$. Conversely, if $Hom_G(V, i(\sigma)) \neq 0$ then $V_N \neq 0$ again by Frobenius reciprocity. ∎

Proposition 2.7. *An irreducible admissible representation is supercuspidal if and only if its matrix coefficients are compactly supported modulo the center.*

Proof. There are several ways to prove the proposition, all relying on the Cartan decomposition $G = KAK$. One, explained in [G, 1.7] for $GL_2(F)$, is to use the Kirillov model. In general, the theorem is due to Harish-Chandra, see [Cass, 5.3.1]. The idea is this. Suppose $V = V(N)$. Let $v \in V$ and $\tilde{v} \in \tilde{V}$. Since the K-orbits of v and $\tilde{v}$ are finite, it is enough to show that

$$\langle \tilde{v}, \rho(a)v \rangle \tag{2.14}$$

has a compact support on A. Let $N_1 \subset N_2 \subset N$ be compact open such that $v \in V(N_2)$ and $\tilde{v} \in \tilde{V}^{N_1}$. Let A^- be the semigroup of all $a = m(t, 1) \in A$ satisfying $|t| \leq 1$. Let $A^-(\varepsilon)$ be the subset where $|t| < \varepsilon$. For small enough ε, if $a \in A^-(\varepsilon)$, then

$$aN_2a^{-1} \subset N_1. \tag{2.15}$$

But now ($*$ denoting a constant depending on the Haar measures)

$$\begin{aligned} \langle \tilde{v}, \rho(a)v \rangle &= * \int_{aN_2a^{-1}} \langle \tilde{\rho}(n)^{-1}\tilde{v},\, \rho(a)v \rangle \, dn \\ &= * \int_{N_2} \langle \tilde{v}, \rho(a)\rho(n)v \rangle \, dn \\ &= 0 \end{aligned} \tag{2.16}$$

since $\mathcal{P}_{N_2}(v) = 0$. Working with the opposite N^w (and using the fact that $V = V(N^w)$) we find the same for $a \in A^+(\varepsilon)$ for ε small enough. Hence the support of

$\langle\tilde{v},\rho(a)v\rangle$ lies in $\varepsilon \le |t| \le |\varepsilon|^{-1}$ which is compact. For the converse direction, either compute directly using the Kirillov model, or look at [Cass, 5.3.1]. ∎

Corollary 2.8. *(i) An irreducible admissible (ρ, V) is supercuspidal if and only if $\tilde{\rho}$ is supercuspidal (ii) [G, 1.18] A supercuspidal representation is unitarizable if and only if its central character is unitary.*

Proof. Since ρ is equal to its double contragredient, the matrix coefficients of ρ and of $\tilde{\rho}$ are the same. For (ii), if the central character is unitary, take a non-zero smooth functional $\tilde{v}$ and define

$$(v_1, v_2) = \int_{C\backslash G} \langle\tilde{v},\rho(g)v_1\rangle\,\overline{\langle\tilde{v},\rho(g)v_2\rangle}dg. \tag{2.17}$$

∎

Proposition 2.9. *Supercuspidal representations are both injective and projective.*

Proof. We prove the proposition when $G = SU$ so the center C is trivial, and leave the easy modification when $C \ne 1$ to the reader (let ω be the central character of ρ, and replace $C_c^\infty(G)$ by $\mathcal{H}_\omega$, the Hecke algebra of locally constant functions on G whose support is compact modulo the center and which transform under C by ω). Let (ρ, V) be irreducible admissible and supercuspidal. Since the matrix coefficients are compactly supported, choosing $\tilde{v}_0 \in \tilde{V}$ we can embed $\iota : V \hookrightarrow C_c^\infty(G)$ (with the right regular representation) via

$$\iota : v \mapsto c_{\tilde{v}_0,v}(g) = \langle\tilde{v}_0,\rho(g)v\rangle\,. \tag{2.18}$$

Choose $v_0 \in V$ and define a projection $\Pi_V : C_c^\infty(G) \to V$

$$\Pi_V\phi = \int_G \phi(g)\rho(g^{-1})v_0dg. \tag{2.19}$$

With a suitable choice of v_0 and $\tilde{v}_0$ we may assume that $\Pi_V \circ \iota$ is the identity. It is enough to note that if $\langle\tilde{v}_0, v_0\rangle \ne 0$, it is not identically 0, since

$$\begin{aligned}\langle\tilde{v},\Pi_V\circ\iota(v)\rangle &= \int_G \langle\tilde{v}_0,\rho(g)v\rangle\left\langle\tilde{v},\rho(g^{-1})v_0\right\rangle dg \\ &= d_\rho^{-1}\langle\tilde{v}_0,v_0\rangle\langle\tilde{v},v\rangle\,. \end{aligned}\tag{2.20}$$

where $d_\rho > 0$ is the *formal degree* of the supercuspidal (or more generally, square-integrable) representation ρ. To check that the expression defining d_ρ is not identically zero, note that we may assume, after a twist by a character, that ρ is unitary, and then the integrand can be written (with suitable u_0 and u in V)

$$\begin{aligned}\langle\tilde{v}_0,\rho(g)v\rangle\left\langle\tilde{v},\rho(g^{-1})v_0\right\rangle &= (u_0,\rho(g)v)(u,\rho(g^{-1})v_0) \\ &= (u_0,\rho(g)v)\overline{(v_0,\rho(g)u)}. \end{aligned}\tag{2.21}$$

Now choose $u_0 = v_0$ and $u = v$.

Let now (U, σ) be a smooth representation and $T : U \to V$ a G-homomorphism. Pick u_0 mapping to v_0 and define similarly $\Pi_U : C_c^\infty(G) \to U$

$$\Pi_U\phi = \int_G \phi(g)\sigma(g^{-1})u_0dg. \tag{2.22}$$

Then $T \circ \Pi_U = \Pi_V$ so $T \circ \Pi_U \circ \iota$ is the identity and $\Pi_U \circ \iota$ splits T. This proves that V is projective in the category of smooth representations.

To prove injectivity, if $V \hookrightarrow U$, then since $\tilde{V}$ is also supercuspidal the projection $\tilde{U} \to \tilde{V}$ splits, hence dualizing again V is a direct summand of U and V is injective. ∎

2.3. **The structure of $i(\sigma)$.** Let σ be a smooth character of M and $I = i(\sigma) = Ind_P^G(\delta^{1/2}\sigma)$. Since $f \in I$ is determined by $f|_K$ it is easily seen that I is admissible. Recall that $\sigma^* = w(\sigma)$ and σ is regular if $\sigma \neq \sigma^*$. Write $I^* = i(\sigma^*)$.

Example 2.1. *$i(\delta^{-1/2}) = Ind_P^G 1 = C^\infty(P\backslash G)$ has an exact sequence*

$$0 \to \mathbb{C} \to Ind_P^G 1 \to St \to 0 \tag{2.23}$$

where St is the Steinberg representation. We then have also

$$0 \to St \to Ind_P^G \delta \to \mathbb{C} \to 0. \tag{2.24}$$

The fact that St is its own contragredient is not hard, but also not obvious. It will follow from Proposition 2.17.

2.3.1. *I_N is always two dimensional.*

Lemma 2.10. *Let $I = (\pi, i(\sigma))$ as before. Then* $\dim I_N \leq 2$.

Proof. Suppose that $\Lambda : I \to \mathbb{C}$ is a linear functional satisfying $\Lambda(\pi(n)f) = \Lambda(f)$ for all $n \in N$. Let

$$\mathcal{P} : C_c^\infty(G) \to I \tag{2.25}$$

be the integral

$$\mathcal{P}\phi(g) = \int_P \phi(p^{-1}g) \cdot \sigma\delta^{1/2}(p) \cdot dp \tag{2.26}$$

where dp is a left invariant Haar measure on P. Then $\mathcal{P}$ is a G-homomorphism (G acting by the right regular representation on $C_c^\infty(G)$) and is onto: If $f \in I$ and $\phi = f \cdot \chi_K$ then, up to a scalar, $\mathcal{P}\phi$ gives us back the function f.

Let $\Delta = \Lambda \circ \mathcal{P}$, a ditstribution on G (a linear functional on $C_c^\infty(G)$). Denoting by $\mathcal{D}(X)$ the algebraic dual of $C_c^\infty(X)$ we have short exact sequences

$$0 \to C_c^\infty(PwP) \to C_c^\infty(G) \to C_c^\infty(P) \to 0 \tag{2.27}$$

$$0 \to \mathcal{D}(P) \to \mathcal{D}(G) \to \mathcal{D}(PwP) \to 0. \tag{2.28}$$

We let λ and ρ be the left and right regular representations of P on these spaces. Then

$$\begin{aligned}(\lambda(p_1)\Delta)(\phi) &= \Delta \circ \lambda(p_1^{-1})\phi \\ &= \Lambda\left(g \mapsto \int_P \phi(p_1 p^{-1} g) \cdot \sigma\delta^{1/2}(p) \cdot dp\right) \\ &= \sigma\delta^{-1/2}(p_1)\Delta(\phi).\end{aligned} \tag{2.29}$$

On the other hand, clearly $\rho(n_1)\Delta = \Delta$ for every $n_1 \in N$, since Λ is N-invariant. It follows that

$$\lambda(p_1)\Delta = \sigma\delta^{-1/2}(p_1)\Delta, \;\; \rho(n_1)\Delta = \Delta. \tag{2.30}$$

To prove the proposition it is enough to prove, since $\mathcal{P}$ is surjective, that the space of distributions Δ satisfying these two conditions is at most 2-dimensional. For that it is enough to show that the space of Δ's satisfying these conditions in $\mathcal{D}(P)$

or in $\mathcal{D}(PwN)$ is at most one-dimensional, but this is essentially the uniqueness of Haar measure. ∎

Lemma 2.11. *If $n = n(b,z)$ then*

$$wn = \begin{pmatrix} \bar{z}^{-1} & -bz^{-1} & 1 \\ & \bar{z}z^{-1} & \bar{b} \\ & & z \end{pmatrix} \begin{pmatrix} 1 & & \\ -\bar{b}\bar{z}^{-1} & 1 & \\ z^{-1} & bz^{-1} & 1 \end{pmatrix}. \tag{2.31}$$

If $v_E(z) = -2r$ then $v_E(b) \geq -r$ and the right matrix is $1 \bmod \pi_E^r$. Fix g. If $f \in I$ then for r large enough

$$f(wng) = \sigma(m(\bar{z}^{-1}, \bar{z}z^{-1}))f(g). \tag{2.32}$$

Corollary 2.12. *If $f(1) = 0$, then $f(wn)$ has a compact support (in n).*

Let

$$I_0 = \ker(f \mapsto f(1)). \tag{2.33}$$

Equivalently, since the support of any f is both open and closed, I_0 is the submodule of functions supported on PwP. We have a short exact sequence

$$0 \to I_0 \to I \to \sigma\delta^{1/2} \to 0 \tag{2.34}$$

where the projection to the one-dimensional $\sigma\delta^{1/2}$ is $f \mapsto f(1)$. We may define the linear functional $\Lambda : I_0 \to \sigma^*\delta^{1/2}$

$$\Lambda f = \int_N f(wn)dn. \tag{2.35}$$

The integrand has compact support by the corollary. Note $f(wnp) = f(wpw^{-1}wp^{-1}np) = \sigma^*\delta^{-1/2}(p)f(wp^{-1}np)$ so for $p \in M$ indeed

$$\begin{aligned} \Lambda\pi(p)f &= \int_N f(wnp)dn \\ &= \sigma^*\delta^{-1/2}(p)\int_N f(wn)d(pnp^{-1}) \\ &= \sigma^*\delta^{1/2}(p)\int_N f(wn)dn. \end{aligned} \tag{2.36}$$

Proposition 2.13. *For any σ, $\dim I_N = 2$ and there exists an exact sequence*

$$0 \to \sigma^*\delta^{1/2} \to I_N \to \sigma\delta^{1/2} \to 0. \tag{2.37}$$

Proof. By the exactness of the Jacquet functor there is such a short exact sequence with $I_{0,N}$ on the left. But $0 \neq \Lambda \in Hom(I_{0,N}, \sigma^*\delta^{1/2})$ and I_N is at most 2-dimensional, so the claim follows. ∎

We remark that under $f \mapsto \phi_f(n) = f(wn)$ we obtain

$$I_0 \simeq c\text{-}Ind_M^P \sigma^*\delta^{-1/2} \tag{2.38}$$

and that in general integration over N yields a functional from $c\text{-}Ind_M^P\chi$ to $\chi\delta$, identifying $\chi\delta$ as the N-coinvariants of $c\text{-}Ind_M^P\chi$.

Corollary 2.14. *$Hom_G(i(\sigma), i(\tau)) \neq 0$ if and only if $\tau = \sigma$ or $\tau = \sigma^*$.*

Proof. We only have to note that if $\sigma^* \neq \sigma$ then $I_N = \sigma^*\delta^{1/2} \oplus \sigma\delta^{1/2}$ so there is a non-zero homomorphism from I_N to $\sigma^*\delta^{1/2}$ as well. The corollary follows then from Frobenius reciprocity. ∎

2.3.2. *I is of always length 1 or 2.*

Lemma 2.15. *Let V be an irreducible subquotient of I. Then V is not supercuspidal.*

Proof. Let $W \subset I$ be such that V is a quotient of W. If V is supercuspidal, it is projective, so it embeds in W, hence in I. But this contradicts the fact that it is supercuspidal. ∎

Corollary 2.16. *If I is reducible, then it is of length 2,*

$$0 \to V \to I \to V' \to 0 \tag{2.39}$$

where V and V' are irreducible and V_N and V'_N are 1-dimensional.

Proof. The Jacquet functor is exact and by the lemma, every subquotient of I must have a non-zero Jacquet module. But we have proved that $\dim I_N = 2$. ∎

2.3.3. *The structure of $i(\sigma)$ for regular σ.* If $\sigma \neq \sigma^*$ then $\dim End_G I = 1$ since $End_G(I) = Hom_M(I_N, \sigma\delta^{1/2})$.

If I is irreducible, then I^* is irreducible too and $I \simeq I^*$. Indeed, there is an intertwining operator $T : I \hookrightarrow I^*$ and it must be onto because it induces an injection, hence by dimension counting an isomorphism, of I_N onto I^*_N, but if I^*/I were non-zero, its Jacquet module would be non-zero.

If, on the other hand, I is reducible, then as we have seen it admits a non-split short exact sequence

$$0 \to V \to I \to V' \to 0 \tag{2.40}$$

with V_N and V'_N 1-dimensional. Recall that $I_N = \sigma\delta^{1/2} \oplus \sigma^*\delta^{1/2}$ in this case, as the two exponents are distinct. We must have $V_N = \sigma\delta^{1/2}$ (since $Hom_G(V, I) \neq 0$) so necessarily $V'_N = \sigma^*\delta^{1/2}$, $Hom_G(V', I) = 0$ and the short exact sequence is non-split. (This can also be seen from the fact that $End_G I$ is 1-dimensional.)

In this case $Hom_G(V', I^*) \neq 0$, so we get $0 \to V' \to I^* \to V'' \to 0$ and arguing as above with I^* instead of I, we get $V'' \simeq V$. Summing up we have proven the following.

Proposition 2.17. *Suppose that $\sigma \neq \sigma^*$. Then either I and I^* are irreducible, and they are then isomorphic, or I admits a non-split short exact sequence as above with V and V' irreducible, and then I^* is expressed in the same way with V and V' interchanged. In this case, $I \not\simeq I^*$.*

Example 2.2. *If $\sigma = \delta^{-1/2}$, then $\sigma^* = \delta^{1/2}$, $V = \mathbb{C}$ and $V' = St$.*

2.3.4. *The intertwining operator in the regular case.* The question remains, how to decide whether I is reducible. As we have seen, there is a unique P-homomorphism $I \to I_N \to \sigma^*\delta^{1/2}$ which on I_0 is given by Λ and corresponds, via Frobenius reciprocity, to an intertwining operator

$$T : I \to I^*. \tag{2.41}$$

If $f \in I_0$ we have

$$\Lambda f = Tf(1) = \int_N f(wn)dn. \tag{2.42}$$

Let $T^* : I^* \to I$ be the corresponding intertwining operator for σ^*. Since $End_G I$ is 1-dimensional, there is a scalar $\gamma(\sigma)$ such that

$$T^* \circ T = \gamma(\sigma). \tag{2.43}$$

Proposition 2.18. *I is irreducible if and only if $\gamma(\sigma) \neq 0$.*

Proof. If I is irreducible, then T and T^* must be isomorphisms. If I is reducible, then as we have seen, T must kill V and map $V' = I/V$ isomorphically onto the $V' \subset I^*$ while T^* kills V', so the composition of the two is 0. ∎

To apply the proposition one needs to be able to compute T (and T^*) on the whole of I. Let

$$N_r = \left\{ n(b,z) |\, b \in \pi_E^r \mathcal{O}_E,\ z \in \pi_E^{2r} \mathcal{O}_E,\ z + \bar{z} = -b\bar{b} \right\}. \tag{2.44}$$

Then N_r is a compact open subgroup of N, and if we normalize the Haar measure of N_0 to be 1, then the Haar measure of N_r is $\omega(\pi_E)^{2r} = q_E^{-2r}$, because $n(b,z) \mapsto n(\pi_E^r b, (\pi_E \bar{\pi}_E)^r z)$ maps N_0 isomorphically onto N_r. It follows that the Haar measure of $N_r \backslash N_{r+1}$ is

$$q_E^{-2r}(1 - q_E^{-2}). \tag{2.45}$$

Moreover if the subset of $N_0 \backslash N_1$ where $v_E(z) = 0$ has measure a and the subset where $v_E(z) = 1$ has measure b then similarly the subsets of $N_{-r-1} \backslash N_{-r}$ where $v_E(z) = -2r-2$ and $-2r-1$ have measure $q_E^{2(r+1)} a$ and $q_E^{2(r+1)} b$.

Write $\sigma = \sigma_0 \omega^s$ where σ_0 is unitary and $s \in \mathbb{R}$. Assume *for simplicity that σ is unramified* and let $\sigma_0(m(\pi_E, 1)) = \lambda$, $|\lambda| = 1$. Then $\sigma\delta^{1/2} = \sigma_0 \omega^{s+1}$. Consider an arbitrary $f \in I$. Then for $n = n(b,z) \in N_{-r-1} \backslash N_{-r}$ and r large, $v_E(z) = -2r-2$ or $-2r-1$ and

$$f(wng) = (\lambda^{-1} q_E^{s+1})^{v_E(z)} f(g) \tag{2.46}$$

so

$$\begin{aligned} &\int_{N_{-r-1}\backslash N_{-r}} f(wng)dn \\ = \;& f(g) \cdot \left\{ q_E^{2(r+1)} a \cdot (\lambda^{-1} q_E^{s+1})^{-2r-2} + q_E^{2(r+1)} b \cdot (\lambda^{-1} q_E^{s+1})^{-2r-1} \right\} \\ = \;& f(g) \cdot \left\{ a\lambda^2 q_E^{-2s} + b\lambda q_E^{-s+1} \right\} (\lambda^2 q_E^{-2s})^r. \end{aligned} \tag{2.47}$$

It follows that if $s > 0$, the integral defining $Tf(g)$

$$Tf(g) = \int_N f(wng)dn \tag{2.48}$$

converges for every f. Moreover, it has analytic continuation in s for all $(\lambda, s) \neq (\pm 1, 0)$. I.e., except if $\sigma^2 = 1$, where there is a pole, the integral makes sense. But if $\sigma^2 = 1$ then σ is irregular, so for all regular unramified σ, the integral for $Tf(g)$ converges, or is defined by analytic continuation, for all f and g.

2.3.5. *The structure of $i(\sigma)$ for irregular σ.* When $\sigma = \sigma^*$ we take advantage of the fact that σ, hence $i(\sigma)$, is unitary, hence completely decomposable.

Lemma 2.19. *Assume that $\sigma = \sigma^*$. The following are equivalent:*

(a) I is reducible (b) $I = V \oplus V'$ a sum of irreducibles, and $V \ncong V'$ (c) the exact sequence

$$0 \to \sigma\delta^{1/2} \to I_N \to \sigma\delta^{1/2} \to 0 \tag{2.49}$$

splits (d) $\dim End_G I = 2$.

Proof. Since I is completely decomposable and of length 1 or 2, and since $End_G I = Hom_G(I_N, \sigma\delta^{1/2})$ is at most 2 dimensional, the equivalence of all four statements is obvious. ∎

The easiest way to decide whether I is reducible is to compute I_N and see if (c) holds. We remark that in the case of GL_2 the irregular principal series are always irreducible, but for SL_2 they can be reducible (see [Cass], Corollary 9.4.6).

2.3.6. *Irreducibility of* $i(\sigma)$ *for* $SU(3)$ *and* $U(3)$. In [Keys, Section 7], David Keys determines which of the representations $i(\sigma)$ are reducible, but he does it for $SU(3)$ rather than $U(3)$. The torus of $SU(3)$ consists of the matrices $m(t) = diag(t, \bar{t}/t, \bar{t}^{-1})$ for $t \in E^\times$. If σ is a smooth character of $E^\times$ we write $\sigma(m(t)) = \sigma(t)$. Write

$$\sigma = \sigma_0 \omega_E^s \tag{2.50}$$

where $|\sigma_0| = 1$ (i.e. is unitary) and $s \in \mathbb{C} \bmod 2\pi i/\log(q_E)$. Note that a change in $\mathrm{Im}(s)$ can be absorbed in σ_0 but $\mathrm{Re}(s)$ is uniquely defined and we call it $\mathrm{Re}(\sigma)$. Clearly $\mathrm{Re}(\tilde{\sigma}) = -\mathrm{Re}(\sigma)$ and if $\mathrm{Re}(\sigma) = 0$ (but not only if), $i(\sigma)$ is unitary. Let $\eta_{E/F}$ be the unique quadratic character of $F^\times$ whose kernel is $N_{E/F}(E^\times)$. Then Keys proves that $i(\sigma)$ is reducible in the following cases.

- If $\sigma = \omega_E^{\pm 1}$ (case of Steinberg representation)
- If $\sigma = \eta\omega_E^{\pm 1/2}$ and $\eta|_{F^\times} = \eta_{E/F}$
- If $\sigma|_{F^\times} = 1$ but $\sigma \neq 1$.

The third type is an irregular σ, the first two are regular. In all other cases $i(\sigma)$ is irreducible.

Rogawski (p. 173) claims that for $U(3)$, if we write

$$\sigma(m(t,s)) = \sigma_1(t)\sigma_2(t\bar{t}^{-1}s) \tag{2.51}$$

then the same condition for reducibility with σ_1 in place of σ, holds.

3. The metaplectic group and the Weil representation

3.1. The Heisenberg group and the Stone-von Neumann theorem.

3.1.1. *The Heisenberg group* $H(W)$. Let $(W, \langle,\rangle)$ be a symplectic space over F (i.e. $\langle,\rangle$ is a non-degenerate alternating bilinear form). The *Heisenberg group* $H = H(W)$ is

$$H = \{(t,w) | t \in F, w \in W\} \tag{3.1}$$

with

$$(t_1, w_1)(t_2, w_2) = (t_1 + t_2 + \langle w_1, w_2 \rangle /2, w_1 + w_2). \tag{3.2}$$

One has then an exact sequence

$$0 \to F \to H \to W \to 0 \tag{3.3}$$

$(t \mapsto (t,0),\ (t,w) \mapsto w)$, F is the center of H, and

$$\begin{aligned} (t_1, w_1)(t_2, w_2)(t_1, w_1)^{-1}(t_2, w_2)^{-1} &= (\langle w_1, w_2 \rangle, 0), \\ (t,w)^{-1} &= (-t, -w). \end{aligned} \tag{3.4}$$

3.1.2. *The isomorphism* $N \simeq H(E)$. View E as a 2-dimensional symplectic space over F with

$$\langle b_1, b_2 \rangle = 2\,\mathrm{Im}(\bar{b}_1 b_2). \tag{3.5}$$

Lemma 3.1. *The map* $n(b,z) \mapsto (\mathrm{Im}(z), b) \in H(E)$ *is an isomorphism* $N \simeq H(E)$.

Proof. Easy computation. ∎

3.1.3. *The Stone-von Neumann theorem.* Let F be a p-adic (local) field, $char.F \neq 2$, and $\psi : F \to \mathbb{C}^\times$ an additive character. We regard F and W as subsets of $H(W)$, but note that W (via $(0,w)$) is not a subgroup. There will be no confusion between the group operation in $H(W)$, which is written multiplicatively, and that of W, written additively. Thus

$$w_1 w_2 = (0, w_1)(0, w_2) = (\langle w_1, w_2 \rangle /2, w_1 + w_2) = \langle w_1, w_2 \rangle\, w_2 w_1. \tag{3.6}$$

Let $H_\psi(W) = H_\psi$ be the extension of W by $\mathbb{C}^\times$ which is the push-out of $H(W)$ via ψ.

Recall that W is self-dual as a l.c.a. group, the duality given by the pairing $\psi(\langle u, v\rangle)$. For any closed subgroup A we let $A^\perp$ be the annihilator under this pairing. A *maximal isotropic subgroup* is an A satisfying $A^\perp = A$. Typical examples are:

- A maximal isotropic subspace Y (then Y is also its own annihilator under $\langle , \rangle$, in the sense of linear algebra).
- A self-dual lattice (open compact subgroup). Use a standard symplectic basis on W to see that self-dual lattices exist.

Theorem 3.2. *There exists a unique (up to isomorphism) smooth irreducible representation* $(\mathcal{S}_\psi, \rho_\psi)$ *of* $H(W)$ *with central character* ψ. *Moreover, this representation is admissible.*

The representation $(\mathcal{S}_\psi, \rho_\psi)$ is called the *Heisenberg representation.*

3.1.4. *Construction.* We begin the proof by constructing a *model* $(\mathcal{S}_{A,\psi}, \rho_{A,\psi})$ for $(\mathcal{S}_\psi, \rho_\psi)$. The model, written also $(\mathcal{S}_A, \rho_A)$ when the reference to ψ is clear, depends on the choice of a maximal isotropic subgroup A in W. The choice of a maximal isotropic subspace leads to the *Schroedinger model* and the choice of a self-dual lattice to the *lattice model*. As a matter of notation, we shall always let $\mathcal{S}_\psi$ stand for an arbitrary model (or for the isomorphism class of the representation), and $\mathcal{S}_{A,\psi}$ for the specific model constructed below.

Let $\tilde{A} = F \times A \subset H$. Note that $\tilde{A}$ is a subgroup, its image in H_ψ is a maximal commutative subgroup, and in the Schroedinger model, even $\tilde{A}$ itself is such. As the commutator subgroup $[\tilde{A}, \tilde{A}]$ of $\tilde{A}$ is equal to $\langle A, A\rangle \subset \ker\psi$, the character ψ extends from F to a unitary character of $\tilde{A}$. Let $\tilde{\psi}$ be such an extension. If $2A = A$ we can take $\tilde{\psi}((t,a)) = \psi(t)$. This will be the case if A is a maximal isotropic subspace or if A is a maximal isotropic lattice and the residual characteristic is not 2. If $A \neq 2A$ the character $\tilde{\psi}$ will have to satisfy

$$\tilde{\psi}(a_1 + a_2) = \psi(\langle a_1, a_2\rangle /2)\tilde{\psi}(a_1)\tilde{\psi}(a_2), \tag{3.7}$$

so can not be trivial on A. Define

$$\mathcal{S}_A = Ind_{\tilde{A}}^H \tilde{\psi}. \tag{3.8}$$

By *Ind* we mean smooth induction. This is the space of $f : H \to \mathbb{C}$ satisfying $f(\tilde{a}h) = \tilde{\psi}(\tilde{a})f(h)$ for all $\tilde{a} \in \tilde{A}$, and which are smooth under right translation by H. We define $\rho_A(h) = \rho(h)$ by

$$(\rho(h)f)(h') = f(h'h). \tag{3.9}$$

Lemma 3.3. *Every $f \in \mathcal{S}_A$ is compactly supported modulo $\tilde{A}$. In other words,*

$$Ind_{\tilde{A}}^{H}\tilde{\psi} = c\text{-}Ind_{\tilde{A}}^{H}\tilde{\psi}. \tag{3.10}$$

Proof. Pick an f. Let $L \subset W$ be a lattice such that $f(h'h) = f(h')$ for all $h \in L$ and $h' \in H$, and $\tilde{\psi}(h) = 1$ for $h \in A \cap L$. If $h \in A \cap L$, $t \in F$ and $w \in W$ then

$$f(tw) = f(twh) = f(t\langle w,h\rangle hw) = \psi(\langle w,h\rangle)f(tw) \tag{3.11}$$

so if $f(tw) \neq 0$, $w \in (A \cap L)^{\perp} = A^{\perp} + L^{\perp} = A + L^{\perp}$. Since $L^{\perp}$ is also a lattice, the support of f is contained in $\tilde{A}L^{\perp}$, so is compact modulo $\tilde{A}$. ∎

Corollary 3.4. *The inner product*

$$(f,g) = \int_{\tilde{A}\backslash H} \overline{f(h)}g(h)dh \tag{3.12}$$

is well-defined and invariant under ρ_A. The Heisenberg representation is unitary.

3.1.5. *The Schroedinger model.* Let $W = X \bigoplus Y$ be a polarization: X and Y are maximal isotropic subspaces. Then $\tilde{X} = FX$ and $\mathcal{S}_X = Ind_{\tilde{X}}^{H}\tilde{\psi}$ is identified with $C_c^{\infty}(Y)$ under restriction: $f \mapsto f|_Y = \phi$.

Lemma 3.5. *The action of H on $C_c^{\infty}(Y)$ is given by the following rules:*
(i) $t.\phi = \psi(t)\phi$ for $t \in F$.
(ii) $(y.\phi)(y') = \phi(y+y')$ for $y \in Y$.
(iii) $(x.\phi)(y') = \psi(\langle y',x\rangle)\phi(y')$ for $x \in X$.

Proof. Exercise. ∎

As a corollary, we have

$$\begin{aligned}(x,y).\phi(y') &= \psi(-\langle y,x\rangle/2)(yx).\phi(y') \\ &= \psi(-\langle y,x\rangle/2)x.\phi(y+y') \\ &= \psi(\langle y',x\rangle + \langle y,x\rangle/2)\phi(y+y').\end{aligned} \tag{3.13}$$

3.1.6. *Irreducibility.* We show that $(\mathcal{S}_A, \rho)$ is irreducible. If $w \in W$ and $L \subset W$ is a small enough lattice in the sense that $\langle w+L, L\rangle \subset \ker\psi$, we define

$$f_{w,L}(\tilde{a}wh) = \tilde{\psi}(\tilde{a}) \text{ if } h \in L \tag{3.14}$$

and $f_{w,L}(h') = 0$ for $h' \notin \tilde{A}wL$. This $f_{w,L}$ is well-defined (check!) and belongs to $\mathcal{S}_A$. If for every $w \in W$ we are given a lattice L_w "small enough" in the above sense, then the functions $\{f_{w,L} | w \in W, L \subset L_w\}$ span $\mathcal{S}_A$ algebraicly over $\mathbb{C}$. This follows at once from Lemma 3.3, by a compactness argument.

Let $0 \neq f \in \mathcal{S}_A$. We shall show that for every w there is an L_w as above such that the H-module spanned by f contains every $f_{w,L}$ with $L \subset L_w$. Pick w. Translating f we may assume $f(w) \neq 0$. Let L_w be a lattice such that

$$f(h'h) = f(h') \tag{3.15}$$

for every $h \in L_w$ and $\langle w + L_w, L_w \rangle \subset \ker \psi$. We fix a Haar measure on A, pick $\varphi \in C_c^\infty(A)$ and consider

$$(\rho(\varphi)f)(h') = \int_A \varphi(a) f(h'a) da, \tag{3.16}$$

which is in the H-span of f. Then for $w' \in W$

$$\begin{aligned}(\rho(\varphi)f)(w') &= \int_A \varphi(a) f(w'a) da \\ &= \int_A \psi(\langle w', a\rangle)\varphi(a) f(aw') da \\ &= \int_A \psi(\langle w', a\rangle)\tilde{\psi}\varphi(a) da \cdot f(w') \\ &= \widehat{\tilde{\psi}\varphi}(w' \bmod A) \cdot f(w')\end{aligned} \tag{3.17}$$

where $\widehat{\tilde{\psi}\varphi} \in C_c^\infty(W/A)$ is the Fourier transform of $\tilde{\psi}\varphi$. (Note that W/A is the dual group of A.) Using Fourier inversion we can find φ such that $\widehat{\tilde{\psi}\varphi}$ is the characteristic function of $w + L \bmod A$ for some lattice $L \subset L_w$. But if $l \in L$ then $f(w+l) = f(wl) = f(w)$, so $\rho(\varphi)f$ is a multiple of $f_{w,L}$, as both are supported on $\tilde{A}(w+L) = \tilde{A}wL$, and are constant on wL.

3.1.7. *Admissibility.* Let L be a lattice in W. We must show that the $f \in \mathcal{S}_A$ which are invariant under L form a finite dimensional space. But the proof of Lemma 3.3 shows that such an f is supported on $\tilde{A}L^\perp$, hence is determined by its values on the finite set $(L + L^\perp)/L$.

3.1.8. *Uniqueness.* Let $(\mathcal{S}, \tau)$ be a smooth irreducible representation of H with central character ψ. Let

$$\begin{aligned}\mathcal{A} = \{&f \in C^\infty(H) |\, f(th) = \psi(t) f(h)\ \forall t \in F, \\ &f \text{ has compact support modulo } F\}.\end{aligned} \tag{3.18}$$

Restriction to W identifies $\mathcal{A}$ with $C_c^\infty(W)$. The group $H \times H$ acts on $\mathcal{A}$ via $\rho_l \times \rho_r$ (left and right translation)

$$((\rho_l \times \rho_r)(h_1, h_2) f)(h) = f(h_1^{-1} h h_2). \tag{3.19}$$

Let $(\mathcal{S}^\vee, \tau^\vee)$ be the contragredient representation of $(\mathcal{S}, \tau)$. If $s \in \mathcal{S}$ and $s^\vee \in \mathcal{S}^\vee$ the *matrix coefficient*

$$f_{s^\vee, s}(h) = s^\vee(hs) \tag{3.20}$$

lies in $\mathcal{A}$. Everything is clear, except that it has a compact support modulo F. But if $L \subset W$ is a lattice fixing both s and $s^\vee$ then for $l \in L$, $h \in W$

$$\begin{aligned}f_{s^\vee,s}(h) &= f_{s^\vee,s}(hl) = s^\vee(\langle h, l\rangle\, lhs) \\ &= \psi(\langle h, l\rangle) f_{s^\vee,s}(h)\end{aligned} \tag{3.21}$$

so if $f_{s^\vee,s}(h) \neq 0$, $h \in L^\perp$, which is compact.

The map $s^\vee \otimes s \mapsto f_{s^\vee,s}$ is a homomorphism $\mathcal{S}^\vee \bigotimes \mathcal{S} \to \mathcal{A}$ respecting the $H \times H$ action.

Let $W = X \bigoplus Y$ be a polarization of W. Then $s \in \mathcal{S}_{X,\psi}$ is identified (via restriction) with $s \in C_c^\infty(Y)$ and $s' \in \mathcal{S}_{Y,\psi^{-1}}$ with $s' \in C_c^\infty(X)$. Moreover, these two representations are set by duality via

$$\langle s', s \rangle = \int_X \int_Y s'(x)s(y)\psi(\langle y, x\rangle)dxdy. \tag{3.22}$$

(Check that this is a pairing of H-modules using Lemma 3.5.) The resulting map

$$C_c^\infty(X \bigoplus Y) = C_c^\infty(X) \bigotimes C_c^\infty(Y) \to \mathcal{A} = C_c^\infty(W) \tag{3.23}$$

given by

$$\begin{aligned} (3.24)\, s' \otimes s &\mapsto f_{s',s} \\ f_{s',s}(x_1, y_1) &= \int_X \int_Y s'(x)s(y+y_1)\psi(\langle y, x_1\rangle + \langle y_1, x_1\rangle/2 + \langle y, x\rangle)dxdy \\ &= \psi(-\langle y_1, x_1\rangle/2) \int_X \int_Y s'(x)s(y)\psi(\langle y, x\rangle)\psi(\langle x+y, x_1+y_1\rangle)dxdy \end{aligned}$$

is the combination of $f(x,y) \mapsto \psi(\langle y, x\rangle)f(x,y)$, the Fourier transform, and the map $f(x_1,y_1) \mapsto \psi(-\langle y_1, x_1\rangle/2)f(x_1,y_1)$, so is an isomorphism.

It follows that $\mathcal{A}$ is irreducible and $\mathcal{S}^\vee \bigotimes \mathcal{S} \simeq \mathcal{A} \simeq \mathcal{S}_{Y,\psi^{-1}} \bigotimes \mathcal{S}_{X,\psi}$ hence $\mathcal{S} \simeq \mathcal{S}_{X,\psi}$.

Exercise 3.1. *Let G_1 and G_2 be two totally disconnected, locally compact, countable-at-infinity groups, and let π_1 and π_2 be smooth representations of G_1 and G_2 respectively. Prove that $\pi_1 \otimes \pi_2$ is a smooth representation of $G_1 \times G_2$, that it is irreducible if both π_i are, and that it then determines the π_i uniquely (up to isomorphism).*

Exercise 3.2. *Give another proof for the irreducibility of $\mathcal{A}$ relating $H \times H$ to the Heisenberg group of a* double *$W \oplus W$ of W with an appropriate alternating form, and $\mathcal{A}$ to the Schroedinger representation of $H(W \oplus W)$.*

3.2. **The metaplectic group and the Weil representation.**

3.2.1. *The metaplectic group.* Fix a model for $(\mathcal{S}_\psi, \rho_\psi)$. Let $Sp = Sp(W)$ be the symplectic group of $(W, \langle,\rangle)$. Every $g \in Sp$ induces an automorphism of $H(W)$ via $g(t,w) = (t, gw)$. The representation

$$g(\rho_\psi) = \rho_\psi \circ g^{-1} : H(W) \to GL(\mathcal{S}_\psi) \tag{3.25}$$

is another irreducible representation of $H(W)$, with central character ψ, *realized on the same space* $\mathcal{S}_\psi$ as ρ_ψ. By the Stone-von Neumann theorem, there exists an automorphism $T_g \in GL(\mathcal{S}_\psi)$ such that

$$\rho_\psi(g^{-1}(h)) = T_g^{-1} \circ \rho_\psi(h) \circ T_g. \tag{3.26}$$

By Schur's lemma T_g is unique up to a scalar from $\mathbb{C}^\times$. Since

$$\begin{aligned} T_{g_2}^{-1}T_{g_1}^{-1}\rho_\psi(h)T_{g_1}T_{g_2} &= T_{g_2}^{-1}\rho_\psi(g_1^{-1}(h))T_{g_2} \\ &= \rho_\psi(g_2^{-1}g_1^{-1}(h)) \\ &= T_{g_1g_2}^{-1}\rho_\psi(h)T_{g_1g_2} \end{aligned} \tag{3.27}$$

we have $T_{g_1g_2} \equiv T_{g_1}T_{g_2} \bmod \mathbb{C}^\times$.

Definition 3.1. *The* metaplectic group $\widetilde{Sp}_\psi$ *is the group consisting of all the possible T_g, for $g \in Sp(W)$. It is automatically equipped with a faithful representation ω_ψ on $\mathcal{S}_\psi$, called the* Weil *(or* metaplectic *or* oscillator*)* representation.

Every model of $\mathcal{S}_\psi$ gives a model of the Weil representation. Note that there is a short exact sequence

$$0 \to \mathbb{C}^\times \to \widetilde{Sp}_\psi \to Sp \to 0. \tag{3.28}$$

3.2.2. *The abstract metaplectic group $\widetilde{Sp}$.* Weil proved in his famous 1964 Acta paper, §43, that $\widetilde{Sp}_\psi$ is the push-out under $\{\pm 1\} \hookrightarrow \mathbb{C}^\times$ of a non-split extension $\widehat{Sp}_\psi$ of Sp by $\{\pm 1\}$. In other words, the 2-cocycle giving the extension (3.28) can be taken to have values in $\{\pm 1\}$. A theorem of Moore (Publ. IHES 35 (1968), 5-70) says that $H^2(Sp(W), \{\pm 1\}) = \mathbb{Z}/2\mathbb{Z}$, hence up to an isomorphism there exists a unique non-split central extension $\widehat{Sp}$ of Sp by $\{\pm 1\}$. We let $\widetilde{Sp} = \widehat{Sp} \times_{\{\pm 1\}} \mathbb{C}^\times$. It follows that there is a unique isomorphism $\widetilde{Sp} \simeq \widetilde{Sp}_\psi$ which is the identity on $\mathbb{C}^\times$ and projects to the identity on Sp. Thus the metaplectic group does not depend on ψ. However, the metaplectic representation ω_ψ depends on ψ, in general (see below).

3.2.3. *The* canonical *intertwining operators $T_{g,A}$ and splittings of the metaplectic group.* We have associated (non unique) $T_g \in GL(\mathcal{S}_\psi)$ with every $g \in Sp(W)$. More generally, let $g \in GSp(W)$ with multiplier $\mu(g) \in F^\times$, i.e. $\langle gw, gw' \rangle = \mu(g) \langle w, w' \rangle$. Then $g(t, w) = (\mu(g)t, gw)$ is an automorphism of $H(W)$, and $g(\rho_\psi) = \rho_\psi \circ g^{-1} : H(W) \to GL(\mathcal{S}_\psi)$ is an irreducible representation of $H(W)$, with central character $\psi \circ \mu(g)^{-1}$, hence is isomorphic to $\mathcal{S}_{\psi \circ \mu(g)^{-1}}$. Thus, *having fixed models* $\mathcal{S}_\psi$ and $\mathcal{S}_{\psi \circ \mu(g)^{-1}}$, there is an intertwining

$$T_g : \mathcal{S}_\psi \longrightarrow \mathcal{S}_{\psi \circ \mu(g)^{-1}} \tag{3.29}$$

satisfying

$$\rho_\psi(g^{-1}(h)) = T_g^{-1} \circ \rho_{\psi \circ \mu(g)^{-1}}(h) \circ T_g. \tag{3.30}$$

If we fix the models $\mathcal{S}_\psi$ for the various ψ, these T_g are only uniquely determined up to a scalar. However, if we allow ourselves to *change the models*, they can be pinned down. Indeed, fix a maximal isotropic subgroup $A \subset W$, and let $g \in GSp(W)$. Then there is a *canonical*

$$T_{g,A} : \mathcal{S}_{A,\psi} \longrightarrow \mathcal{S}_{g(A), \psi \circ \mu(g)^{-1}} \tag{3.31}$$

satisfying the last formula. Note, however, that even if $\mu(g) = 1$, $g(A)$ may be different from A, so this $T_{g,A}$ can not be considered as an element of $\widetilde{Sp}_\psi$, except for the special case where $g(A) = A$ (see below). The construction of the canonical $T_{g,A}$ is simple. For $f \in \mathcal{S}_{A,\psi}$ let

$$T_{g,A} f(h) = f(g^{-1}(h)). \tag{3.32}$$

Exercise 3.3. *Check that (3.30) holds. Show the "cocycle relation"*

$$T_{g_1, g_2(A)} \circ T_{g_2, A} = T_{g_1 g_2, A}. \tag{3.33}$$

Theorem 3.6. *Suppose $G \subset Sp(W)$ is a subgroup stabilizing a maximal isotropic subgroup A. Then the short exact sequence (3.28) splits canonically over G.*

Proof. The splitting is supplied by $g \mapsto T_{g,A}$. ∎

Examples: (i) A is a maximal isotropic subspace. Then G is a parabolic subgroup of $Sp(W)$.

(ii) A is a self-dual lattice ($p \neq 2$) and $G = Sp(A)$ an integral form of Sp, or any compact subgroup.

3.2.4. *Dependence on ψ.* The metaplectic representation ω_ψ depends on ψ. Nevertheless, the ideas of the last paragraph can be used to prove the following fact.

Proposition 3.7. *Let $\mu \in F^\times$. If μ is a square in F, then the representations ω_ψ and $\omega_{\psi\circ\mu}$ are isomorphic.*

Proof. Let $A = X$ be a maximal isotropic subspace and $t \in F^\times \subset GSp$ (a scalar matrix). Then $t(X) = X$ and $\mu(t) = t^2$. The isomorphism

$$T_{t,X} : \mathcal{S}_{X,\psi} \to \mathcal{S}_{X,\psi\circ t^{-2}} \tag{3.34}$$

intertwines the Weil representation of $\widetilde{Sp}_\psi$ and that of $\widetilde{Sp}_{\psi\circ t^{-1}}$. Indeed, if $T_g \in \widetilde{Sp}_\psi$ then $T_{t,X} \circ T_g \circ T_{t,X}^{-1} \in \widetilde{Sp}_{\psi\circ t^{-2}}$ and conjugation by $T_{t,X}$ induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{C}^\times & \to & \widetilde{Sp}_\psi & \to & Sp & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & \mathbb{C}^\times & \to & \widetilde{Sp}_{\psi\circ t^{-2}} & \to & Sp & \to & 0 \end{array}. \tag{3.35}$$

∎

We conclude that there are (at most) $[F^\times : F^{\times 2}]$ possibilities for ω_ψ, up to isomorphism, as abstract representations of $\widetilde{Sp}$.

3.2.5. *Formulae in the Schroedinger model.* Fix a polarization $W = X \bigoplus Y$ and a symplectic basis $e_1, \dots, e_{2n}$ with respect to which

$$\langle w, w' \rangle = {}^t w \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} w'. \tag{3.36}$$

Here X is spanned by the first n basis vectors, and Y by the last ones. Then $Sp(W)$ is generated by matrices of the form

$$m(g) = \begin{pmatrix} g & \\ & {}^t g^{-1} \end{pmatrix} \tag{3.37}$$

for $g \in GL_n(F)$, matrices of the form

$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \tag{3.38}$$

for $b = {}^t b$, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.39}$$

We identify $\mathcal{S}_{X,\psi}$ with $C_c^\infty(Y)$ via restriction. As we have seen in Theorem 3.6, the Weil representation splits over the parabolic subgroup stabilizing X, generated by the $m(g)$ and $n(b)$. Moreover, the proof of that theorem gave a canonical splitting, which in our case is given by ($\phi \in C_c^\infty(Y)$)

$$\begin{aligned} T_{m(g),X}\phi(y) &= \phi({}^t g y) \\ T_{n(b),X}\phi(y) &= \psi(\frac{{}^t yby}{2}) \cdot \phi(y). \end{aligned} \tag{3.40}$$

It is convenient to twist this splitting over the parabolic subgroup $P(X)$ generated by the $m(g)$ and $n(b)$ by the character which is 1 on the unipotent radical and $|\det(g)|_F^{1/2}$ on $m(g)$. Here $|a|_F = \omega_F(a)$ gives $|\pi_F|_F = q_F^{-1}$. We denote it by

$$\begin{aligned} M[m(g)]\phi(y) &= |\det(g)|_F^{1/2}\phi({}^t gy) \\ M[n(b)]\phi(y) &= \psi(\frac{{}^t yby}{2}) \cdot \phi(y). \end{aligned} \tag{3.41}$$

The advantage that this normalization has is that it is *unitary* with respect to the standrad inner product on $C_c^\infty(Y)$

$$(\phi_1, \phi_2) = \int_Y \overline{\phi_1(y)}\phi_2(y)dy. \tag{3.42}$$

To fix ideas, we normalize the Haar measure dy to give total mass 1 to the $\mathcal{O}_F$-lattice spanned by $e_{n+1}, \ldots, e_{2n}$.

In contrast, $T_{J,X}$ is an isomorphism of $\mathcal{S}_{X,\psi}$ with $\mathcal{S}_{Y,\psi}$ since $J(X) = Y$. To find a formula for $M[J] = T_J$ on $\mathcal{S}_{X,\psi}$ we must find an isomorphism between $\mathcal{S}_{X,\psi}$ and $\mathcal{S}_{Y,\psi}$ as representations of $H(W)$. This is given by the Fourier transform. Fix Haar measures on X and Y so that they give the lattice spanned by $e_1, \ldots, e_n$ (resp. $e_{n+1}, \ldots, e_{2n}$) total mass 1, and assume that ψ is normalized so that $\mathcal{O}_F$ is self-dual. (These normalizations are not essential here, and are made just to fix ideas.)

Lemma 3.8. *Identify $\mathcal{S}_{X,\psi}$ with $C_c^\infty(Y)$ as in Lemma 3.5, and similarly $\mathcal{S}_{Y,\psi}$ with $C_c^\infty(X)$ reversing the roles of X and Y. Let*

$$\mathcal{F} : C_c^\infty(Y) \to C_c^\infty(X) \tag{3.43}$$

be the map

$$\mathcal{F}\phi(x) = \int_Y \psi(\langle y', x\rangle)\phi(y')dy'. \tag{3.44}$$

Then $\mathcal{F}$ is a unitary *H-isomorphism.*

Proof. Exercise. It is enough to check the claim separately for each of the three subgroups F, X, Y of H. For F ths claim is trivial. For X and Y, this is the well-known fact that $\mathcal{F}$ intertwines translation with multiplication by an exponential. One only has to be careful about signs. Finally the fact that $\mathcal{F}$ is unitary follows from the normalizations of Haar measures and of ψ. ∎

Corollary 3.9. *We have the formula (up to $\mathbb{C}^\times$)*

$$M[J]\phi(y) = T_J\phi(y) = \int_Y \psi(-{}^t y' \cdot y)\phi(y')dy'. \tag{3.45}$$

Proof. A possible choice for T_J is $T_{J,Y} \circ \mathcal{F}$. Here $T_{J,Y} : C_c^\infty(X) = \mathcal{S}_{Y,\psi} \to \mathcal{S}_{X,\psi} = C_c^\infty(Y)$ is the map $(T_{J,Y}\phi)(h) = \phi(J^{-1}(h))$, and restricted to $X \subset H$, $J^{-1} : X \to Y$ is just the identity (when both spaces are identified with F^n). ∎

3.2.6. *Formulae in the lattice model.* [To be completed]

3.2.7. *Properties of the Weil representation.*

- It is smooth (exercise, use the lattice model).
- The element $-1 \in Sp$ is in the center of Sp. Let $\mathcal{S}^{\pm}$ be the $\pm$ eigenspaces. They are irreducible (exercise, use the Schroedinger model).
- Let A be a self-dual lattice in W and K_A the stabilizer of A in Sp. Then, as we have seen, $\mathcal{S}_\psi$ splits over K_A. The subspace $\mathcal{S}^{K_A}$ of K_A-invariants vectors is 1-dimensional and lies in $\mathcal{S}^+$.
- Let W_1 and W_2 be two symplectic spaces with forms $\langle , \rangle_1$ and $\langle , \rangle_2$. Let $W = W_1 \bigoplus W_2$ be their orthogonal direct sum. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be models for the Heisenberg representations of $H(W_1)$ and $H(W_2)$, and identify $H(W)$ with the quotient of $H(W_1) \times H(W_2)$ by the anti-diagonal $\{(t,-t)|t \in F\}$. Then the Heisenberg representation of $H(W)$ is $\mathcal{S} = \mathcal{S}_1 \bigotimes \mathcal{S}_2$. One has an embedding

$$Sp(W_1) \times Sp(W_2) \hookrightarrow Sp(W) \tag{3.46}$$

and a homomorphism

$$GL(\mathcal{S}_1) \times GL(\mathcal{S}_2) \to GL(\mathcal{S}) \tag{3.47}$$

with kernel $\left\{(z, z^{-1})|z \in \mathbb{C}^\times\right\}$. Combining the two and noting that, by its very definition, $\widetilde{Sp}_\psi$ is a subgroup of $Sp \times GL(\mathcal{S}_\psi)$, we get a homomorphism

$$\widetilde{Sp}_\psi(W_1) \times \widetilde{Sp}_\psi(W_2) \to \widetilde{Sp}_\psi(W) \tag{3.48}$$

with kernel $\left\{(z, z^{-1})|z \in \mathbb{C}^\times\right\}$. This homomorphism commutes with the projection to the symplectic groups. It is then easy to check that the restriction of the Weil representation ω_ψ of W to $\widetilde{Sp}_\psi(W_1) \times \widetilde{Sp}_\psi(W_2)$ is nothing but $\omega_{\psi,1} \bigotimes \omega_{\psi,2}$.

3.2.8. *Combining the Heisenberg and the Weil representations.* Consider

$$\mathcal{G} = \widetilde{Sp}(W) \ltimes H(W) \tag{3.49}$$

(a semi-direct product) where $ghg^{-1} = g(h)$ $(g \in \widetilde{Sp}(W),\ h \in H(W))$. Then we can combine ρ_ψ and ω_ψ into one representation of $\mathcal{G}$ on $\mathcal{S}_\psi$

$$\omega_\psi \ltimes \rho_\psi : gh \mapsto \omega_\psi(g) \circ \rho_\psi(h). \tag{3.50}$$

3.3. A "baby example": representations of R.

3.3.1. *The lattice model for $H(W)$ when $W = E$ is a quadratic extension of F.* Let as before

- $\psi : F \to \mathbb{C}^1$ so that $\psi(a\mathcal{O}_F) = 1$ if and only if $a \in \mathcal{O}_F$.
- $E = F + F\iota$, $\bar{\iota} = -\iota$, ι a generator of the different
- $\nu : E^\times \to \mathbb{Z}$ normalized valuation
- $(u, v) = \bar{u}v$, $\langle u, v\rangle = Tr_{E/F}(\iota^{-1}\bar{u}v) = 2\,\mathrm{Im}(u, v)$
- $A = \mathcal{O}_E = \mathcal{O}_E^\perp$ a self-dual lattice in the 2-dimensional symplectic space $W = (E, \langle .,. \rangle)$. Then A is a maximal isotropic subgroup w.r.t. $\psi(\langle .,. \rangle) : W \times W \to \mathbb{C}^1$.
- $H(W)$ the Heisenberg group, $\pi : H(W) \to W$ the projection
- $\tilde{\psi} : \tilde{A} \to \mathbb{C}^1$ is an extension of ψ to $\tilde{A} = \pi^{-1}(A)$. If $p \neq 2$ we take $\tilde{\psi}((t, u)) = \psi(t)$. We assume that this is the case and let the reader make the necessary adjustments when $p = 2$.

Let $\mathcal{F} = \mathcal{S}_A$ be the *lattice model* of the Heisenberg representation ρ_ψ of $H(W)$. Recall

(3.51)
$$\mathcal{F} = \{\phi : H(W) \to \mathbb{C} |\ \phi \text{ smooth}, \ \phi((t_1, u_1)(t, u)) = \psi(t_1)\phi((t, u)) \ \forall u_1 \in \mathcal{O}_E\}.$$

For $(t, u) \in H(W)$

(3.52) $$(\rho_\psi(t, u)\phi)(t', u') = \phi((t', u')(t, u)).$$

For $\alpha \in E$ let $1_\alpha(u)$ be the characteristic function of $\alpha + \mathcal{O}_E$ and

(3.53) $$\phi_\alpha(t, u) = \psi(t + \langle \alpha, u\rangle /2) \cdot 1_\alpha(u).$$

Then if $u_1 \in \mathcal{O}_E$

(3.54)
$$\begin{aligned}
\phi_\alpha((t_1, u_1)(t, u)) &= \phi_\alpha(t + t_1 + \langle u_1, u\rangle /2, u_1 + u) \\
&= \psi(t + t_1 + \langle u_1, u\rangle /2 + \langle \alpha, u_1 + u\rangle /2) \cdot 1_\alpha(u + u_1) \\
&= \psi(t + t_1 + \langle u_1, \alpha\rangle /2 + \langle \alpha, u_1 + u\rangle /2) \cdot 1_\alpha(u) \\
&= \psi(t + t_1 + \langle \alpha, u\rangle /2) \cdot 1_\alpha(u) \\
&= \psi(t_1)\phi_\alpha(t, u).
\end{aligned}$$

Thus $\phi_\alpha \in \mathcal{F}$ and if $\mathcal{W}$ is a system of representatives for $E/\mathcal{O}_E$, then

(3.55) $$\mathcal{F} = \bigoplus_{\alpha \in \mathcal{W}} \mathbb{C}\phi_\alpha.$$

A simple computation gives

(3.56)
$$\begin{aligned}
\left(\rho_\psi(0, v)\phi_\alpha\right)(0, u) &= \phi_\alpha((0, u)(0, v)) \\
&= \psi(\langle u, v\rangle /2)\phi_\alpha(0, u + v).
\end{aligned}$$

We sometimes write $\phi(u) = \phi(0, u)$. Then

(3.57) $$\rho_\psi((0, v))\phi_\alpha(u) = \psi(\langle u, v\rangle /2)\phi_\alpha(u + v).$$

This is a "mixture" of translation and multiplication by an exponential. Note that if $\beta \equiv \alpha \bmod \mathcal{O}_E$

(3.58) $$\phi_\beta(u) = \psi(\langle \beta, \alpha\rangle /2)\phi_\alpha(u).$$

In particular, if $(\beta - \alpha)\alpha \in \mathcal{O}_E$ then $\phi_\alpha = \phi_\beta$.

Let $\varepsilon \in E^1 \subset Sp(W)$. Then $\varepsilon A = A$ so a possible splitting for the metaplectic representation of $\widetilde{Sp}_\psi(W)$ over E^1 is given by

(3.59) $$(\omega_\psi(\varepsilon)\phi)(t, u) = \phi(t, \varepsilon^{-1}u).$$

It is easy to compute now

(3.60)
$$\begin{aligned}
\omega_\psi(\varepsilon)\phi_\alpha(u) &= \phi_\alpha(\varepsilon^{-1}u) \\
&= \psi(\left\langle \alpha, \varepsilon^{-1}u\right\rangle /2) \cdot 1_\alpha(\varepsilon^{-1}u) \\
&= \psi(\langle \varepsilon\alpha, u\rangle /2) \cdot 1_{\varepsilon\alpha}(u) = \phi_{\varepsilon\alpha}(u).
\end{aligned}$$

Thus

(3.61) $$\omega_\psi(\varepsilon)\phi_\alpha = \phi_{\varepsilon\alpha}.$$

We can now decompose

(3.62) $$\mathcal{F} = \bigoplus_{\mu \in \widehat{E^1}} \mathcal{F}_\mu$$

with respect to the action of $\omega_\psi(E^1)$. We shall prove later:

Proposition 3.10. *For any μ, $\dim \mathcal{F}_\mu \le 1$.*

Remark 3.1. *This proposition is elementary, although it will be derived later from high-powered techniques. Consider, in $\mathbb{C}[E]$, with basis e_α, the subgroup generated by $e_\beta - \{\beta,\alpha\} e_\alpha$ whenever $\beta \equiv \alpha \bmod \mathcal{O}_E$ (here we used $\{\beta,\alpha\} = \psi(\langle\beta,\alpha\rangle/2)$). Let $\mathcal{F}$ be the quotient by this subgroup. Since $\{\varepsilon\beta,\varepsilon\alpha\} = \{\beta,\alpha\}$ for $\varepsilon \in E^1$ the action of E^1 on $\mathbb{C}[E]$ induces a smooth action on $\mathcal{F}$, and the claim is that every eigenspace appears with multiplicity at most 1. Try to prove it directly! You may want to assume first that E/F is unramified and $p \neq 2$.*

Remark 3.2. *It is* not true *that any μ appears! See [Ge-Ro1], Section 5.2, where the question which μ appear in the decomposition (they call this "baby" Weil representation ω^1_ψ) intervene in the question which Weil representations of $U(3)$ are supercuspidal.*

3.3.2. *The group R.* Recall our unitary group G (denoted by U in the first section). Recall that $N \simeq H(E)$ under

$$n(b,z) \mapsto (\mathrm{Im}(z), b). \tag{3.63}$$

Let $R = D \ltimes N$ where

$$D = \{m(1,s) |\, s \in E^1\}. \tag{3.64}$$

If $C = \{m(t,t) |\, t \in E^1\}$ is the center of G then CR is the largest subgroup of G containing N in which Z is central.

3.3.3. *Irreducible representations of R.* Suppose (X,τ) is an irreducible representation of R, and assume that the center Z of R acts via the character ψ. If ψ is non-trivial then as a representation of N, τ must be isotypical, isomorphic to a number of copies of the Heisenberg representation ρ_ψ.

Since $m(1,s)n(b,z) = n(s^{-1}b,z)m(1,s)$ we may embed $D \subset Sp(E)$, $m(1,s)$ mapping to $b \mapsto s^{-1}b$. Since D is compact, we may fix a splitting $\omega^i_\psi = \omega_\psi \circ i$ of the metaplectic representation of $\widetilde{Sp}_\psi$ over D (see 3.6), i.e. we can lift

$$\begin{array}{ccc} & & \widetilde{Sp}_\psi \\ & \overset{i}{\nearrow} & \downarrow \\ D & \subset & Sp(E) \end{array} . \tag{3.65}$$

Specifically, if we use the lattice model $\mathcal{F}$ described above, then we choose i so that $\omega^i_\psi(m(1,s))$ maps ϕ_α to $\phi_{s^{-1}\alpha}$. We let

$$\tau_\psi(m(1,s)n(b,z)) = \omega^i_\psi(m(1,s)) \circ \rho_\psi(n(b,z)). \tag{3.66}$$

Any other splitting differs by a character χ of D and we let

$$\tau_{\psi,\chi}(dn) = \chi(d)\tau_\psi(dn) \tag{3.67}$$

$(d \in D, n \in N)$.

Proposition 3.11. *Every irreducible representation τ of R in which Z acts via $\psi \neq 1$ is a $\tau_{\psi,\chi}$ for a unique χ.*

Proof. Let

$$Y = Hom_N(\tau|_N, \tau_\psi|_N) = Hom_N(\tau|_N, \rho_\psi). \tag{3.68}$$

Since $\tau|_N$ is ρ_ψ-isotypical, Y is non-zero. The group D acts on Y by $\omega(d)A = \tau_\psi(d) \circ A \circ \tau(d)^{-1}$. Since D is compact, Y breaks up as a sum of D-eigenspaces. If the χ^{-1}-eigenspace is non-zero, there exists an $A \in Y$ with $\omega(d)A = \chi^{-1}(d)A$, i.e.

$$A \circ \tau(d) = \tau_{\psi,\chi}(d) \circ A. \tag{3.69}$$

This means that A is intertwining between τ and $\tau_{\psi,\chi}$ as R-representations, and from the irreducibility of τ, $\tau \simeq \tau_{\psi,\chi}$. ∎

We let $\widehat{R}(\psi)$ stand for the representations $\tau_{\psi,\chi}$ of R in which Z acts via a non-trivial ψ. If we consider irreducible representations of CR, then one has to specify also the character of C.

3.4. **Weil representations of unitary groups.**

3.4.1. *Unitary groups inside symplectic groups.* Let E/F be a quadratic extension, assumptions and notation as in Section 1.1.1. Let $(V, (,))$ be a non-degenerate hermitian vector space over E and $\mathbf{G} = \mathbf{U}(V)$ the associated unitary group (for the moment we do not assume that V contains an isotropic vector). Let $n = \dim_E V$.

Let W be the vector space V, regarded as a $2n$ dimensional vector space over F, equipped with the symplectic form

$$\langle w_1, w_2 \rangle = 2\,\mathrm{Im}(w_1, w_2). \tag{3.70}$$

Then

$$\mathbf{G} = \mathbf{U}(V) \subset \mathbf{Sp}(W). \tag{3.71}$$

Let $\mathbf{C} = \mathbf{E}^1$ be the center of $\mathbf{G}$. The pair $(\mathbf{G}, \mathbf{C})$ is a *dual reductive pair* in $\mathbf{Sp}(W)$. This means

- Both $\mathbf{G}$ and $\mathbf{C}$ are reductive groups (obvious),
- Each is the centralizer of the other in $\mathbf{Sp}(W)$.

In fact, if $g \in Sp(W)$ commutes with C then it is an E-linear endomorphism, so preserves also $\langle w_1, \iota w_2 \rangle$, hence g preserves

$$2(w_1, w_2) = \langle w_1, \iota w_2 \rangle + \iota \langle w_1, w_2 \rangle,$$

and belongs to G. On the other hand if $g \in Sp(W)$ commutes with G then since $C \subset G$ it is again E-linear, and since V is an irreducible G-module, must be scalar, hence in C.

3.4.2. *Splitting of the metaplectic representation over $U(V)$.*

Theorem 3.12. *(Kazhdan) Let F be a p-adic field. Then there exists a splitting $i : G \to \widetilde{Sp}_\psi$ of the metaplectic extension over G.*

Remark 3.3. *(i) If i' is another splitting then $i'i^{-1}$ is a character of G, so factors through the determinant, i.e. for some $\nu \in \widehat{E^1}$*

$$i'(g) = \nu(\det g) \cdot i(g) \tag{3.72}$$

In particular, the splitting is unique over $SU(V)$.

(ii) Caution! The cover $\widehat{Sp} \to Sp$ splits over $SU(V)$, but not over $U(V)$. In fact, for $G = Sp$ or SU, the map

$$H^2(G, \mu_2) \to H^2(G, \mathbb{C}^\times) \tag{3.73}$$

is injective, because its kernel is $H^1(G, \mathbb{C}^\times)/2H^1(G, \mathbb{C}^\times)$ and $H^1(G, \mathbb{C}^\times) = 0$. But for $G = U$ this group does not vanish.

For a splitting i as in the theorem, write $\omega_\psi^i = \omega_\psi \circ i$ and call $(\mathcal{S}_\psi, \omega_\psi^i)$ the metaplectic (or Weil, or *oscillator*) representation of G associated with i.

We have seen that as representations of $\widetilde{Sp}_\psi(W)$, ω_ψ and $\omega_{\psi\circ a}$ are equivalent when a is a square in F. For the restriction to G we can have a better result.

Theorem 3.13. *(i) If $a \in N_{E/F}(E^\times)$ then ω_ψ^i and $\omega_{\psi\circ a}^i$ are equivalent.*

(ii) Let $C = E^1$ be the center of G. Let $(\mathcal{S}_\psi(\chi), \omega_\psi^i(\chi))$ be the subrepresentation of $(\mathcal{S}_\psi, \omega_\psi^i)$ on which C acts via χ. Then

$$\mathcal{S}_\psi = \bigoplus_{\chi \in \widehat{E^1}} \mathcal{S}_\psi(\chi), \tag{3.74}$$

each $\omega_\psi^i(\chi)$ is irreducible (as a representation of G), and an irreducible representation of G occurs at most once among the $\omega_\psi^i(\chi)$.

(iii) If we replace the splitting i by i' (3.72) then $\mathcal{S}_\psi(\chi)$ becomes $\mathcal{S}_\psi(\chi\nu^n)$.

Compare with the statement that $\mathcal{S}_\psi^\pm$ are irreducible as representations of $Sp(W)$. The center of $Sp(W)$ is $\{\pm 1\}$, and the *dual reductive pair* there is $(Sp(W), \{\pm 1\})$. One should think of the theorem, or of Howe's conjecture, of which (ii) is a special case, in general, as a testimony to the "smallness" of the metaplectic representation.

The smooth irreducible $\omega_\psi^i(\chi)$ are called the *Weil representations* of $G = U(V)$.

Proof. For (iii) note that if $v \in \mathcal{S}_\psi(\chi)$ and we change i to i' then for $\zeta \in C$

$$\omega_\psi^{i'}(\zeta)v = \nu(\zeta^n) i(\zeta) v = \nu^n \chi(\zeta) v. \tag{3.75}$$

Point (ii) is a special case of Howe's conjecture and follows from [MVW]. For (i) fix a model $\mathcal{S}_{A,\psi}$ for ω_ψ and let $\omega_\psi^i : G \to GL(\mathcal{S}_{A,\psi})$ be the metaplectic representation of G, realized in this model. Let $\gamma \in GSp(W)$ have multiplier $\mu(\gamma)$. Then

$$T_{\gamma,A} \circ \omega_\psi(i(g)) \circ T_{\gamma,A}^{-1} \in GL(\mathcal{S}_{\gamma A, \psi\circ\mu(\gamma)^{-1}}) \tag{3.76}$$

intertwines $\rho_{\psi\circ\mu(\gamma)^{-1}}(h)$ and $\rho_{\psi\circ\mu(\gamma)^{-1}}((\gamma g \gamma^{-1})(h))$. If γ centralizes G, this gives a splitting of the metaplectic representation associated to $\psi \circ \mu(\gamma)^{-1}$, so may be called $\omega_{\psi\circ\mu(\gamma)^{-1}}(i(g))$. But then

$$\omega_{\psi\circ\mu(\gamma)^{-1}}(i(g)) = T_{\gamma,A} \circ \omega_\psi(i(g)) \circ T_{\gamma,A}^{-1} \tag{3.77}$$

which means that ω_ψ^i and $\omega_{\psi\circ\mu(\gamma)^{-1}}^i$ are equivalent. Now apply this to $\gamma \in E^\times$, noting that γ centralizes G and $\mu(\gamma) = N_{E/F}(\gamma)$. ∎

3.4.3. *A natural parametrization of the splittings and of the Weil representations.* For later purposes, when we globalize the metaplectic representation, we would like to know that its local splittings over $G = U(V)$ at the various places of the number field F can be chosen in such a way, that the resulting splitting over $\mathbf{G}(\mathbb{A})$ agrees on the global points $\mathbf{G}(F)$ with the canonical splitting of the adelic metaplectic extension over $\mathbf{Sp}(F)$ given by Weil. For that purpose it is important to have a *canonical* set parametrizing the splittings, with local-global compatability.

As we have seen in (3.72), the splittings $i : G \to \widetilde{Sp}_\psi$ of $\widetilde{Sp}_\psi \to Sp$ form a torsor under $\widehat{E^1}$. Consider the short exact sequence

$$0 \to F^\times \to E^\times \xrightarrow{\alpha} E^1 \to 0 \tag{3.78}$$

with $\alpha(x) = x/\bar{x}$ (which is surjective by Hilbert's theorem 90). Let $\eta_{E/F}$ be the quadratic character of the extension E/F (i.e. the unique non-trivial character of $F^\times/N_{E/F}(E^\times)$). Then the set of characters of $E^\times$ whose restriction to $F^\times$ is $\eta_{E/F}$ is also a torsor under $\widehat{E^1}$. One can show that these two torsors are canonically isomorphic. In other words, one can attach to $\gamma \in \widehat{E^\times}$, whose restriction to $F^\times$ is $\eta_{E/F}$, a splitting i_γ in a canonical way which is compatible with twisting by $\widehat{E^1}$. See [Ge-Ro, remark on p. 457]. If $\nu \in \widehat{E^1}$ and $\nu_E = \nu \circ \alpha$ then $i_{\nu_E \gamma} = (\nu \circ \det) \cdot i_\gamma$.

We denote the resulting representation of G on $\mathcal{S}_\psi$, denoted previously by $\omega_\psi^{i_\gamma}$, by $\omega(\gamma, \psi)$, and the (irreducible) component on which the center C acts via χ by $\omega(\gamma, \psi, \chi)$. For future reference we record the following.

Proposition 3.14. *(i) The Weil representations $\omega(\gamma, \psi, \chi)$ are irreducible. Here γ varies over the characters of $E^\times$ whose restriction to $F^\times$ is $\eta_{E/F}$ and χ over $\widehat{E^1}$.*

(ii) Two Weil representations $\omega(\gamma, \psi, \chi)$ and $\omega(\gamma', \psi', \chi')$ are equivalent if and only if $\gamma' = \gamma, \chi' = \chi$ and $\psi' = \psi_a$ for $a \in N_{E/F}(E^\times)$.

Proof. For (ii) see [Ge-Ro2], Remark on the bottom of p.461 and [Ge-Ro1], Proposition 5.1.1. It follows by showing that Weil representations are *exceptional*, i.e. admit a τ-Heisenberg model for a unique $\tau \in \widehat{R}(\psi)$, for ψ representing only one of the two classes modulo $N_{E/F}(E^\times)$, see below. ∎

3.4.4. *The mixed model.* Assume from now on that $n = 3$ as in the introduction, and $V = E^3$, $(u, v) = {}^t\bar{u}Sv$.

We let W be the vector space $V = E^3$ regarded as a vector space over F. Let $V_1 = Ee_1$ etc. and $V_{13} = V_1 \bigoplus V_3$. Note that V_{13} is a hermitian hyperbolic plane over E, while V_2 is anisotropic. For the symplectic structure, $W_{13} = W_1 \bigoplus W_3$ is a complete polarization, while W_2 can not be polarized by E-subspaces, only by F-subspaces (obviously, since it is 1-dimensional over E...).

Let $(\mathcal{F}, \rho_\psi^2)$ be any model for the Heisenberg representation of $H(W_2) = H(E)$ (with $2\,\mathrm{Im}(\bar{z}_1 z_2)$ as the symplectic form). Let $\mathcal{S}_{W_3,\psi} = \mathcal{S}(W_1)$ (we use $\mathcal{S}$ for C_c^∞) be the Schroedinger model of the Heisenberg representation ρ_ψ^{13} of $H(W_{13})$. Then $H(W_{13}) \times H(W_2)$ maps surjectively onto $H(W)$ with $\{(z, -z)\}$ as kernel, and we may take

$$\rho_\psi = \rho_\psi^{13} \bigotimes \rho_\psi^2 \tag{3.79}$$

as a model of the Heisenberg representation of $H(W)$. The underlying space is $\mathcal{S}(W_1, \mathcal{F})$, the space of locally constant functions of compact support on W_1 with values in the model $\mathcal{F}$ of ρ_ψ^2. We call such a model *mixed*.

The Weil representation $\omega_\psi(g)$ ($g \in \widetilde{Sp}_\psi$) is an isomorphism between $(\rho_\psi, \mathcal{S}(W_1, \mathcal{F}))$ and $(\rho_\psi \circ g, \mathcal{S}(W_1, \mathcal{F}))$. The following diagram commutes

$$\begin{array}{ccc} \mathcal{S} & \overset{\rho_\psi(h)}{\rightarrow} & \mathcal{S} \\ \downarrow \omega_\psi(g) & & \downarrow \omega_\psi(g) \\ \mathcal{S} & \overset{\rho_\psi(g(h))}{\rightarrow} & \mathcal{S} \end{array} . \tag{3.80}$$

Proposition 3.15. *We have the following formulae in the mixed model [to be completed].*

4. Whittaker and Heisenberg (generalized Whittaker) models

4.1. Whittaker models for $GL(2)$.

4.1.1. *Whittaker models and Whittaker functionals.* Let $G = GL_2(F)$ and let (V, π) be a smooth irreducible representation of G. Let N be the upper unipotent radical, M the torus and $P = MN$ the Borel subgroup. Recall that a ψ-Whittaker functional is a linear functional (not necessarily smooth!)

$$\lambda : V \to \mathbb{C} \tag{4.1}$$

such that

$$\lambda(\pi(n)v) = \psi(n)\lambda(v) \tag{4.2}$$

for all $n \in N$. Thus

$$\lambda \in Hom_N(\pi|_N, \psi) \simeq Hom_G(\pi, Ind_N^G \psi) \tag{4.3}$$

(Frobenius reciprocity). The space $\mathcal{W} = Ind_N^G \psi$ of functions $W : G \to \mathbb{C}$ which are right invariant under some open subgroup U and satisfy $W(ng) = \psi(n)W(g)$ for $n \in N$ is called the space of Whittaker functions. It is a (huge) smooth representation of G under right translation, and λ can be seen as an embedding of the irreducible π in $\mathcal{W}$. Specifically, the relation between the *Whittaker functional* λ and the *Whittaker model* $v \mapsto W_v$ is given by

$$W_v(g) = \lambda(\pi(g)v). \tag{4.4}$$

4.1.2. *Uniqueness.* The following theorem is well-known, and is the basis for the construction of L functions for GL_2 (see the book by Jacquet and Langlands).

Theorem 4.1. *Either π is a character or it is infinite dimensional. In the latter case the space of Whittaker functionals is one-dimensional. Equivalently, the π-isotypical subspace $\mathcal{W}(\pi)$ in $\mathcal{W}$ is irreducible.*

4.2. Whittaker models for $U(3)$.

4.2.1. *The irreducible representations of N.* Let now $G = U(V)$ as before. As we have seen, $N \simeq H(E)$ and sits in an exact sequence

$$0 \to Z \to N \to E \to 0 \tag{4.5}$$

with $Z = \iota F$. The irreducible smooth representations σ of N are therefore of two types

(i) σ is 1-dimensional: there is then a character χ of E such that $\sigma(n(b,z)) = \chi(b)$. We denote this σ by χ_N.

(ii) σ is infinite dimensional. It must have a non-trivial central character ψ on $Z \simeq F$, and then by Stone-von-Neumann

$$\sigma = \rho_\psi = Ind_{N'}^N \psi_{N'} \tag{4.6}$$

where

$$N' = \left\{ \begin{pmatrix} 1 & b & z \\ & 1 & -b \\ & & 1 \end{pmatrix} | b \in F,\, z + \bar{z} = -b^2 \right\} \tag{4.7}$$

is the pre-image in N of a maximal isotropic subgroup of E. Here

$$\psi_{N'}(n(b,z)) = \psi(z + b^2/2) \tag{4.8}$$

is the extension of ψ to a character of N' which is trivial on the subgroup $\{n(b, -b^2/2)\}$. Note that $z + b^2/2 \in \iota F$. This ρ_ψ is the Schroedinger model for $H(E)$. We can also look at the lattice model (see below).

4.2.2. *Ordinary Whittaker models.* Call the irreducible representation (V, π) of G *non-degenerate* if for some non-trivial character χ of E, it has a χ_N-Whittaker functional, i.e. a

$$\lambda \in Hom_N(\pi|_N, \chi_N) = Hom_G(\pi, Ind_N^G \chi_N). \tag{4.9}$$

As before, it admits then a Whittaker model $v \mapsto W_v(g) = \lambda(\pi(g)v)$.

Lemma 4.2. *If π has a χ_N-Whittaker functional for one χ, then it has a χ_N-Whittaker functional for every non-trivial χ.*

Proof. The torus M normalizes N, and $m(t,1)n(b,z)m(t,1)^{-1} = n(tb, t\bar{t}z)$, so if $\lambda \in Hom_N(\pi|_N, \chi_N)$, $\lambda_t = \lambda \circ m(t,1)$ is a $(\chi \circ t)_N$-Whittaker functional. But as t runs over $E^\times$ we get in this way all the non-trivial characters of E. ∎

Theorem 4.3. *(Kazhdan) If π is non-degenerate, then its Whittaker model is unique, i.e. the space of χ_N-Whittaker functionals, for a given χ_N, is 1-dimensional.*

4.2.3. *Generalized Whittaker models (Heisenberg models).* Degenerate representations exist, so we look for different models. Recalling the isomorphism $N \simeq H(E)$, the group $D = E^1$ may be identified with $U(E) \subset Sp(E) \simeq SL_2(F)$ and R with the semi-direct product

$$R = U(E) \ltimes H(E) \subset Sp(E) \ltimes H(E). \tag{4.10}$$

Since we have identified N with $H(E)$, and E with $W_2 = Ee_2$ (the hermitian form being induced from the original one on $V = E^3$), we denote the Heisenberg representation of $H(E)$ by $(\mathcal{F}, \rho_\psi^2)$. Kazhdan's theorem 3.12 (or the compactness of D) guarantees that there is a splitting $i : D = U(E) \to \widetilde{Sp}_\psi(E)$, hence, as we have seen, the representations

$$\tau_{\psi,\chi} = \chi\omega_\psi^i \ltimes \rho_\psi^2 \tag{4.11}$$

of R on the space $\mathcal{F}$ of ρ_ψ^2. We denote these representations by $\widehat{R}(\psi)$.

The main theorem is the following.

Theorem 4.4. *(Piatetski-Shapiro). 1) Let (π, V) be an irreducible smooth infinite dimensional representation of G. Then for* some *irreducible representation (σ, X) of R such that $\sigma|_Z$ is a non-trivial character ψ, there exists a non-trivial linear map*

$$\lambda_\sigma \in Hom_R(\pi|_R, \sigma). \tag{4.12}$$

2) The linear map λ_σ is unique up to a scalar.

The map λ_σ is called a generalized Whittaker functional. By Frobenius reciprocity λ_σ corresponds to an embedding

$$\pi \hookrightarrow Ind_R^G \sigma. \tag{4.13}$$

This embedding sends $v \mapsto W_v$ where $W_v(g) = \lambda_\sigma(\pi(g)v)$ is a function $G \to X$ smooth for right translation, and satisfying $W_v(rg) = \sigma(r)W_v(g)$. We denote the totality of these functions by $\mathcal{W}(\pi,\sigma)$ and call it the *Heisenberg model* (or generalized Whittaker model) of the representation π.

Remark 4.1. *(i) Existence and uniqueness of generalized Whittaker models holds for* any *infinite dimensional* π*, whether it has an ordinary Whittaker model (i.e. is non-degenerate) or not.*

(ii) The space X *is the space* $\mathcal{F}$ *of the Heisenberg representation of* $N = H(E)$*, and the generalized Whittaker functions are elements of*

$$Ind_N^G \rho_\psi^2 = Ind_N^G Ind_{N'}^N \psi_{N'} \simeq Ind_{N'}^G \psi_{N'}. \tag{4.14}$$

However, we also have to tell how $d \in D$ *acts. If* $\sigma = \tau_{\psi,\chi}$ *then* d *acts via* $\chi(d)\omega_\psi^i(d)$*. We can make this explicit working with the lattice model for* $\mathcal{F}$*.*

(iii) The set of σ *for which* π *embeds in* $Ind_R^G\sigma$ *(or the set of* χ *if we fix the splitting* i*) is interesting. It depends of course on the choice of* ψ*. We will discuss it later.*

4.3. **The existence theorem for Whittaker and generalized Whittaker models.**

4.3.1. *Construction of* l_ψ*.* Let $G = GL(2)$ or $U(3)$ and $Z = N$ or Z respectively. Let ψ be a non-trivial character of F. In both cases, we regard ψ as a representation of Z. Let (π, V) be an irreducible admissible infinite dimensional representation of G.

We first show that there exists a non-zero functional

$$l_\psi : V \to \mathbb{C} \tag{4.15}$$

satisfying

$$l_\psi(\pi(z)v) = \psi(z) l_\psi(v) \tag{4.16}$$

($v \in V, z \in Z$). In the $GL(2)$ case this is already a Whittaker functional. Let $V(Z,\psi)$ be the subspace of V spanned by the vectors

$$\pi(z)v - \psi(z)v \tag{4.17}$$

for $z \in Z$, $v \in V$ and

$$V_{Z,\psi} = V/V(Z,\psi). \tag{4.18}$$

The existence of l_ψ is equivalent to $V_{Z,\psi} \neq 0$.

Lemma 4.5. *A vector* $v \in V(Z,\psi)$ *if and only if for some (equiv. every sufficiently large) open compact* $U \subset Z$

$$\int_U \psi^{-1}(z)\pi(z)v\,dz = 0. \tag{4.19}$$

Proof. Clear. ∎

Lemma 4.6. *We have*

$$\bigcap_{\psi\neq 1} V(Z,\psi) = V^Z. \tag{4.20}$$

Proof. If $v \in V^Z$ then by the previous lemma it is in every $V(Z,\psi)$. Conversely, suppose v is in the LHS. Let $L : V \to \mathbb{C}$ be an arbitrary functional and consider $f(z) = L(\pi(z)v)$, a smooth function. Let $Z = \bigcup Z_n$ where $Z_n = \pi_F^{-n}\mathcal{O}_F$. If f is not constant, then by Fourier theory, for some n_0, for each $n \geq n_0$ there exists a character $\psi_n \neq 1$ such that

$$\int_{Z_n} \psi_n^{-1}(z)f(z)dz \neq 0. \tag{4.21}$$

We may furthermore take the ψ_n so that $\psi_{n+1}|_{Z_n} = \psi_n$. Then $\psi = \lim \psi_n$ satisfies

$$L(\int_{Z_n} \psi^{-1}(z)\pi(z)vdz) \neq 0 \tag{4.22}$$

for all sufficiently large n, contradicting the previous lemma, unless f is constant. But if f is constant for all L, $v \in V^Z$. ∎

Lemma 4.7. *Let (π, V) be an irreducible admissible infinite dimensional representation. Then $V^Z = 0$.*

Proof. We leave the $GL(2)$ case as an exercise. The key point is that $SL_2(F)$ is generated by Z and any open subgroup. In the $U(3)$ case, take $v \in V^Z$. By the $GL(2)$ case, v is fixed by the $SL_2(F)$ embedded in the four corners of G. In particular, by all $m(t,1)$ for $t \in F^\times$. The stabilizer of v in N is then an open subgroup fixed by conjugation by $m(t,1)$ for all t, hence must be all of N. Let G_0 be the closed subgroup of G generated by N and wNw^{-1} (where w is the Weyl element). Then G_0 fixes v, but since $G = P \cup NwP$ (Bruhat decomposition) G_0 is normal in G, hence must equal $G' = SU(3)$. This means that $V^{G'} \neq 0$, but then this is a submodule on which G acts via its quotient $G/G' \simeq F^\times$, and from irreducibility $V = V^{G'}$ is one-dimensional. ∎

Corollary 4.8. *There exists a $\psi \neq 1$ for which there is an $l_\psi \neq 0$.*

4.3.2. *Dependence on ψ.* Next we consider the dependence on ψ. In the $GL(2)$ case, given one ψ, any other is ψ_a for $a \in F^\times$, so defining

$$l'(v) = l(\pi\begin{pmatrix} a & \\ & 1 \end{pmatrix}v) \tag{4.23}$$

we see that if $l(\pi(z)v) = \psi(z)v$ then $l'(\pi(z)v) = \psi_a(z)v$.

In the $U(3)$ case the same idea, using $m(t,1)$ for $t \in E^\times$, shows that if we have an l for ψ, we have one for $\psi_{t\bar{t}}$. Since $[F^\times : N_{E/F}(E^\times)] = 2$, there are two classes of ψ's modulo the torus action.

4.3.3. *The existence theorem.* In the $GL(2)$ case, we already have the existence of a Whittaker model. In the $U(3)$ case, we start with an

$$l_\psi \in Hom_Z(\pi, \psi). \tag{4.24}$$

We may replace it with

$$l_\psi^U(v) = \int_U l_\psi(\pi(u)v)du \tag{4.25}$$

over a small open $U \subset R$ such that $U \cap Z \subset \ker\, \psi$. If we define $\psi_{UZ}(uz) = \psi(z)$, this is then well-defined. Furthermore:

1. If U is small enough so that it fixes v_0 for which $l_\psi(v_0) \neq 0$, also $l_\psi^U(v_0) \neq 0$.

2. The functional l_ψ^U is fixed by U: $l_\psi^U \circ \pi(u) = l_\psi^U$ for all $u \in U$.
3. Since Z and U commute, $l_\psi^U(\pi(z)v) = \psi(z) l_\psi^U(v)$.

Thus without loss of generality, l_ψ itself is already fixed by U. In other words,

$$l_\psi \in Hom_{UZ}(\pi, \psi_{UZ}) = Hom_R(\pi, Ind_{UZ}^R \psi_{UZ}). \tag{4.26}$$

However, just as in the case of the ordinary Heisenberg representation, one proves that $Ind_{UZ}^R \psi_{UZ} = c\text{-}Ind_{UZ}^R \psi_{UZ}$ (the problem is only in inducing to N, since D is compact anyhow) and as a result, $Ind_{UZ}^R \psi_{UZ}$ is unitarizable, hence completely reducible. This means that for *some* irreducible $(\sigma, X) \in \widehat{R}(\psi)$ there exists a non-zero

$$\mathcal{L}_\psi \in Hom_R(\pi, \sigma) = Hom_G(\pi, Ind_R^G \sigma). \tag{4.27}$$

This gives the σ-Whittaker model.

4.4. **Uniqueness of the generalized Whittaker model (the Gelfand-Kazhdan method).**

4.4.1. *The spaces $Hom^\infty(V, X)$.* Quite generally let G and H be groups, (X, τ) a smooth representation of H and (V, π) a smooth representation of G. Then $H \times G$ acts on $Hom(V, X)$ via

$$(h, g) : A \mapsto \tau(h) \circ A \circ \pi(g^{-1}). \tag{4.28}$$

We write $Hom^\infty(V, X)$ for the subspace of A 's on which this action is smooth. Note that if A is *any* homomorphism, $\phi_1 \in \mathcal{S}(H)$ and $\phi_2 \in \mathcal{S}(G)$ then $\tau(\phi_1) \circ A \circ \pi(\phi_2)$ is in $Hom^\infty(V, X)$.

In particular we have $End^\infty(X)$ and $\tau(\phi)$ belongs to it for every $\phi \in \mathcal{S}(H)$. The following lemma is clear.

Lemma 4.9. *If $A \in End^\infty(X)$ and ${}^tA$ is the transpose of A, then ${}^tA$ preserves $\tilde{X}$ (the smooth dual) and defines $\tilde{A} \in End^\infty(\tilde{X})$.*

4.4.2. *Contragredients of representations of G.* Let G be the unitary group, and consider the automorphism $g^\alpha = \bar{g}$ and the anti-automorphism $g^\theta = \bar{g}^{-1}$. Note that they preserve the subgroup R. If π is a smooth representation of G we let $\pi^\alpha(g) = \pi(g^\alpha)$.

Lemma 4.10 (MVW, p.91)**.** *Let (π, V) be an irreducible admissible representation of G. We have $(\pi^\alpha, V) \simeq (\tilde{\pi}, \tilde{V})$.*

Proof. Let $\phi \in \mathcal{S}(G)$. We let $\phi^\alpha(g) = \phi(g^\alpha)$, $\dot{\phi}(g) = \phi(g^{-1})$ and $\phi^\theta(g) = \phi(g^\theta)$. The operator $\pi(\phi)$ is of finite rank (by the admissibility of π) and we let $Tr_\pi(\phi) = Tr\,\pi(\phi)$. This is a distribution on G (i.e. a linear functional on $\mathcal{S}(G)$). Moreover, it is invariant under conjugation, i.e. if $\gamma \in G$ and $C_\gamma(\phi)(g) = \phi(\gamma^{-1} g \gamma)$ then $Tr_\pi \circ C_\gamma = Tr_\pi$. In addition, it satisfies $Tr_\pi(\phi_1 * \phi_2) = Tr_\pi(\phi_2 * \phi_1)$.

It is well-known that to prove that π^α and $\tilde{\pi}$ are isomorphic, it is enough to prove that $Tr_{\pi^\alpha} = Tr_{\tilde{\pi}}$ (see [Bump], Theorem 4.2.1). Now

$$Tr_{\tilde{\pi}}(\phi) = Tr_\pi(\dot{\phi}) \tag{4.29}$$

and $Tr_{\pi^\alpha}(\phi) = Tr_\pi(\phi^\alpha)$. Thus we are reduced to showing that $Tr_\pi(\phi^\theta) = Tr_\pi(\phi)$. Now g^θ is conjugate in G to g. This is non-trivial, see [MVW, Prop. I.2 p.79] and the remark below. The claim follows now from [K-G, Theorem 1a on p.101], where it is shown that if T is a distribution which is invariant under conjugation, then $T(\phi^\theta) = T(\phi)$. ∎

Remark 4.2. *It is a rewarding exercise to check that g^θ and g are conjugate. For example, for*

$$g = m(t,s)n(b,z) = \begin{pmatrix} t & tb & tz \\ 0 & s & -s\bar{b} \\ & & \bar{t}^{-1} \end{pmatrix} \tag{4.30}$$

($z + \bar{z} = -b\bar{b}$, $s\bar{s} = 1$), we have

$$g^\theta = \begin{pmatrix} \bar{t}^{-1} & -s\bar{b} & tz \\ 0 & s & tb \\ & & t \end{pmatrix}. \tag{4.31}$$

One has to distinguish two cases: if $t\bar{t} = 1$ conjugation by $m(r,1)$ for an appropriate $r \in E^1$ does the job. Otherwise, find the eigenvector of g with eigenvalue $\bar{t}^{-1}$ and check that it is isotropic. Then everything follows (why?).

4.4.3. *Gelfand datum.* We assume that the group G is unimodular. A *pre-Gelfand datum* $(G, H, (X,\tau), {}^\theta, {}^t)$ is given by:

- $H \subset G$ a closed subgroup
- (X,τ) a countable-dimensional irreducible admissible representation of H
- $g \mapsto g^\theta$ is an involution of G, preserving H and the (left) Haar measures on G and on H. For $\phi \in \mathcal{S}(G)$ let $\phi^\theta(g) = \phi(g^\theta)$.
- $A \mapsto {}^tA$ is an involution of $End^\infty(X)$, satisfying

$$\tau(\phi^\theta) = {}^t\tau(\phi) \tag{4.32}$$

  for $\phi \in \mathcal{S}(H)$.

Example 4.1. *(i) $(R, D, \mu, \theta(g) = \bar{g}^{-1}, {}^ta = a)$*
(ii) $(GL_2(F), Z, \psi, \theta(g) = w({}^tg)w^{-1}, {}^ta = a)$ where $w = (0,1;1,0)$
(iii) $(G = U(3), R, (X,\tau), \theta(g) = \bar{g}^{-1}$, see below for ${}^t)$

Let $\mathcal{S}(G) = C_c^\infty(G)$, with

$$\phi_1 * \phi_2(g) = \int_G \phi_1(gh^{-1})\phi_2(h)dh \tag{4.33}$$

as product. Define, for $h_1, h_2 \in H$

$$S_{h_1,h_2}\phi(g) = \phi(h_1^{-1}gh_2^{-1}). \tag{4.34}$$

An $End^\infty(X)$*-valued distribution* on G is a linear map T from $\mathcal{S}(G)$ to $End^\infty(X)$. It is called *quasi-invariant* if

$$T(S_{h_1,h_2}\phi) = \tau(h_1) \circ T(\phi) \circ \tau(h_2) \tag{4.35}$$

for all $h_1, h_2 \in H$, $\phi \in \mathcal{S}(G)$. The *convolution* $T_1 * T_2$ is defined as follows. Let

$$\phi(g_1g_2) = \sum \phi_{1,i}(g_1)\phi_{2,i}(g_2). \tag{4.36}$$

Then

$$T_1 * T_2(\phi) = \sum T_1(\phi_{1,i}) \circ T_2(\phi_{2,i}). \tag{4.37}$$

If both T_i are quasi-invariant, so is $T_1 * T_2$.

Define

$$T^\theta(\phi) = {}^tT(\phi^\theta). \tag{4.38}$$

Then (check!)

$$(\phi_1 * \phi_2)^\theta = \phi_2^\theta * \phi_1^\theta, \;\; (T_1 * T_2)^\theta = T_2^\theta * T_1^\theta. \tag{4.39}$$

Lemma 4.11. *If T is quasi-invariant, so is T^θ.*

Proof. One checks easily that $(S_{h_1,h_2}\phi)^\theta = S_{h_2^\theta,h_1^\theta}\phi^\theta$. Then

$$\begin{aligned} T^\theta(S_{h_1,h_2}\phi) &= {}^t[T(S_{h_2^\theta,h_1^\theta}\phi^\theta)] \\ &= {}^t[\tau(h_2^\theta)\circ T(\phi^\theta)\circ\tau(h_1^\theta)] \\ &= \tau(h_1)\circ T^\theta(\phi)\circ\tau(h_2). \end{aligned} \tag{4.40}$$

∎

A pre-Gelfand datum is called a *Gelfand datum* if any $End^\infty(X)$-valued quasi-invariant distribution is θ-invariant.

4.4.4. *Two lemmas on duals.* Let (π, V) be an admissible (not necessarily irreducible, in this subsection) representation of a group G. Recall that $(\tilde\pi, \tilde V)$ denotes the smooth dual, and that (thanks to the admissibility of π), $\tilde{\tilde V} = V$ (i.e. the canonical embedding of V in $\tilde V' = Hom(\tilde V, \mathbb{C})$ identifies V with the smooth vectors in $\tilde V'$). Recall that for $\phi \in C_c^\infty(G)$

$$\pi(\phi)(v) = \int_G \phi(g)\pi(g)v dg. \tag{4.41}$$

We clearly have, for $\tilde v \in \tilde V$

$$\begin{aligned} \langle \pi(\phi)v, \tilde v\rangle &= \int_G \phi(g)\left\langle v, \tilde\pi(g^{-1})\tilde v\right\rangle dg \\ &= \left\langle v, \int_G \dot\phi(g)\tilde\pi(g)\tilde v dg\right\rangle = \left\langle v, \tilde\pi(\dot\phi)\tilde v\right\rangle \end{aligned} \tag{4.42}$$

where $\dot\phi(g) = \phi(g^{-1})$ (the last step used the unimodularity of G).

Lemma 4.12. *Let $\xi \in \tilde V'$. The definition*

$$\langle \pi(\phi)\xi, \tilde v\rangle = \left\langle \xi, \tilde\pi(\dot\phi)\tilde v\right\rangle \tag{4.43}$$

defines a map $\pi(\phi) : \tilde V' \to V$.

Proof. This map defines $\pi(\phi)\xi$ as an element of $\tilde V'$. It remains to show that it is smooth, i.e. $\pi(\phi)\xi \in \tilde{\tilde V} = V$. Let U be an open subgroup such that $\phi(hg) = \phi(g)$ for $h \in H$. Then

$$\begin{aligned} \langle \pi(\phi)\xi, \tilde\pi(h)\tilde v\rangle &= \left\langle v, \int_G \dot\phi(g)\tilde\pi(gh)\tilde v dg\right\rangle \\ &= \langle \pi(\phi)\xi, \tilde v\rangle. \end{aligned} \tag{4.44}$$

∎

Next, consider quasi-invariant V-valued distributions on G, i.e.

$$T \in Hom_{\mathcal{S}(G)}(\mathcal{S}(G), V), \tag{4.45}$$

or $T : \mathcal{S}(G) \to V$ satisfying $T(\phi_1 * \phi) = \pi(\phi_1)T(\phi)$. Since $\mathcal{S}(G)$ is not a ring with 1, this turns out to be larger than V.

Lemma 4.13. *We have*

$$Hom_{\mathcal{S}(G)}(\mathcal{S}(G), V) = \tilde{V}'. \tag{4.46}$$

Proof. Define $i : Hom_{\mathcal{S}(G)}(\mathcal{S}(G), V) \to \tilde{V}'$ by $i(T)(\tilde{v}) = \langle T(\phi), \tilde{v}\rangle$, for any ϕ such that $\tilde{\pi}(\dot{\phi})\tilde{v} = \tilde{v}$ (for example we may take $\phi = \dot{\phi}$ the characteristic function of a compact open subgroup U stabilizing $\tilde{v}$, divided by its Haar measure). In the opposite direction define $j : \tilde{V}' \to Hom_{\mathcal{S}(G)}(\mathcal{S}(G), V)$ by $j(\xi)(\phi) = \pi(\phi)(\xi) \in \tilde{\tilde{V}} = V$. It is easy to check that these two maps are inverse to each other. ∎

4.4.5. *The main theorem.*

Theorem 4.14. *Assume that we are given a Gelfand datum $(G, H, (X, \tau), ^\theta, ^t)$.*

(i) For any countable dimensional irreducible admissible representation (π, V) of G we have

$$\dim Hom_H(\pi|_H, \tau) \cdot \dim Hom_H(\tilde{\pi}|_H, \tilde{\tau}) \leq 1. \tag{4.47}$$

(ii) If both dimensions are non-zero (hence 1), $(\tilde{\pi}, \tilde{V}) \simeq (\pi^\alpha, V)$ where $\pi^\alpha(g) = \pi((g^\theta)^{-1})$.

Remark. In the application to $U(3)$ we shall have $\pi^\alpha(g) = \pi(\bar{g})$ where $\bar{g}$ is complex conjugation applied to a matrix g. In the application to $GL(2)$ we shall have $\pi^\alpha(g) = \pi(w({}^t g)^{-1}w^{-1}) \sim \pi({}^t g^{-1})$.

Given $\lambda \in Hom_H(\pi|_H, \tau)$ we define a homomorphism

$$\mathcal{S}(G) \to Hom^\infty(V, X), \quad \phi \mapsto \lambda_\phi = \lambda \circ \pi(\phi) \tag{4.48}$$

where, as usual, $\pi(\phi)v = \int_G \phi(g)\pi(g)v\, dg$. Note that $\phi \mapsto \lambda_\phi$ intertwines the $G \times H$ action on both sides, i.e. if $g_1 \in G$ and $h_1 \in H$ then

$$[(g_1, h_1)\phi](g) = \phi(h_1^{-1}gg_1), \quad (g_1, h_1)A = \tau(h_1) \circ A \circ \pi(g_1)^{-1} \tag{4.49}$$

and then $\lambda_{(g_1,h_1)\phi} = (g_1, h_1)\lambda_\phi$.

Lemma 4.15. *The kernel*

$$J(\lambda) = \{\phi \in \mathcal{S}(G) |\, \lambda_\phi = 0\} \tag{4.50}$$

determines λ up to a scalar.

Proof. The module $Hom^\infty(V, X)$ is an irreducible admissible $G \times H$-module (see [B-Z]). Hence $\phi \mapsto \lambda_\phi$ is surjective. Assume $J(\lambda) = J(\lambda')$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \to & J(\lambda) & \to & \mathcal{S}(G) & \xrightarrow{\lambda} & Hom^\infty(V, X) & \to & 0 \\ & & || & & || & & \downarrow u & & \\ 0 & \to & J(\lambda') & \to & \mathcal{S}(G) & \xrightarrow{\lambda'} & Hom^\infty(V, X) & \to & 0 \end{array}. \tag{4.51}$$

Then u is an isomorphism, but by Schur's lemma it must be a scalar c. It follows that $(\lambda' - c\lambda)_\phi = 0$ for every ϕ, hence $\lambda' = c\lambda$. ∎

Let $\phi \in \mathcal{S}(G)$. Given $\lambda \in Hom_H(\pi|_H, \tau)$ and $\mu \in Hom_H(\tilde{\pi}|_H, \tilde{\tau})$ we consider the *generalized Bessel distribution*

$$T(\phi) = \lambda \circ \pi(\phi) \circ \mu^{\vee} \in End^{\infty}(X). \tag{4.52}$$

Here, if $\tilde{V}$ is the space of $\tilde{\pi}$ (the smooth G-dual of π), $\mu^{\vee} : X \to \tilde{X}' \to \tilde{V}'$ is the algebraic dual of μ (restricted to X). Note that even if we use the admissibility of τ to identify X with the smooth H-dual of $\tilde{X}$, $\pi|_H$ is not admissible and the smooth H-dual of $(\tilde{V}, \tilde{\pi})$ may be larger than V. However, $\pi(\phi)$ smoothens $\tilde{V}'$ and maps it back to V. Thus $T(\phi)$ maps X to X. It is a smooth endomorphism because if ϕ is bi-invariant under translation by an open $U \subset H$ then both $\mu^{\vee}$ and λ being H-homomorphisms, $T(\phi)$ is also bi-invariant under U.

Lemma 4.16. *The distribution T is quasi-invariant (for H and τ).*

Proof. Clear. ∎

Consider the bilinear form

$$B : \mathcal{S}(G) \times \mathcal{S}(G) \to End^{\infty}(X) \tag{4.53}$$

defined by

$$B(\phi_1, \phi_2) = T(\phi_1 * \dot{\phi}_2). \tag{4.54}$$

By the assumptions of the theorem, since T is quasi-invariant, $T = T^{\theta}$. Hence

$$\begin{aligned} B(\phi_1, \phi_2) &= T^{\theta}(\phi_1 * \dot{\phi}_2) = {}^tT((\phi_1 * \dot{\phi}_2)^{\theta}) \\ &= {}^tT(\overset{\cdot\,\theta}{\phi_2} * \phi_1^{\theta}) = {}^tT(\phi_2^{\alpha} * \overset{\cdot\,\alpha}{\phi_1}) \\ &= {}^tB(\phi_2^{\alpha}, \phi_1^{\alpha}). \end{aligned} \tag{4.55}$$

Note that $B(\phi_1, \phi_2) = \lambda \circ \pi(\phi_1) \circ \pi(\dot{\phi}_2) \circ \mu^{\vee}$ and the image of $\pi(\dot{\phi}_2) \circ \mu^{\vee}$, as ϕ_2 runs over $\mathcal{S}(G)$, spans V. Thus $J(\lambda)$ is the left kernel of B. Similarly, $\mu \circ \tilde{\pi}(\phi_2) = 0$ if and only if $\pi(\dot{\phi}_2) \circ \mu^{\vee} = 0$, so the right kernel of B is $J(\mu)$. But the formula $B(\phi_1, \phi_2) = {}^tB(\phi_2^{\alpha}, \phi_1^{\alpha})$ implies that

$$J(\lambda) = J(\mu)^{\alpha}. \tag{4.56}$$

For another λ' we have $J(\lambda') = J(\mu)^{\alpha} = J(\lambda)$ hence λ and λ' are proportional. Note that for the proof to work we *need* to have a non-zero μ. Thus $\dim Hom_H(\pi|_H, \tau) = 1$. Similarly $\dim(\tilde{\pi}|_H, \tilde{\tau}) = 1$. Finally, to get (ii) of the main theorem, the identity $J(\mu) = J(\lambda)^{\alpha}$ yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & J(\lambda) & \to & \mathcal{S}(G) & \xrightarrow{\lambda} & Hom^{\infty}(V, X) & \to & 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow \alpha & & \\ 0 & \to & J(\mu) & \to & \mathcal{S}(G) & \xrightarrow{\mu} & Hom^{\infty}(\tilde{V}, \tilde{X}) & \to & 0 \end{array} \tag{4.57}$$

where the vertical arrows are isomorphisms induced by α. For any representation W of G or $G \times H$ let W^{α} be the representation obtained by first applying the automorphism α to the group, following it by the action on W ("twisting" the representation by α). Then the meaning of the diagram is that as $G \times H$ modules

$$Hom^{\infty}(V^{\alpha}, X^{\alpha}) \simeq Hom^{\infty}(\tilde{V}, \tilde{X}). \tag{4.58}$$

But this implies, that as G-modules, $V^{\alpha} \simeq \tilde{V}$.

4.4.6. *Application to representations of R.* We apply the main theorem to the pre-Gelfand datum $(R, D, \mu, \theta(g) = \bar{g}^{-1}, {}^t a = a)$.

Lemma 4.17. *This is a Gelfand datum, i.e. any quasi-invariant ditribution T is θ-invariant.*

Proof. In this case $X = \mathbb{C}$ so a quasi-invariant distribution is a linear functional $T : \mathcal{S}(R) \to \mathbb{C}$ satisfying

$$T(S_{d_1,d_2}\phi) = \mu(d_1 d_2)T(\phi). \tag{4.59}$$

Now $D \times D$ ($D = E^1$) is a compact group, so $\mathcal{S}(R)$ decomposes as a direct sum with respect to the characters of $D \times D$ (acting via left and right translations). A quasi-invariant distribution is nothing else but a linear functional on the Hecke algebra

$$\mathcal{H}(R, \mu) = \left\{\phi \in \mathcal{S}(R) |\, \phi(d_1 g d_2) = \mu^{-1}(d_1 d_2)\phi(g)\right\}. \tag{4.60}$$

As $T^\theta(\phi) = T(\phi^\theta)$, all that we have to show is that $\phi \in \mathcal{H}(R, \mu)$ is θ-invariant, where $\phi^\theta(g) = \phi(\bar{g}^{-1})$. But

$$m(1, s)n(b, z)m(1, s)^{-1} = n(s^{-1}b, z) \tag{4.61}$$

so if $\phi \in \mathcal{H}(R, \mu)$, $\phi(n(a, z)) = \phi(n(b, z))$ whenever $N_{E/F}(a) = N_{E/F}(b)$. Now

$$n(a, z)^\theta = n(-\bar{a}, z),\ m(1, t)^\theta = m(1, t). \tag{4.62}$$

It follows that $\phi(g^\theta) = \phi(g)$ for $g = n(a, z)$ or $g = m(1, t)$ hence for every $g \in R$. ∎

Corollary 4.18. *(see 3.10) In the decomposition of an irreducible representation (X, τ) of R under D*

$$X = \bigoplus_{\mu \in \widehat{D}} X_\mu \tag{4.63}$$

every X_μ is at most one-dimensional. As a result, X is D-admissible.

Proof. Note that if $Hom_D(\tau, \mu) \neq 0$ then also $Hom_D(\tilde{\tau}, \tilde{\mu}) \neq 0$. But the main theorem now gives that both dimensions are 1. ∎

Corollary 4.19. *If $g^\alpha = \bar{g}$ ($= (g^\theta)^{-1}$) then $(\tilde{X}, \tilde{\tau})$ is isomorphic to (X, τ^α).*

Proof. This follows from the main theorem. It also follows from the existence of an invariant inner product on (X, τ), together with the fact that $\phi \mapsto \overline{\phi(\bar{g})}$ is a conjugate-linear automorphism of X. In other words, if $\phi \in X$, recall that ϕ is a certain function on N, with N acting by right translation and D via ω_ψ^i. Then $\phi^*(g) = \overline{\phi(\bar{g})}$ also lies in X. The existence of an inner product implies that $\tilde{\tau}$ is conjugate-isomorphic to τ, and the map $\phi \mapsto \phi^*$ is a conjugate-isomorphism between τ and τ^α, thus $\tilde{\tau} \simeq \tau^\alpha$. ∎

Corollary 4.20. *There exists a well-defined involution ^{t} on $End^\infty(X)$ such that*

$${}^t\tau(\phi) = \tau(\phi^\theta) \tag{4.64}$$

for $\phi \in \mathcal{S}(R)$.

Proof. Fix an isomorphism $i : X \to \tilde{X}$ such that $\tilde{\tau}(g) \circ i = i \circ \tau(\bar{g})$ and define

$$ {}^tA = i^{-1} \circ \tilde{A} \circ i \tag{4.65} $$

where $\tilde{A}$ is the dual of A on $\tilde{X}$. Then

$$ \begin{aligned} {}^t\tau(\phi) &= i^{-1} \circ \widetilde{\tau(\phi)} \circ i \\ &= i^{-1} \circ \tilde{\tau}(\dot{\phi}) \circ i \\ &= \tau(\phi^\theta) \end{aligned} \tag{4.66} $$

since $\phi^\theta(g) = \phi(\bar{g}^{-1}) = \dot{\phi}(\bar{g})$. ∎

4.4.7. *Conclusion of the proof of uniqueness.* To prove the uniqueness of a generalized Whittaker model for $U(3)$ we apply the main theorem to the pre-Gelfand datum $(G = U(3), R, (X, \tau), \theta(g) = \bar{g}^{-1}, {}^t)$. The following lemma is the Key Lemma.

Lemma 4.21. *This is a Gelfand datum.*

Proof. We must show that if $T(S_{h_1,h_2}\phi) = \tau(h_1) \circ T(\phi) \circ \tau(h_2)$ for all $h_1, h_2 \in R$ then $T^\theta = T$, i.e. ${}^tT(\phi^\theta) = T(\phi)$ for every $\phi \in \mathcal{S}(G)$. Equivalently, we have to show that if

$$ T(\phi_1 * \phi * \phi_2) = \tau(\phi_1) \circ T(\phi) \circ \tau(\phi_2) \tag{4.67} $$

for all $\phi \in \mathcal{S}(G)$ and $\phi_i \in \mathcal{S}(R)$, then ${}^tT(\phi) = T(\phi^\theta)$. Note that τ satisfies this relation (when $\phi \in \mathcal{S}(R)$), so the essence of the lemma is that the quasi-invariance forces this relation to hold on the larger Hecke algebra $\mathcal{S}(G)$.

Consider the Bruhat decomposition $G = P \cup PwN$. Then the double cosets $N\backslash G/N$ are represented by the matrices $m(t,s)$ and $m'(t,s) = m(t,s)w$. Call these representatives (when t runs over $E^\times$ and s over E^1) Δ and Δ'. We have an exact sequence

$$ 0 \to \mathcal{S}(PwN) \to \mathcal{S}(G) \to \mathcal{S}(P) \to 0 \tag{4.68} $$

and a dual exact sequence for distributions. If T is quasi-invariant so is T^θ. Considering $T - T^\theta$ instead of T it is enough to show that if T is quasi-invariant and $T^\theta = -T$, then $T = 0$. By the just-mentioned exact sequence, it is enough to do it separately for distributions on P and distributions supported on the big Bruhat cell. We do it first for the latter.

Since $\mathcal{S}(PwN) = \mathcal{S}(N) \otimes \mathcal{S}(\Delta') \otimes \mathcal{S}(N)$ we may fix $\phi \in \mathcal{S}(\Delta')$ and consider

$$ T_\phi : \mathcal{S}(N) \times \mathcal{S}(N) \to End^\infty(X) \tag{4.69} $$

given by $T_\phi(\phi_1, \phi_2) = T(\phi_1 \otimes \phi \otimes \dot{\phi}_2)$ where $(n_1, n_2 \in N)$

$$ \phi_1 \otimes \phi \otimes \dot{\phi}_2(n_1 m'(t,s) n_2) = \phi_1(n_1)\phi(m'(t,s))\phi_2(n_2). \tag{4.70} $$

This becomes then a *quasi-invariant* $End^\infty(X)$-valued distribution on $N \times N$, where we consider $End^\infty(X)$ as an admissible $N \times N$-module via left and right τ-action. As we have seen, there exists an $\alpha(\phi) \in \widetilde{End^\infty(X)}'$ such that

$$ T(\phi_1 \otimes \phi \otimes \phi_2) = \tau(\phi_1) \circ \alpha(\phi) \circ \tau(\phi_2). \tag{4.71} $$

Now $(\phi_1 \otimes \phi \otimes \phi_2)^\theta = \phi_2^\theta \otimes \phi^\theta \otimes \phi_1^\theta$, so from the known properties of τ

$$ {}^tT((\phi_1 \otimes \phi \otimes \phi_2)^\theta) = \tau(\phi_1) \circ {}^t\alpha(\phi^\theta) \circ \tau(\phi_2) \tag{4.72} $$

and (the key point!) for $\phi \in \mathcal{S}(\Delta')$, $\phi^\theta = \phi$ since $^\theta$ is the identity on Δ'. We must show then that $^t\alpha(\phi) = \alpha(\phi)$.

Recall that $X = \bigoplus_{\mu \in \widehat{D}} X_\mu$ with each $\dim X_\mu \leq 1$, and the transpose on $End^\infty(X)$ was obtained from the identification of the dual of X_μ with X_μ^α. Thus $^tA = A$ for $A \in End^\infty(X)$ (or in its double-dual) whenever A commutes with $\tau(d)$ for $d \in D$. It is now an easy consequence of the fact that Δ' commute with D, that $\alpha(\phi)$ commutes with $\tau(d)$, hence $^t\alpha(\phi) = \alpha(\phi)$. Note that here we use the fact that T is also D-quasi-invariant. Previously we only used quasi-invariance with respect to N, and $R = DN$.

Finally, we consider distributions supported on P. The same analysis shows that we may look at $\phi_1 \otimes \phi$ with $\phi_1 \in \mathcal{S}(N)$ and $\phi \in \mathcal{S}(\Delta)$, and that

$$T(\phi_1 \otimes \phi) = \tau(\phi_1) \circ \alpha(\phi) \tag{4.73}$$

for some $\alpha(\phi) \in \widetilde{End^\infty(X)}'$, that we now have to show satisfies $^t\alpha(\phi^\theta) = \alpha(\phi)$. We may assume that ϕ is the characteristic function of an open compact subset of Δ. However, $m(t,s) \in \Delta$ are not anymore invariant under θ, unless $t\bar{t} = 1$, so we can not say that $\phi^\theta = \phi$. Luckily, the relation

$$m(t,s)n(0,z) = n(0,t\bar{t}z)m(t,s) \tag{4.74}$$

together with the equivariance (on both sides!) of T imply that if ϕ is supported on $m(t,s)$ with $t\bar{t} \neq 1$ we must have $\alpha(\phi) = 0$. ∎

Lemma 4.22. *We have*

$$Hom_R(\pi|_R, \tau) \neq 0 \Leftrightarrow Hom_R(\tilde{\pi}|_R, \tilde{\tau}) \neq 0. \tag{4.75}$$

Proof. We have already noticed that $\tilde{\pi} \simeq \pi^\alpha$ and that $\tilde{\tau} \simeq \tau^\alpha$. But clearly, any R-homomorphism between π and τ is also an R-homomorphism between π^α and τ^α. ∎

Given the two lemmas, the uniqueness of the generalized Whittaker model follows immediately from the main theorem.

4.5. **Kirillov models and exceptional representations.**

4.5.1. *Exceptional representations.* Let (π, V) be an irreducible infinite-dimensional admissible representation of $G = U(3)$. Let $\Lambda(\pi,\psi)$ be the set of $(\tau, X) \in \widehat{R}(\psi)$ for which there exists a τ-Heisenberg functional

$$\lambda_\tau : \pi|_R \to \tau \tag{4.76}$$

(hence a τ-Heisenberg model). Let $\Lambda(\pi) = \bigcup_{\psi \neq 1} \Lambda(\pi,\psi)$.

The torus $T = \{m(t,1) | \, t \in E^\times\}$ acts on $\widehat{R}$, the element $m(t,1)$ sending (τ, X) to (τ^t, X) where

$$\tau^t(r) = \tau(m(t,1)rm(t,1)^{-1}). \tag{4.77}$$

Note that E^1 acts trivially, and in general $m(t,1)$ maps $\widehat{R}(\psi)$ to $\widehat{R}(\psi_{t\bar{t}})$. This action preserves $\Lambda(\pi)$, because if λ_τ is a τ-Heisenberg functional then

$$\lambda_{\tau^t}(v) = \lambda_\tau(\pi(m(t,1))v) \tag{4.78}$$

is a τ^t-Heisenberg functional. Let $\Omega(\pi)$ be a set of representatives for the quotient of $\Lambda(\pi)$ by this action. If ψ_1 and ψ_2 are representatives of the $\psi's$ modulo the action of $N_{E/F}(E^\times)$ then $\Omega(\pi)$ is represented by $\Lambda(\pi,\psi_1) \cup \Lambda(\pi,\psi_2)$.

Definition 4.1. *We call* π exceptional *if* $|\Omega(\pi)| = 1$. *As we shall see below, the Weil representations* $\omega(\gamma, \psi, \chi)$ *are exceptional.*

In general, $\Omega(\pi)$ may be finite or infinite. The following facts are proved in Piatetski-Shapiro's Yale lectures:

- If $\Omega(\pi)$ is finite, then π does not have an (ordinary) Whittaker model, i.e. it is degenerate.
- If π is supercuspidal, this condition is also necessary, i.e. π is non-degenerate if and only if $\Omega(\pi)$ is infinite.

Thus Weil representations are "the most degenerate" representations. Note that Weil representations may be supercuspidal or (constituents of) principal series alike.

4.5.2. *Kirillov models.* Let $(\tau, X) \in \Lambda(\pi, \psi)$. Let $\lambda_\tau \in Hom_R(\pi|_R, \tau)$ be a Heisenberg functional. Then

$$W_v^\tau(g) = \lambda_\tau(\pi(g)v) \tag{4.79}$$

is a function from G to X, which transforms like τ under left translation by R. It is uniquely determined by τ, up to a scalar. Define a function $E^\times \to X$

$$\varphi_v^\tau(t) = W_v^\tau(m(t,1)). \tag{4.80}$$

Lemma 4.23. *Let* $\Omega(\pi)$ *be a set of representatives as above for the action of* T *on* $\Lambda(\pi)$. *If* $\varphi_v^\tau = 0$ *identically for each* $\tau \in \Omega(\pi)$ *then* $v = 0$.

Proof. If this is the case, then φ_v^τ vanishes for every $\tau \in \Lambda(\pi)$, because conjugating τ by $m(t_1, 1)$ has the effect of translating the function φ_v^τ by t_1. But the proof of the existence of Heisenberg models showed that for every $v \neq 0$ there exists a ψ such that $v \notin V(Z, \psi)$. Under this condition we have constructed a functional λ_τ for some $\tau \in \Lambda(\pi, \psi)$, not vanishing on v, and proved that it is unique up to a scalar. Hence $\varphi_v^\tau \neq 0$. ∎

The linear map assigning the collection of functions $\{\varphi_v^\tau | \, \tau \in \Omega(\pi)\}$ to v is therefore injective. It is called the Kirillov model of π. If π is exceptional, it consists of a single function. The nature of these functions is captured by the following proposition.

Proposition 4.24. *(i) For every* v *there exists a* $c(v) > 0$ *such that* $\varphi_v^\tau(t) = 0$ *if* $|t| > c(v)$, *for every* $\tau \in \Omega(\pi)$.

(ii) For every v *and* $(\tau, X) \in \Omega(\pi)$ *the space of* X *spanned by* $\varphi_v^\tau(t)$ *(for all* $t \in T$*) is finite dimensional.*

(iii) Assume $T_0 \subset T$ *is compact. Then for every* v, *the set of* $\tau \in \Omega(\pi)$ *where* $\varphi_v^\tau|_{T_0} \neq 0$, *is finite.*

(iv) Let $(\tau, X) \in \Omega(\pi)$. *Then for every* $\varphi \in C_c^\infty(E^\times, X)$ *there exists a* $v \in V$ *such that* $\varphi_v^\tau = \varphi$ *and* $\varphi_v^{\tau'} = 0$ *for every* $\tau \neq \tau' \in \Omega(\pi)$.

Proof. (i) The proof is based on the identity

$$m(t,1)n(0,z) = n(0, t\bar{t}z)m(t,1). \tag{4.81}$$

Let $U \subset F$ be small enough open compact so that $\pi(n(0,z))v = v$ for $z \in U$. Then

$$\begin{aligned} \varphi_v^\tau(t) &= W_v^\tau(m(t,1)n(0,z)) \\ &= W_v^\tau(n(0,t\bar{t}z)m(t,1)) \\ &= \psi(t\bar{t}z)\varphi_v^\tau(t). \end{aligned} \tag{4.82}$$

For t large $\psi(t\bar{t}z) \neq 1$ for some $z \in U$, proving $\varphi_v^\tau(t) = 0$.

(ii) Fix v and τ. If $U \subset D$ is open compact such that $\pi(U)$ fixes v, then for $u \in U$, $t \in T$

$$\begin{aligned} \tau(u)\varphi_v^\tau(t) &= \lambda_\tau(\pi(ut)v) \\ &= \lambda_\tau(\pi(tu)v) \\ &= \lambda_\tau(\pi(t)v) = \varphi_v^\tau(t). \end{aligned} \tag{4.83}$$

Thus $\varphi_v^\tau(t)$ is fixed by U, and part (ii) follows from the fact that X is D-admissible.

(iii) See P.-S. lectures, p. 32.

(iv) See P.-S. lectures, p. 33. ∎

4.5.3. *The modules $V(N)$ and $V(Z)$.* Let (π, V) be admissible irreducible representation of G. Recall the definition

$$\begin{aligned} V(N) &= Span\,\{\pi(n)v - v|\, v \in V,\, n \in N\} \\ V(Z) &= Span\{\pi(z)v - v|\, v \in V,\, z \in Z\}. \end{aligned} \tag{4.84}$$

Clearly $V(Z) \subset V(N) \subset V$. Recall that $V_N = V/V(N)$ is the Jacquet module, and that it is at most 2 dimensional, and vanishes if and only if π is supercuspidal.

Lemma 4.25. *(i) If $v \in V(N)$ then every φ_v^τ $(\tau \in \Omega(\pi))$ has a compact support.*

(ii) A vector $v \in V(Z)$ if and only if every φ_v^τ $(\tau \in \Omega(\pi))$ has compact support, and only finitely many φ_v^τ do not vanish.

Proof. (i) Let $v_1 = \pi(n)v - v$. Then

$$\begin{aligned} \varphi_{v_1}^\tau(t) &= \lambda_\tau(\pi(tnt^{-1})\pi(t)v) - \lambda_\tau(\pi(t)v) \\ &= (\tau(tnt^{-1}) - 1)\varphi_v^\tau(t). \end{aligned} \tag{4.85}$$

But when τ and v are fixed, $\varphi_v^\tau(t)$ belong to a finite dimensional subspace of X, so if $|t|$ is small, tnt^{-1} is close to 1 in N and fixes it, hence $\varphi_{v_1}^\tau(t) = 0$. On the other hand we have seen that when $|t|$ is large, $\varphi_v^\tau(t) = 0$ too.

(ii) Let $v_1 = \pi(z)v - v$. Fix ψ and let $\tau \in \Lambda(\pi, \psi)$. Then

$$\varphi_{v_1}^\tau(t) = (\psi(t\bar{t}z) - 1)\varphi_v^\tau(t). \tag{4.86}$$

When $|t|$ is small this is 0 *independently of* τ. Thus there exists a compact T_0 where all the $\varphi_{v_1}^\tau$ are supported. By the previous proposition, part (iii) only finitely many of them do not vanish. Conversely, suppose the conclusion of (ii) holds. To show that $v \in V(Z)$ we must show that

$$\int_U \pi(z)vdz = 0 \tag{4.87}$$

for some (large enough) open compact U. Let $\{\psi_1, \psi_2\}$ be representatives for the ψ's. If the integral is not 0, then there exists a $j \in \{1, 2\}$, a $\tau \in \Lambda(\pi, \psi_j)$ and a

$t \in T$ such that

$$\begin{aligned} 0 &\neq \lambda_{\tau^t}\left(\int_U \pi(z) v dz\right) \\ &= \int_U W_v^\tau(m(t,1)n(0,z))dz \\ &= \int_U W_v^\tau(n(0,t\bar{t}z)m(t,1))dz \\ &= \int_U \psi(t\bar{t}z)dz \cdot \varphi_v^\tau(t). \end{aligned} \tag{4.88}$$

Since there are only finitely many τ to consider, and for each τ the support of $\varphi_v^\tau(t)$ is compact, there is a compact set $T_0 \subset T$ to consider altogether. But then we can find a large enough U so that $\psi(t\bar{t}z)$ is a non-trivial character on U for all $t \in T_0$, hence the integral vanishes for all choices of τ and t. ∎

Corollary 4.26. *If $\Omega(\pi)$ is finite, then $V(Z) = V(N)$.*

Proof. Clear. ∎

Corollary 4.27. *For any (π, V) the map $\varphi \mapsto (\varphi_v^\tau)_{\tau \in \Omega(\pi)}$ defines an isomorphism of B-modules (where $B = TR$)*

$$V(Z) \simeq \bigoplus_{\tau \in \Omega(\pi)} c\text{-}Ind_R^{TR}\tau. \tag{4.89}$$

Proof. The map is well defined by part (ii) of the lemma, and respects the TR-action (note that T acts on the right also on τ, i.e. we have to identify $c\text{-}Ind_R^{TR}\tau$ with $c\text{-}Ind_R^{TR}\tau^t$). The injectivity follows from the fact that if all φ_v^τ ($\tau \in \Omega(\pi)$) vanish, $v = 0$ (the Kirillov model). The surjectivity follows from part (iv) of the proposition. ∎

Theorem 4.28. *(i) The representation π does not have an ordinary Whittaker model if and only if $V(Z) = V(N)$ (e.g. if $\Omega(\pi)$ is finite, and in particular if π is exceptional).*

(ii) If π is supercuspidal ($V(N) = V$) then this happens if and only if $\Omega(\pi)$ is finite.

4.5.4. *Weil representations are exceptional.* We prove together the following two theorems.

Theorem 4.29. *The Weil representation $\pi = \omega(\gamma, \psi, \chi)$ is exceptional. Moreover, let $\tau' = \chi \otimes \tau(\gamma, \psi)$ be the representation of $R' = CR$ where we have labelled the representations in $\widehat{R}(\psi)$ by γ (the parameter used in labelling splitting of the metaplectic group extension over unitary groups). Then $\Omega(\pi) = \{\tau'\}$.*

Theorem 4.30. *There is a bijection $\omega(\gamma, \psi, \chi) \leftrightarrow \tau(\gamma, \psi, \chi) = \chi \otimes \tau(\gamma, \psi)$ between Weil representations and T-orbits in $\widehat{R'}$ (recall $C \simeq E^1$ is the center).*

5. Global theta functions on the unitary group (survey)

5.1. The global metaplectic group.

5.1.1. *Notation.* F a number field, $\mathbb{A}$ the adeles of F, F_v the completion, $\mathcal{O}_v$ the ring of integers ($v \nmid \infty$), $\psi = \prod_v \psi_v$ a non trivial character of $F \backslash \mathbb{A}$. Let $(W, \langle , \rangle)$ be a symplectic space of dimension $2n$ over F, and for each v fix a model $(\mathcal{S}_v, \rho_v)$ of the Heisenberg representation of $H(W, F_v)$ associated to ψ_v. At $v|\infty$ this is a Hilbert space and we let $\mathcal{S}_v^\infty$ be the space of smooth Schwartz (rapidly decaying) functions in the Schroedinger model.

Let L be a lattice in $W(F)$. We say that v is *unramified* if

- $v \nmid 2\infty$
- ψ_v is unramified (of conductor 1)
- L_v is self dual.

These conditions hold for all but finitely many v. If they hold, $H(W, \mathcal{O}_v)$, an extension of L_v by $\mathcal{O}_v$, is an integral structure in $H(W, F_v)$, and the subspace of vectors fixed by it in $\mathcal{S}_v$ is one-dimensional. Fix a generator s_v^0 for the $H(W, \mathcal{O}_v)$-invariants. Let

$$\mathcal{S}^\infty = \bigotimes_{v|\infty} \mathcal{S}_v^\infty \otimes {\bigotimes_{v\nmid\infty}}' \mathcal{S}_v \tag{5.1}$$

where the restricted tensor product is taken with respect to the s_v^0. This is the space of Schwartz-Bruhat functions. Then the group

$$H(W, \mathbb{A}) = {\prod_v}' H(W, F_v) \tag{5.2}$$

acts on $\mathcal{S}^\infty$ via $\rho_\psi = \bigotimes' \rho_v$.

For v unramified, the stabilizer K_v of s_v^0 in $\widehat{Sp}(W, F_v)$ is a compact open subgroup lifting $Sp(W, \mathcal{O}_v)$. Let N be the kernel of the product map $\prod_v' \{\pm 1\} \to \{\pm 1\}$ and define

$$\widehat{Sp}(W, \mathbb{A}) = {\prod_v}' \widehat{Sp}(W, F_v)/N. \tag{5.3}$$

The tensor product representation

$$\omega_\psi = {\bigotimes_v}' \omega_v \tag{5.4}$$

(on the space $\mathcal{S}^\infty$), where ω_v is the metaplectic representation of $\widehat{Sp}(W, F_v)$ associated with ψ_v, factors through $\widehat{Sp}(W, \mathbb{A})$. We call it the *global metaplectic representation associated with* ψ. We let $\widetilde{Sp}(W, \mathbb{A}) = \mathbb{C}^\times \times_{\{\pm 1\}} \widehat{Sp}(W, \mathbb{A})$ and extend ω_ψ naturally to it.

5.1.2. *The theta functional.* The following theorem is the corner-stone of the modern theory of theta functions. It is implicit in Weil's Acta paper, but appears for the first time in Howe's Corvalis lecture, where he does not give a proof, but calls it a "strong form of the Poisson summation formula".

Theorem 5.1. *There exists a unique-up-to-scalar distribution*

$$\Theta : \mathcal{S}^\infty \to \mathbb{C} \tag{5.5}$$

which is invariant under the action of $H(W,F) \subset H(W,\mathbb{A})$.

Proof. Fix a complete polarization $W = X \oplus Y$ of W w.r.t. $\langle,\rangle$. Choose the model $\mathcal{S}_{Y_v} = \mathcal{S}(X_v)$ for $\mathcal{S}_v$. Then $\mathcal{S}^\infty$ is identified with the space $\mathcal{S}(X_\mathbb{A})$ of Schwarz-Bruhat functions on $X_\mathbb{A}$. In terms of this model we can define

$$\Theta(\phi) = \sum_{x \in X_F} \phi(x). \tag{5.6}$$

The sum converges absolutely and is clearly invariant under $H(W,F)$.

To prove uniqueness we eventually have to prove (a generalization of) the following lemma. ∎

Lemma 5.2. *Let* $\Theta : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ *be a tempered distribution which is invariant under translation by* 1 *and also under multiplication by* $e(x) = \exp(2\pi i x)$. *Then, up to a scalar*

$$\Theta(\phi) = \sum_{n \in \mathbb{Z}} \phi(n). \tag{5.7}$$

Equivalently, both Θ and $\Theta \circ \mathcal{F}$ where $\mathcal{F}$ is the Fourier transform, are invariant under translation by $\mathbb{Z}$. This lemma is standard Fourier analysis, the only subtle point being the topological issues hidden in the definition of tempered distributions. The lemma implies the Poisson summation formula (since if Θ satisfies it, so does $\Theta \circ \mathcal{F}$) hence is often considered as a "strong version of Poisson summation formula".

5.1.3. *Splitting over rational points.* As a corollary of the uniqueness of the theta functional, Weil proved the following.

Theorem 5.3. *The covering* $\widetilde{Sp}(W,\mathbb{A}) \to Sp(W,\mathbb{A})$ *splits uniquely over* $Sp(W,F)$.

Proof. Uniqueness follows from the fact that $Sp(W,F)' = \{1\}$. To prove that the covering splits, note that since $\widetilde{Sp}(W,F)$ normalizes $H(W,F)$, for any $g \in \widetilde{Sp}(W,F)$, the distribution

$$\phi \mapsto \Theta(g\phi) \tag{5.8}$$

($\phi \in \mathcal{S}^\infty$) is also $H(W,F)$-invariant, hence is equal to $\lambda(g) \cdot \Theta$, and λ is a character splitting the exact sequence

$$0 \to \mathbb{C}^\times \to \widetilde{Sp}(W,F) \to Sp(W,F) \to 0. \tag{5.9}$$

∎

Note that the proof also shows that for every $g \in Sp(W,F)$, $\Theta(g\phi) = \Theta(\phi)$. Indeed, the splitting is *defined* by letting $\lambda(g) = 1$ for $g \in Sp(W,F)$.

5.1.4. *The global Weil representation as an automorphic representation of* $\widetilde{Sp}(W,\mathbb{A})$. From a different perspective, the theta distribution embeds $(\mathcal{S}^\infty, \omega_\psi)$ uniquely in $\mathcal{A}(Sp(W,F)\backslash\widetilde{Sp}(W,\mathbb{A}))$. Just as we did for Whittaker models locally, when we interpreted a Whittaker functional as giving rise to a Whittaker model (Frobenius duality), we can define for $\phi \in \mathcal{S}^\infty$ and $g \in \widetilde{Sp}(W,\mathbb{A})$

$$\theta_\phi(g) = \Theta(\omega_\psi(g)\phi). \tag{5.10}$$

The function θ_ϕ is *left invariant* under $Sp(W,F)$ and can be proved to be *automorphic*: slowly increasing, K-finite, Z-finite, where Z is the center of the universal enveloping algebra of $\widetilde{Sp}(W,F_\infty)$.). It is also a *genuine* function on $\widetilde{Sp}(W,\mathbb{A})$, meaning that $\mathbb{C}^\times$ acts on it (via right translation) as scalars, so in particular it does not descend to $Sp(W,\mathbb{A})$.

Clearly, $\phi \mapsto \theta_\phi$ intertwines the action of $\widetilde{Sp}(W,\mathbb{A})$ (via ω_ψ) on ϕ with right translation. Thus the *existence of* Θ means that ω_ψ is automorphic and the *uniqueness* is a multiplicity 1 result.

Note that for different ψ, the resulting subspaces of $\mathcal{A}(Sp(W,F)\backslash\widetilde{Sp}(W,\mathbb{A}))$ are disjoint.

The functions θ_ϕ are called *theta functions*. In the case of $SL_2 = Sp_2$ one can get in this way the classical Jacobi theta function as an automorphic form.

The key point to remember is that ω_ψ is a very special, in fact very "small" automorphic representation. In a certain precise sense (??) it is the smallest automorphic representation after the one-dimensional ones. However, when we restrict it to the preimage $\tilde{G}_1 \times \tilde{G}_2$ in $\widetilde{Sp}(W,\mathbb{A})$ of a dual reductive pair $G_1 \times G_2$ in $Sp(W,\mathbb{A})$ (Note: the $\tilde{G}_i$ will still commute), and in particular if the metaplectic group splits over $G_1 \times G_2$, as is often the case, we get very interesting representations of the smaller groups G_i and a correspondence between such representations.

5.2. **Splitting over $G_\mathbb{A}$.**

5.2.1. *Compatible adelic splitting.* Let E/F be a quadratic extension of number fields, $z \mapsto \bar{z}$ denoting the non-trivial element in $Gal(E/F)$. We borrow the notation from the beginning of the notes and let $\mathbf{G} = U(V,(,))$ where $V = E^3$ etc. It is an algebraic group over F. We let W be the restriction of scalars of V (from E to F) and $\langle,\rangle = \mathrm{Im}(,)$ the symplectic form on W. Then $\mathbf{G} \subset \mathbf{Sp}(W)$. Fix ψ as before. Let $\mathbf{C}$ be the center of $\mathbf{G}$ (isomorphic to $\mathbf{E}^1 = \ker(\mathbf{E}^\times \to \mathbf{F}^\times)$ as an algebraic group).

Since every $G_v = \mathbf{G}(F_v)$ lifts to $\widetilde{Sp}(W_v)$, and for almost all v we can choose the embedding to respect the compact subgroups $\mathbf{G}(\mathcal{O}_v)$ and $Sp(W(\mathcal{O}_v))$, it is possible to lift $G_\mathbb{A}$ to $\widetilde{Sp}(W_\mathbb{A})$. However, the global rational points G_F sit inside $Sp(W_F)$, which already lifts uniquely, by Weil, to $\widetilde{Sp}(W_\mathbb{A})$, and we would like to know that these liftings are compatible.

Recall that the liftings $s(\gamma_v) : G_v \to \widetilde{Sp}(W_v)$ are classified by characters $\gamma_v : E_v^\times \to \mathbb{C}^\times$ whose restriction to $F_v^\times$ is η_{E_v/F_v}. If v is inert or ramified in E this was mentioned before. If v is split, then $E_v = F_v \times F_v$, $G_v = GL_3(F_v)$ and the same holds true, if we interprete the groups correctly. This is valid also for $v|\infty$.

Proposition 5.4 (Ge-Ro2). *Suppose that the γ_v are the local components of an idele class character*

$$\gamma : E^\times\backslash E_\mathbb{A}^\times \to \mathbb{C}^\times \tag{5.11}$$

whose restriction to $F^\times\backslash F_\mathbb{A}^\times$ is $\eta_{E/F}$. Then the resulting embedding $s(\gamma)$ of $G_\mathbb{A}$ in $\widetilde{Sp}(W_\mathbb{A})$ is compatible over G_F with the (unique) splitting of the metaplectic group over $Sp(W_F)$.

If γ^* is another such idele class character, then $\gamma^*(z) = \gamma(z)\nu(z/\bar{z})$ where ν is a character of $E^1\backslash E_\mathbb{A}^1$ and

$$s(\gamma^*)(g) = s(\gamma)(g) \cdot \nu(\det g). \tag{5.12}$$

5.2.2. *Global Weil representations and theta functions on the unitary group.* Given an idele class character γ and an additive character ψ on $F\backslash F_{\mathbb{A}}$ as before, we get the representation $\omega_{\psi,\gamma}$ of $G_{\mathbb{A}}$ by

$$\omega_{\psi,\gamma}(g) = \omega_{\psi}(s(\gamma)(g)) \tag{5.13}$$

on the space $\mathcal{S}^{\infty}$ of Schwartz-Bruhat functions Φ. As an explicit model for $\mathcal{S}^{\infty}$ it is convenient to take, in the case of $U(3)$, the *mixed model* described above. At any given v we take $\mathcal{S}_v = \mathcal{S}(E_v, \mathcal{F}_v)$, Schwartz functions on E_v with values in $\mathcal{F}_v$, the *lattice* model for the Heisenberg representation of $H(E_v)$, with the "small" metaplectic representation ω_{ψ}^1 of $\widetilde{Sp}(E_v)$ acting on $\mathcal{F}_v$. We have worked it out for finite non-split v. One can do it also for $v|\infty$ or finite split v, where the group G_v becomes $GL_3(F_v)$.

In any case, given a Schwartz-Bruhat function

$$\Phi \in \bigotimes_{v}{}' \mathcal{S}(E_v, \mathcal{F}_v) \tag{5.14}$$

we can associate to it a *theta function* on $g \in G_{\mathbb{A}}$

$$\theta_{\Phi}(g) = \Theta(\omega_{\psi}(s(\gamma)(g))\Phi). \tag{5.15}$$

The fact that $s(\gamma)$ maps G_F to $Sp(W_F)$ and that $g \mapsto \Theta(g\phi)$ is left invariant under $Sp(W_F)$ implies that θ_{Φ} is a well-defined *automorphic form* on $G_F\backslash G_{\mathbb{A}}$. Clearly, the map $\Phi \mapsto \theta_{\Phi}$ intertwines the representation $\omega_{\psi,\gamma}$ with the right regular representation of $G_{\mathbb{A}}$ on $\mathcal{A}(G_F\backslash G_{\mathbb{A}})$.

Caution. The space of theta functions that we have produced depends both on ψ and on γ. It will be denoted

$$\Theta(\psi, \gamma) \subset \mathcal{A}(G_F\backslash G_{\mathbb{A}}). \tag{5.16}$$

5.2.3. *Explicit formulas for $\omega_{\psi,\gamma}$.* Working in the mixed model, Gelbart and Rogawski gave the following explicit formulas. Recall that Φ is a Schwartz-Bruhat function on $E_{\mathbb{A}}$ with values in $\mathcal{F}_{\mathbb{A}} = \otimes_v' \mathcal{F}_v$, and for $\mathcal{F}_v$ we can take the lattice model: nice functions on E_v satisfying $\phi(a + x) = \psi_v(-\langle a, x\rangle_v/2)\phi(x)$ for $a \in \mathcal{O}_v$. We assume that γ is unitary.

- For $s \in E^1_{\mathbb{A}}$

$$\omega_{\psi,\gamma}(m(1, s))\Phi(w) = \omega^1_{\psi,\gamma}(s)(\Phi(w)). \tag{5.17}$$

Here $\omega^1_{\psi,\gamma}$ is the "small" metaplectic representation of $E^1_{\mathbb{A}}$ on $\mathcal{F}_{\mathbb{A}}$. The splitting $s(\gamma)$, when restricted to matrices of the form $m(1, s)$, provides a lifting of $U(E) = E^1$ to $\widetilde{Sp}(E) \subset \widetilde{Sp}(W)$. (Note that the preimage of $Sp(E) \subset Sp(W)$, $E = Ee_2 \subset W$, in $\widetilde{Sp}(W)$, can be canonically identified with $\widetilde{Sp}(E)$.) Thus γ gives also $\omega^1_{\psi,\gamma} = \omega^1_{\psi} \circ s(\gamma)|_{E^1}$.

- For $t \in E^{\times}_{\mathbb{A}}$

$$\omega_{\psi,\gamma}(m(t, 1))\Phi(w) = \gamma(t)|t|^{1/2}\Phi(\bar{t}w). \tag{5.18}$$

- For $n(b, z) \in N_{\mathbb{A}}$

$$\omega_{\psi,\gamma}(n(b, z))\Phi(w) = \psi(zw\bar{w})\rho^1_{\psi}(n(bw, 0))\Phi(w). \tag{5.19}$$

6. Fourier analysis on $G_{\mathbb{A}}$ (survey)

6.1. **Fourier expansion along Z.**

6.1.1. *Fourier-Jacobi coefficients along Z.* Let $\varphi \in \mathcal{A}(G_F\backslash G_{\mathbb{A}})$. Since $Z_F\backslash Z_{\mathbb{A}}$ is compact we may define, for any additive character ψ of $F\backslash F_{\mathbb{A}}$, the ψ-th Fourier(-Jacobi) coefficient *along* Z to be

$$\varphi_\psi(g) = \int_{Z_F\backslash Z_{\mathbb{A}}} \varphi(zg)\overline{\psi(z)}dz. \tag{6.1}$$

Then $\varphi_\psi(zg) = \psi(z)\varphi_\psi(g)$ for $z \in Z_{\mathbb{A}}$. When $\psi = 1$ we denote this integral by

$$\varphi_Z(g) = \int_{Z_F\backslash Z_{\mathbb{A}}} \varphi(zg)dz..$$

We say that φ is *hypercuspidal* if $\varphi_Z \equiv 0$ and *cuspidal* if $\varphi_N \equiv 0$, where

$$\varphi_N(g) = \int_{N_F\backslash N_{\mathbb{A}}} \varphi(ng)dn. \tag{6.2}$$

We have

$$\varphi(g) = \varphi_Z(g) + \sum_{\psi\neq 1}\varphi_\psi(g). \tag{6.3}$$

Since the matrices $m(t,1)$ of the torus T_F (i.e. $t \in E^\times$) normalize $Z_F\backslash Z_{\mathbb{A}}$ it is easy to verify that

$$\varphi_{\psi_{t\bar{t}}}(g) = \varphi_\psi(m(t,1)g). \tag{6.4}$$

Lemma 6.1. *Assume that $\varphi_\psi = 0$ for every $\psi \neq 1$. Then φ factors through the map*

$$\det : G_F\backslash G_{\mathbb{A}} \to E^\times\backslash E^\times_{\mathbb{A}}. \tag{6.5}$$

In particular, if φ belongs to an infinite dimensional discrete automorphic unitary representation (see below) then some $\varphi_\psi \neq 0$.

6.1.2. *Fourier expansion of φ_Z along N.* Since φ_Z is left-invariant under $N_FZ_{\mathbb{A}}$, and $N_FZ_{\mathbb{A}}\backslash N_{\mathbb{A}} \simeq E\backslash E_{\mathbb{A}}$ we may define, for any character ξ of $E\backslash E_{\mathbb{A}}$

$$W^\xi_\varphi(g) = \int_{N_FZ_{\mathbb{A}}\backslash N_{\mathbb{A}}} \varphi_Z(ng)\overline{\xi(n)}dn. \tag{6.6}$$

This is just an (ordinary) Whittaker ξ-function on $G_{\mathbb{A}}$:

$$W^\xi_\varphi(ng) = \xi(n)W^\xi_\varphi(g). \tag{6.7}$$

Note that if ξ is non-trivial, all the others are ξ_a $(a \in E^\times)$. We have

$$\varphi_Z(g) = \varphi_N(g) + \sum_{\xi\neq 1} W^\xi_\varphi(g). \tag{6.8}$$

Thus φ is hypercuspidal if and only if it is cuspidal, and all its ordinary Whittaker functions (Fourier coefficients along N) $W^\xi_\varphi = 0$.

Since the torus T_F normalizes $N_FZ_{\mathbb{A}}\backslash N_{\mathbb{A}}$ it is easy to check that

$$W^\xi_\varphi(m(t,1)g) = W^{\xi_t}_\varphi(g). \tag{6.9}$$

Thus one Whittaker function of φ vanishes if and only if they all vanish.

6.1.3. *Classification of unitary automorphic representation on G.* A unitary representation π of $G_{\mathbb{A}}$ is called a *discrete* representation if it can be realized on a closed subspace V_π of $L^2(G_F\backslash G_{\mathbb{A}})$ ($G_{\mathbb{A}}$ acting by right translation). The automorphic forms in π (i.e. functions which are smooth, of moderate growth, K-finite and $\mathfrak{Z}$-finite where $\mathfrak{Z}$ is the center of the enveloping algebra of G_∞) form a dense subspace V_π^0. When we write integrals involving functions from V_π we often assume implicitly that they belong to V_π^0. The integrals, often over a set of measure zero in $G_{\mathbb{A}}$, do not make sense for L^2 functions.

Let π be a discrete representation. We say that π is *cuspidal* if $\varphi_N = 0$ for all $\varphi \in V_\pi^0$.

We say that π is *hypercuspidal* if $\varphi_Z = 0$ for all $\varphi \in V_\pi^0$. We thus have (denoting hypercuspidal spectrum by 00)

$$L^2(G_F\backslash G_{\mathbb{A}})_{00} \subset L^2(G_F\backslash G_{\mathbb{A}})_0 \subset L^2(G_F\backslash G_{\mathbb{A}})_d. \tag{6.10}$$

6.1.4. *Representations of R.* In applying Fourier analysis to the study of cuspidal representations it is not enough to do harmonic analysis on N, because for hypercuspidal functions, all W_φ^ξ vanish.

Let $\mathbf{R}$ be the centralizer of $\mathbf{Z}$ in $\mathbf{G}$. It is *larger* than the group $\mathbf{DN}$ which was called $\mathbf{R}$ in the local sections, but not much:

$$\mathbf{R} = \mathbf{C} \times \mathbf{D} \ltimes \mathbf{N} \tag{6.11}$$

and both $\mathbf{C}$ and $\mathbf{D}$ are isomorphic to $\mathbf{E}^1$. Its center is $\mathbf{C} \times \mathbf{Z}$.

The importance of the group $\mathbf{R}$ lies in the fact that because it commutes with $\mathbf{Z}$, left integration against $\psi(z)$ over $Z_F\backslash Z_{\mathbb{A}}$ does not destroy automorphy with respect to R_F, i.e. the functions (here $\varphi \in \mathcal{A}(G_F\backslash G_{\mathbb{A}})$ is a fixed automorphic function)

$$r \mapsto \varphi_\psi(rg) \tag{6.12}$$

are functions on $R_F\backslash R_{\mathbb{A}}$. Now this space is *compact,* so we have a discrete decomposition

$$L^2(R_F\backslash R_{\mathbb{A}}) = \bigoplus_{\tau\in\widehat{R}} L^2(R_F\backslash R_{\mathbb{A}})_\tau \tag{6.13}$$

where $\widehat{R}$ is (by definition) the *automorphic spectrum* of R (which is, by comactness, all discrete). Gelbart and Rogawski [Ge-Ro2, Theorem 2.2.1] proved:

Theorem 6.2. *Every $\tau \in \widehat{R}$ appears with multiplicity one, i.e. $L^2(R_F\backslash R_{\mathbb{A}})_\tau$ is irreducible.*

The central character of τ is given by

$$(\chi, \psi) \tag{6.14}$$

where χ and ψ are automorphic characters of C and Z, i.e. characters of $C_F\backslash C_{\mathbb{A}} = E_F^1\backslash E^1$ and $Z_F\backslash Z_{\mathbb{A}}$ as usual. Gelbart and Rogawski gloss over $\psi = 1$ saying that τ is one-dimensional then. I do not understand it, as the group R/Z is not abelian, essentially "dihedral". Is it possible that all its *automorphic* representations are 1-dimensional?

In any case, we are interested in the τ whose restriction to $Z_{\mathbb{A}}$ is ψ. As E_F is a maximal isotropic subgroup of $E_{\mathbb{A}}$, these can be realized on the space of L^2 functions on $N_F\backslash N_{\mathbb{A}}$ which transform under $Z_F\backslash Z_{\mathbb{A}}$ via ψ, and $N_{\mathbb{A}}$ acts via the adelic Heisenberg representation ρ_ψ (right translation). After we have decomposed w.r.t. the characters χ of $C_F\backslash C_{\mathbb{A}}$, multiplicity one now is the global analogue of the multiplicity one result that we had for the local R.

The space parametrizing the τ is the same as the space parametrizing the Weil representations - triples (γ, ψ, χ) where γ is an idele class character of $E^\times\backslash E_{\mathbb{A}}^\times$ whose restriction to $F^\times\backslash F_{\mathbb{A}}^\times$ is the quadratic Hecke character $\eta_{E/F}$, ψ and χ as before.

6.1.5. *Primitive Fourier coefficients.* Let $\psi \neq 1$ as before, fix $g \in G_{\mathbb{A}}$ and decompose the function of r

$$\varphi_\psi(rg) = \sum_{\tau\in\widehat{R}(\psi)} \varphi_\tau(r;g) \tag{6.15}$$

according to the spectral decomposition of $L^2(R_F\backslash R_{\mathbb{A}})$. Then for a fixed $r_1 \in R_{\mathbb{A}}$

$$\varphi_\tau(rr_1;g) \in L^2(R_F\backslash R_{\mathbb{A}})_\tau \tag{6.16}$$

hence $\varphi_\tau(rr_1;g) = \varphi_\tau(r;r_1g)$ and we may write unambiguously $\varphi_\tau(r;g) = \varphi_\tau(rg)$. We then substitute $r = 1$ and get the decomposition

$$\varphi_\psi(g) = \sum_{\tau\in\widehat{R}(\psi)} \varphi_\tau(g). \tag{6.17}$$

Of course, we may further decompose with respect to the automorphic characters χ of C. The φ_τ are called the *primitive Fourier-Jacobi coefficients.* Rogawski and Glauberman (J. Crelle XXX) worked them out in the classical language of FJ expansions and proved multiplicity one there, although it is not clear to me in what sense this is different from the representation theoretic multiplicity one result (except for the language). As usual, there are issues of level appearing in the classical language (related essentially the conductor of γ) and "new versus old" questions.

Once again, the torus T_F normalizes R_F and we have

$$\varphi_{\tau^t}(g) = \varphi_\tau(m(t,1)g). \tag{6.18}$$

Thus if the primitive FJ coefficient corresponding to $\tau = \tau(\gamma, \psi, \chi)$ is non-zero, so is the one corresponding to τ^t. Note that the parameters of τ^t are $(\gamma, \psi_{t\bar{t}}, \chi)$.

6.1.6. *Exceptional automorphic representations.* As in the local case, we denote by $\Lambda(\pi)$ the set of $\tau \in \widehat{R}$ for which some $\varphi_\tau \neq 0$, for $\varphi \in V_\pi$. Once again, this collection is a union of orbits of T_F and we call π *exceptional* if

- $\Lambda(\pi)$ consists of a single orbit under T_F
- For every $\varphi \in V_\pi$, $\varphi_N = \varphi_Z$.

Thus a *cuspidal* π is exceptional if it is hypercuspidal and $\Lambda(\pi)$ consists of one orbit only. But non-cuspidal exceptional representations exist (see below).

Proposition 6.3. *If π is globally exceptional, all its local components are exceptional.*

For this one would have to discuss Heisenberg models for local GL_3 at the split primes, and the archimedean places, which we have not had time to cover.

6.2. **Irreducible Weil representations.**

6.2.1. *The $\omega(\psi,\gamma,\chi)$ and global theta lifting.* Recall the space $\Theta(\psi,\gamma)\subset\mathcal{A}(G_F\backslash G_{\mathbb{A}})$ of the functions

$$\theta_\Phi(g)=\Theta(\omega_{\psi,\gamma}(g)\Phi) \tag{6.19}$$

where Φ runs over the Schwartz-Bruhat functions in the metaplectic representation of $\widetilde{Sp}(W_{\mathbb{A}})$. We write

$$\Theta(\psi,\gamma,\chi) \tag{6.20}$$

for the subspace on which $C_F\backslash C_{\mathbb{A}}$ acts via χ. Note that as $C_{\mathbb{A}}=E^1_{\mathbb{A}}$ is compact, the χ are unitary.

Theorem 6.4. *(i) The space $\Theta(\psi,\gamma)$ is contained in $L^2(G_F\backslash G_{\mathbb{A}})_d$ (but not necessarily in the cuspidal part).*

(ii) The closure of $\Theta(\psi,\gamma,\chi)$ in L^2 is an irreducible unitary representation *denoted by $\omega(\psi,\gamma,\chi)$ (these are the global Weil representations).*

(iii) The representation $\omega(\psi,\gamma,\chi)$ is exceptional*. the unique T_F-orbit in $\Lambda(\pi)$ is the one containing $\tau(\psi,\gamma,\chi)$.*

Since $C_F\backslash C_{\mathbb{A}}$ is compact the projection onto the χ-component is given by

$$\begin{aligned}\theta_\Phi &\mapsto \int_{C_F\backslash C_{\mathbb{A}}}\theta_\Phi(ug)\overline{\chi(u)}du\\ &= \int_{C_F\backslash C_{\mathbb{A}}}\Theta_\Phi(\omega_{\psi,\gamma}(ug))\overline{\chi(u)}du\end{aligned} \tag{6.21}$$

where Θ_Φ is the theta function associated to Φ on $\widetilde{Sp}(W_{\mathbb{A}})$. This is an instance of the *global theta correspondence* on the dual reductive pair $(\mathbf{C},\mathbf{G})$ (theta series lifting from $U(1)$ to $U(3)$). For any automorphic character χ of $C=E^1$, the corresponding space $\Theta(\psi,\gamma,\chi)$ (which of course depends on γ and ψ) is non-zero. However, it is not always cuspidal.

Remark 6.1. *(i) The question whether $\omega(\psi,\gamma,\chi)$ is* cuspidal *(hence, being exceptional, also hypercuspidal) is equivalent to the* vanishing *of a theta series lifting from $U(1)$ to $U(1)$. In turn, the* existence *of a $\psi\neq 1$ for which such a theta series lifting does not vanish is equivalent to $L(\frac{1}{2},\gamma^{-1}\chi_E^{-1})\neq 0$ (here $\chi_E(z)=\chi(z/\bar z)$). I may be mistaken about the precise Hecke character of E figuring out in the L-value...*

(ii) The theta liftings, where non-zero, are compatible with the local theta liftings (local Howe conjecture is known here).

7. The Shimura integral and $L(\pi,s)$ (survey)

8. Appendices

8.1. The Leray invariant.

8.1.1. *Leray invariants of mutually transversal maximal isotropic subspaces.* Let $(W,\langle,\rangle)$ be a symplectic space over a field F, $char.F\neq 2$. Let $\Omega^3(W)$ be the set of ordered triples (X,Y,Z) of maximal isotropic subspaces, and $\Omega^3(W)_0$ the triples which are mutually transversal. Let $G=Sp(W)$. While G acts transitively on $\Omega^2(W)_0$, it does not act transitively on $\Omega^3(W)_0$, and the Leray invariant $L(X,Y,Z)$ is an invariant which completely classifies the G-orbits on $\Omega^3(W)_0$.

Let $(X, Y, Z) \in \Omega^3(W)_0$. Let $u : W = X \bigoplus Y \to X$ be the projection along Y and $q(z_1, z_2)$ the bilinear form on Z

$$q(z_1, z_2) = \langle z_1, u(z_2)\rangle . \tag{8.1}$$

Then q is symmetric and non-degenerate. In fact, if $z_i = x_i + y_i$ then

$$q(z_1, z_2) = \langle y_1, x_2\rangle = - \langle x_1, y_2\rangle = \langle y_2, x_1\rangle = q(z_2, z_1). \tag{8.2}$$

The Leray invariant $L(X, Y, Z)$ is the isomorphism type of the quadratic space (Z, q).

Lemma 8.1. *The Leray invariant classifies G-orbits on $\Omega^3(W)_0$.*

Proof. If $g \in G$ then clearly $L(X, Y, Z) = L(gX, gY, gZ)$ by transport of structure. Conversely, suppose (X, Y, Z) and (X', Y', Z') are in $\Omega^3(W)_0$ and have the same Leray invariant. Since G acts transitively on $\Omega^2(W)_0$ we may assume $X = X'$ and $Y = Y'$. Fixing a symplectic basis subordinate to $W = X \bigoplus Y$ we have

$$n = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \in Sp_{2n}(F) \tag{8.3}$$

$(b = {}^t b)$ which is the identity on X, mapping Y to Z, and similarly n' mapping Y to Z'. Note that b and b' are non-singular, by tranversality. By assumption $(q \simeq q')$ there is an $h \in GL_n(F)$ with $b' = {}^t hbh$. But then

$$\begin{pmatrix} {}^t h & \\ & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \begin{pmatrix} {}^t h & \\ & h^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & b' \\ & 1 \end{pmatrix} \tag{8.4}$$

so $g = \begin{pmatrix} {}^t h & \\ & h^{-1} \end{pmatrix}$ maps (X, Y, Z) to (X', Y', Z'). ∎

8.1.2. *Leray invariants in general.* We now attach an invariant to any triple $(X, Y, Z) \in \Omega^3(W)$. Let

$$R = X \cap Y + X \cap Z + Y \cap Z. \tag{8.5}$$

Then R is an isotropic subspace,

$$R^\perp = (X + Y) \cap (X + Z) \cap (Y + Z) \tag{8.6}$$

and $R^\perp/R$ is a symplectic space with the induced alternating form. If $w \in R^\perp$ and we write $w = x + y$ ($x \in X$ and $y \in Y$), then both x and y belong to $R^\perp$. It follows that $R^\perp = (R^\perp \cap X) + (R^\perp \cap Y)$ and modulo R we get maximal isotropic subspaces

$$X_R = (R^\perp \cap X + R)/R \tag{8.7}$$

and similarly Y_R and Z_R which are mutually transversal. We define $L(X, Y, Z)$ as $L(X_R, Y_R, Z_R)$.

8.2. **Weil index.**

8.2.1. *A new class of functions.* Let F be a locally compact non-archimedean field and $\psi : F \to \mathbb{C}^1$ a non trivial additive character. Let X be a finite dimensional vector space over F and $X^* = Hom(X, F)$ its dual. We write x^*x for the pairing of $x \in X$ and $x^* \in X^*$ and

$$\langle x^*, x\rangle = \psi(x^*x). \tag{8.8}$$

A lattice in X is an open compact $\mathcal{O}_F$-submodule.

Let $\mathcal{S}(X)$ and $\mathcal{S}(X^*)$ be the spaces of Schwartz functions on X and X^* respectively, i.e. locally constant complex-valued functions of compact support.

Let dx and dx^* be dual Haar measures. By this we mean that if $f \in \mathcal{S}(X)$ and

$$g(x^*) = \mathcal{F}f(x^*) = \int_X \langle x^*, x\rangle\, f(x)dx \tag{8.9}$$

then Fourier inversion holds:

$$f(-x) = \mathcal{F}g(x) = \int_{X^*} \langle x^*, x\rangle\, g(x^*)dx^*. \tag{8.10}$$

For example, if ψ is normalized so that $\mathcal{O}_F$ is its own dual, and X_0 and X_0^* are dual lattices in X and X^* then we may normalize dx and dx^* by agreeing that X_0 and X_0^* have measure 1.

It is then well known that $\mathcal{F}$ is a unitary isomorphism of $\mathcal{S}(X)$ with $\mathcal{S}(X^*)$, so extends to an isomorphism of $L^2(X, dx)$ with $L^2(X^*, dx^*)$.

We want to introduce another class of functions[2] $\mathcal{W}(X)$ containing $\mathcal{S}(X)$ which is of interest, and to which $\mathcal{F}$ extends naturally. The advantage of this class is that it makes no use of the topology of $\mathbb{C}$. In fact, we may continue to assume (as we could clearly do up till now) that $\mathbb{C}$ is an arbitrary algebraicly closed field of characteristic 0. Another advantage of this class is that if Q is a quadratic form on X then $\phi_Q(x) = \psi(Q(x))$ belongs to $\mathcal{W}(X)$.

Definition 8.1. *A function $f : X \to \mathbb{C}$ belongs to $\mathcal{W}(X)$ if it is locally constant, and there exists a lattice $L_f \subset W$ with the property that for every lattice $L \subset L_f$ there exists a compact set $K \subset X$ such that for every $x \notin K$*

$$\int_L f(x+u)du = 0. \tag{8.11}$$

Definition 8.2. *A pair of lattices $L \subset K$ is called* good *for f if for every $x \notin K$ (8.11) holds.*

If (L, K) is good for f, so is (L, K') for every $K \subset K'$. Every function in $\mathcal{W}(X)$ has good pairs of lattices with arbitrarily small L. As a last piece of terminology, call a function $h : X \to \mathbb{C}$ *uniformly locally constant* if it is invariant under translation by some lattice.

Proposition 8.2. *(i) $\mathcal{S}(X) \subset \mathcal{W}(X)$.*

(ii) If Q is a quadratic form on X then $\phi_Q(x) = \psi(Q(x)) \in \mathcal{W}(X)$.

(iii) If $f \in \mathcal{W}(X)$ and h is uniformly locally constant, then $hf \in \mathcal{W}(X)$.

Proof. Points (i) and (iii) are clear. Point (ii) follows from the fact that if $(u, v) = Q(u+v) - Q(u) - Q(v)$

$$\phi_Q(x+u) = \phi_Q(x)\psi((x,u))\phi_Q(u). \tag{8.12}$$

[2] $\mathcal{W}$ for Weil

Let L be small enough so that $\phi_Q(u) \equiv 1$ for $u \in L$. Then for $x \notin L^\perp$, $\int_L \psi((x,u))du = 0$. ∎

Definition 8.3. *If $f \in \mathcal{W}(X)$ then*

$$\int_X f(x)dx \tag{8.13}$$

is defined by fixing a good pair of lattices $L \subset K$ and letting the integral equal $\int_K f(x)dx$.

This integral is well-defined. First note that replacing K by a larger lattice K' does not change the integral because $K' - K$ is a finite disjoint union of cosets of L. Next, if $L' \subset K'$ is another good pair of lattices, then by the first remark $\int_K = \int_{K+K'} = \int_{K'}$.

8.2.2. *The Fourier transform.* Let $f \in \mathcal{W}(X)$ and $x^* \in X^*$. Since $\langle x^*, x\rangle$ is a uniformly locally constant function of x, the integral

$$\mathcal{F}f(x^*) = \int_X \langle x^*, x\rangle\, f(x)dx \tag{8.14}$$

is defined. If $f \in \mathcal{S}(X)$ it agrees with the Fourier transform of f, so belongs to $\mathcal{S}(X^*)$.

Lemma 8.3. *The function $\mathcal{F}f(x^*)$ is locally constant.*

Proof. Let K^* be a lattice in X^* containing x^* and let $L \subset K$ be a good pair of lattices for f with $\langle K^*, L\rangle = 1$. Let $L^* \subset K^*$ be a lattice satisfying $\langle L^*, K\rangle = 1$. Let $u^* \in L^*$. Then, as a function of x, $\langle x^* + u^*, x\rangle$ is invariant under L, so $L \subset K$ is good for the function $\langle x^* + u^*, x\rangle\, f(x)$ and

$$\mathcal{F}f(x^* + u^*) = \int_K \langle x^* + u^*, x\rangle\, f(x) = \int_K \langle x^*, x\rangle\, f(x) = \mathcal{F}f(x^*) \tag{8.15}$$

since $\langle u^*, x\rangle = 1$ for $u^* \in L^*$ and $x \in K$. ∎

Lemma 8.4. *The function $\mathcal{F}f \in \mathcal{W}(X^*)$.*

Proof. Let L^* be a lattice in X^*. Let $(L^*)^\perp$ be the dual lattice in X and $L \subset (L^*)^\perp$ a lattice such that $f|_{(L^*)^\perp}$ is invariant under translation by $u \in L$. Let $L^* \subset K^* = L^\perp$. We claim that if $x^* \notin K^*$ then

$$\int_{L^*} \mathcal{F}f(x^* + u^*)du^* = 0. \tag{8.16}$$

Let $x^* \notin K^*$. As $\mathcal{F}f$ is locally constant and L^* is compact, there is a large enough lattice K in X such that

$$\mathcal{F}f(x^* + u^*) = \int_K \langle x^* + u^*, x\rangle\, f(x)dx \tag{8.17}$$

for all $u^* \in L^*$. Without loss of generality, $(L^*)^\perp \subset K$. Now

$$\begin{aligned} \int_{L^*} \mathcal{F}f(x^* + u^*)du^* &= \int_{L^*} \left(\int_K \langle x^* + u^*, x\rangle\, f(x)dx \right) du^* \\ &= \int_K \langle x^*, x\rangle\, f(x) \left(\int_{L^*} \langle u^*, x\rangle\, du^* \right) dx \\ &= c \int_{(L^*)^\perp} \langle x^*, x\rangle\, f(x)dx \end{aligned} \tag{8.18}$$

with c equal to the measure of L^*. But over $(L^*)^\perp$ the function $f(x)$ is invariant under L, while $\langle x^*, x\rangle$ is not L-invariant, since $x^* \notin K^* = L^\perp$, so the integral vanishes. ∎

Theorem 8.5. *The Fourier transform is an isomorphism of $\mathcal{W}(X)$ with $\mathcal{W}(X^*)$ and satisfies $\mathcal{F}\mathcal{F}f(x) = f(-x)$.*