p-ADIC PROPERTIES OF PICARD MODULAR SCHEMES AND MODULAR FORMS

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The p-adic theory of modular curves and modular forms, as developed in the early 70's by Serre, Katz, Mazur, Manin and many others, inspired over the last thirty years an extensive study of higher dimensional Shimura varieties, with spectacular applications to almost every area in number theory and representation theory.

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In the course of these vast generalizations, some of the more special phenomena pertaining to modular curves did not find appropriate analogues, and are only beginning to get addressed in recent years. In this work we study what we believe to be the simplest Shimura varieties beyond modular curves (and Shimura curves), namely Picard modular surfaces. For arithmetic questions over quadratic imaginary fields they play a role similar to the role played by modular curves over $\mathbb{Q}$. Our goal is to explore the arithmetic and geometry of these surfaces, and of modular forms over them, concentrating on aspects that have been only recently discovered, or appear to be new. Our hope is that staying within the bounds of $U(2,1)$ will keep the notation simple, and our tool-box relatively light, yet at the same time will allow for the exposition of new principles.

Our work relies on important work of Larsen, Bellaiche, Vollaard, Bültel, and Wedhorn. The modulo-p geometry of Picard modular surfaces, and, more generally, Shimura varieties asscoiated to unitary groups, has seen important contributions recently also by Goldring and Nicole, Koskivirta, Kudla, Rapoport, Terstiege, Howard and Pappas, and maybe others of which we are anaware. For completeness, we have included more than usual background material. We hope that we have not neglected to give appropriate credit when citing others' results.

We have chosen to work with a prime p which is inert in the underlying quadratic imaginary field, as this is the more interesting case, both from the algebro-geometric point of view, where the geometry modulo p is more challenging, and from the representation theoretic point of view, where the unitary group is non-split. We do not touch upon questions of automorphic L-functions or representation theory. For these see the comprehensive volume [L-R] and the references therein. We do not touch upon Galois representations either, although eventually we hope that our results will be relevant to both these areas.

This paper concerns the algebraic geometry of Picard modular schemes and modular forms modulo p. In a subsequent work we hope to treat p-adic modular forms in the style of Serre and Katz, the canonical subgroup, overconvergence, and further topics. We shall now explain the main results and the structure of the paper.

Chapters 1-2 summarize known results, and serve as a systematic introduction on which chapters 3-4 and future work will be based. In particular, sections 1.4 and 2.4 rely on the theses of Larsen [La1] and Bellaiche [Bel]. We recommend the latter for its very readable exposition. Sections 1.2.4, 1.6, 2.1, 2.5 and 2.6 contain results or computations for which we did not find adequate references in the literature. Chapter 2 introduces the two basic automorphic vector bundles $\mathcal{P}$ (a *plane* bundle) and $\mathcal{L}$ (a *line* bundle). Modular forms are sections of vector bundles from the tensor algebra generated by these two, but we contend ourselves with scalar valued modular forms, which are sections of $\mathcal{L}^k$ (k being the weight).

Chapter 3 contains new results. There are three strata to the Picard surface S modulo p: the μ-ordinary (open and dense) locus S_μ, the general supersingular (one dimensional) locus S_{gss}, and the superspecial points S_{ssp} (see [Bu-We],[V]). After recalling this, we find various relations between $\mathcal{P}$ and $\mathcal{L}$ that are peculiar to characteristic p. We study the Verschiebung homomorphism $V_{\mathcal{L}}$ from $\mathcal{L}$ to $\mathcal{P}^{(p)}$ and $V_{\mathcal{P}}$ from $\mathcal{P}$ to $\mathcal{L}^{(p)}$. We find that outside the superspecial points, both maps have rank one, but that $\mathrm{Im}(V_{\mathcal{L}}) = \ker(V_{\mathcal{P}}^{(p)})$ is the *defining equation* of the general supersingular stratum S_{gss}. We prove that the line bundle $\mathcal{P}_0 = \ker(V_{\mathcal{P}})$, defined over $S_\mu \cup S_{gss}$, does not extend across the superspecial points. Using these relations we define a Hasse invariant $h_{\bar{\Sigma}}$, a section of $\mathcal{L}^{p^2-1}$, whose zero divisor is precisely the supersingular locus $S_{ss} = S_{gss} \cup S_{ssp}$. Although the same definition has been already given by Goldring and Nicole in [Go-Ni] for all unitary Shimura varieties, and recently generailzed to all Shimura varieties of Hodge type by Koskivirta and Wedhorn [Ko-We], in the special case of Picard surfaces our analysis goes deeper.

Following the definition of $h_{\bar{\Sigma}}$ we define on S_{ss} a *secondary* Hasse invariant h_{ssp}, a section of $\mathcal{L}^{p^3+1}|_{S_{ss}}$, whose zeroes are the superspecial points S_{ssp} (albeit with high multiplicity). This secondary Hasse invariant is closely related to recent work of Boxer [Bo], and points at a general phenomenon that deserves further exploration. It seems reasonable that Boxer's method will generalize to Shimura varieties of PEL type and will produce invariants which agree with the partial Hasse invariants appearing in [Gor]. As an application of our analysis of the secondary Hasse invariant we derive a striking formula, expressing the number of irreducible components of the supersingular stratum in terms of the Euler number of $\bar{S}_{\mathbb{C}}$. By well-known results, due in this case to Holzapfel, this number is given essentially by an L-value.

As for modular curves, one can introduce an Igusa scheme $Ig(p^n)$ of any level p^n. We focus on $Ig = Ig(p)$, but note in passing that the higher $Ig(p^n)$ will become intsrumental in the theory of p-adic modular forms, deferred to future work. Although initially defined only over the μ-ordinary stratum, Ig can be compactified over the whole of S (and over the cuspidal divisors in $\bar{S}$ as well) by "extracting a $p^2 - 1$ root of $h_{\bar{\Sigma}}$". The line bundle $\mathcal{L}$ is trivialized over the μ-ordinary locus Ig_μ, and this allows us to regard modular forms as functions on Ig, with prescribed poles along Ig_{ss}, the supersingular locus. This is the geometric basis for developing an analogue of Serre's theory of "modular forms modulo p" in our context. In particular, we study their Fourier-Jacobi expansions (complex FJ expansions are q-expansions with theta functions as coefficients, but here we employ the arithmetic analysis of Bellaiche, explained in Section 2.4). This leads to the notion of a "filtration" of a modular form mod p, as in the case of elliptic modular forms.

Chapter 4 seems to be entirely new. The theory becomes richer once we bring in the Gauss-Manin connection and the Kodaira-Spencer isomorphism. We define a *theta operator* on modular forms mod p (in weight k) as follows. We first divide our modular form f by the k-th power of the canonical section $a(1)$ of $\mathcal{L}$ on Ig, to get a *function* $g = r(f)$ on Ig, with a pole of order k along Ig_{ss}. We apply the (inverse of the) Kodaira-Spencer isomorphism to dg to get a section of $\mathcal{P} \otimes \mathcal{L}$, which we map via $V_{\mathcal{P}} \otimes 1$ to get a meromorphic section of $\mathcal{L}^{p+1}$. Multiplying by $a(1)^k$ allows us to descend back to S, so we obtain a meromorphic modular form $\Theta(f)$ of weight $k + p + 1$. This construction is motivated by [An-Go], although substantial new phenomena appear in the present case. We study this operator. On the one hand, near the cusps we relate the *retraction* of a formal neighborhood of the cuspidal divisor, which was introduced by Bellaiche in [Bel], to the complex computations of section 1.6 and 2.6. This allows us to show that the effect of Θ on FJ expansions, is, as in the classical case, a "Tate twist". On the other hand, we study $\Theta(f)$ along the supersingular locus, and show that, thanks to the fact that we have divided out by $\mathcal{P}_0 = \ker V_{\mathcal{P}}$, $\Theta(f)$ is in fact everywhere holomorphic! We derive some interesting consequences for "theta cycles", where there are similarities, but also surprising deviations from the classical theory. We end the paper with a comparison of our theta operator with the Serre-Katz theta operator on modular curves, embedded in $\bar{S}$. Theta operators on classical modular forms have been instrumental in studying congruences between them, with applications to Galois representations. We expect the same to be true for Picard modular forms.

1. Background

1.1. **The unitary group and its symmetric space.**

1.1.1. *Notation.* Let $\mathcal{K}$ be an imaginary quadratic field, contained in $\mathbb{C}$. We denote by $\Sigma : \mathcal{K} \hookrightarrow \mathbb{C}$ the inclusion and by $\bar{\Sigma} : \mathcal{K} \hookrightarrow \mathbb{C}$ its complex conjugate. We use the following notation:

- $d_\mathcal{K}$ - the square free integer such that $\mathcal{K} = \mathbb{Q}(\sqrt{d_\mathcal{K}})$.
- $D_\mathcal{K}$ - the discriminant of $\mathcal{K}$, equal to $d_\mathcal{K}$ if $d_\mathcal{K} \equiv 1 \bmod 4$ and $4d_\mathcal{K}$ if $d_\mathcal{K} \equiv 2, 3 \bmod 4$.
- $\delta_\mathcal{K} = \sqrt{\mathcal{D}_\mathcal{K}}$ - the square root with positive imaginary part, a generator of the different of $\mathcal{K}$, sometimes simply denoted δ.
- $\omega_\mathcal{K} = (1 + \sqrt{d_\mathcal{K}})/2$ if $d_\mathcal{K} \equiv 1 \bmod 4$, otherwise $\omega_\mathcal{K} = \sqrt{d_\mathcal{K}}$, so that $\mathcal{O}_\mathcal{K} = \mathbb{Z} + \mathbb{Z}\omega_\mathcal{K}$.
- $\bar{a}$ - the complex conjugate of $a \in \mathcal{K}$.
- $\mathrm{Im}_\delta(a) = (a - \bar{a})/\delta$, for $a \in \mathcal{K}$.

We fix an integer $N \geq 3$ (the "tame level") and let $R_0 = \mathcal{O}_\mathcal{K}[1/2d_\mathcal{K}N]$. This is our *base ring*. If R is any R_0-algebra and M is any R-module with $\mathcal{O}_\mathcal{K}$-action, then M becomes an $\mathcal{O}_\mathcal{K} \otimes R$-module and we have a canonical *type decomposition*

$$M = M(\Sigma) \oplus M(\bar{\Sigma}) \tag{1.1}$$

where $M(\Sigma) = e_\Sigma M$ and $M(\bar{\Sigma}) = e_{\bar{\Sigma}} M$, and where the idempotents e_Σ and $e_{\bar{\Sigma}}$ are defined by

$$e_\Sigma = \frac{1 \otimes 1}{2} + \frac{\delta \otimes \delta^{-1}}{2}, \quad e_{\bar{\Sigma}} = \frac{1 \otimes 1}{2} - \frac{\delta \otimes \delta^{-1}}{2}. \tag{1.2}$$

Then $M(\Sigma)$ (resp. $M(\bar{\Sigma})$) is the part of M on which $\mathcal{O}_\mathcal{K}$ acts via Σ (resp. $\bar{\Sigma}$). The same notation will be used for sheaves of modules on R-schemes, endowed with an $\mathcal{O}_\mathcal{K}$ action. If M is locally free, we say that it has *type* (p, q) if $M(\Sigma)$ is of rank p and $M(\bar{\Sigma})$ is of rank q.

We denote by

$$\mathbf{T} = res_\mathbb{Q}^\mathcal{K} \mathbb{G}_m \tag{1.3}$$

the non-split torus whose $\mathbb{Q}$-points are $\mathcal{K}^\times$, and by ρ the non-trivial automorphism of $\mathbf{T}$, which on $\mathbb{Q}$-points induces $\rho(a) = \bar{a}$. The group $\mathbb{G}_m$ embeds in $\mathbf{T}$ and the homomorphism $a \mapsto a \cdot \rho(a)$ from $\mathbf{T}$ to itself factors through a homomorphism

$$N : \mathbf{T} \to \mathbb{G}_m, \tag{1.4}$$

the *norm homomorphism*. Its kernel $\ker(N)$ is denoted $\mathbf{T}^1$.

1.1.2. *The unitary group.* Let $V = \mathcal{K}^3$ and endow it with the hermitian pairing

$$(u, v) = {}^t\bar{u} \begin{pmatrix} & & \delta^{-1} \\ & 1 & \\ -\delta^{-1} & & \end{pmatrix} v. \tag{1.5}$$

We identify $V_{\mathbb{R}}$ with $\mathbb{C}^3$ ($\mathcal{K}$ acting via the natural inclusion Σ). It then becomes a hermitian space of signature $(2,1)$. Conversely, any 3-dimensional hermitian space over $\mathcal{K}$ whose signature at the infinite place is $(2,1)$ is isomorphic to V after rescaling the hermitian form by a positive rational number.

Let

$$\mathbf{G} = \mathbf{GU}(V, (,)) \tag{1.6}$$

be the general unitary group of V, regarded as an algebraic group over $\mathbb{Q}$. For any $\mathbb{Q}$-algebra A we have

$$\mathbf{G}(A) = \left\{(g,\mu) \in GL_3(A \otimes \mathcal{K}) \otimes A^\times |\ (gu, gv) = \mu \cdot (u,v)\ \forall u, v \in V_A\right\}. \tag{1.7}$$

We write $G = \mathbf{G}(\mathbb{Q})$, $G_\infty = \mathbf{G}(\mathbb{R})$ and $G_p = \mathbf{G}(\mathbb{Q}_p)$. A similar notational convention will apply to any algebraic group over $\mathbb{Q}$ without further ado. If p splits in $\mathcal{K}$, $\mathbb{Q}_p \otimes \mathcal{K} \simeq \mathbb{Q}_p^2$ and G_p becomes isomorphic to $GL_3(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$. The isomorphism depends on the embedding of $\mathcal{K}$ in $\mathbb{Q}_p$, i.e. on the choice of a prime above p in $\mathcal{K}$. For a non-split prime p the group G_p, like G_∞, is of (semisimple) rank 1.

As μ is determined by g we often abuse notation and write g for the pair (g,μ) and $\mu(g)$ for the *multiplier* μ. It is a character of algebraic groups over $\mathbb{Q}$, $\mu : \mathbf{G} \to \mathbb{G}_m$. Another character is $\det : \mathbf{G} \to \mathbf{T}$, defined by $\det(g,\mu) = \det(g)$. If we let

$$\nu = \mu^{-1} \det : \mathbf{G} \to \mathbf{T} \tag{1.8}$$

then both μ and det are expressible in terms of ν, namely $\mu = \nu \cdot (\rho \circ \nu)$ and $\det = \nu^2 \cdot (\rho \circ \nu)$. The first relation is a consequence of the relation $\det \cdot (\rho \circ \det) = \mu^3$, and the second is a consequence of the first and the definition of ν.

The groups

$$\mathbf{U} = \ker \mu, \quad \mathbf{SU} = \ker \nu = \ker \mu \cap \ker(\det) \tag{1.9}$$

are the unitary and the special unitary group, respectively.

We also introduce an alternating $\mathbb{Q}$-linear pairing $\langle , \rangle : V \times V \to \mathbb{Q}$ defined by $\langle u, v \rangle = \mathrm{Im}_\delta(u,v)$. We then have the formulae

$$\langle au, v \rangle = \langle u, \bar{a}v \rangle, \quad 2(u,v) = \langle u, \delta v \rangle + \delta \langle u, v \rangle. \tag{1.10}$$

We call $\langle , \rangle$ the *polarization form*, for a reason that will become clear soon.

1.1.3. *The hermitian symmetric domain.* The group $G_\infty = \mathbf{G}(\mathbb{R})$ acts on $\mathbb{P}^2_{\mathbb{C}} = \mathbb{P}(V_{\mathbb{R}})$ by projective linear transformations and preserves the open subdomain $\mathfrak{X}$ of negative definite lines (in the metric $(,)$). If we switch to coordinates in which the hermitian quadratic form (u,u) assumes the standard shape $x\bar{x} + y\bar{y} - z\bar{z}$, it becomes evident that $\mathfrak{X}$ is biholomorphic to the open unit ball in $\mathbb{C}^2$, hence is connected, and G_∞ acts on it transitively. We shall nevertheless stick to the coordinates introduced above. Every negative definite line is represented by a unique vector ${}^t(z,u,1)$ and such a vector represents a negative definite line if and only if

$$\lambda(z,u) \stackrel{\text{def}}{=} \mathrm{Im}_{\ \delta}(z) - u\bar{u} > 0. \tag{1.11}$$

One refers to the realization of $\mathfrak{X}$ as the set of points $(z,u) \in \mathbb{C}^2$ satisfying this inequality as a *Siegel domain of the second kind*. It is convenient to think of the point $x_0 = (\delta/2, 0)$ as the "center" of $\mathfrak{X}$.

If we let K_∞ be the stabilizer of x_0 in G_∞, then K_∞ is compact modulo center ($K_\infty \cap \mathbf{U}(\mathbb{R})$ is compact and isomorphic to $U(2) \times U(1)$). Since G_∞ acts transitively on $\mathfrak{X}$, we may identify $\mathfrak{X}$ with G_∞/K_∞.

The space $\mathfrak{X}$ carries a Riemannian metric which is invariant under the action of G_∞, the Bergmann metric. It may be described as follows. Switching once again to coordinates on $\mathbb{C}^3$ in which the hermitian quadratic form (u,u) becomes $x\bar{x}+y\bar{y}-z\bar{z}$, the real symmetric bilinear form $\mathrm{Re}(u,v)$ has signature $(4,2)$, and the manifold given by the equation $x\bar{x}+y\bar{y}-z\bar{z}=-1$ becomes a circle bundle over $\mathfrak{X}$. The restriction of $\mathrm{Re}(u,v)$ to the tangent bundle of this manifold has signature $(4,1)$ and when we take the quotient by the circle action we get an invariant Riemannian metric on $\mathfrak{X}$.

The usual upper half plane embeds in $\mathfrak{X}$ (holomorphically and isometrically) as the set of points where $u=0$.

1.1.4. *The cusps of $\mathfrak{X}$.* The boundary $\partial\mathfrak{X}$ of $\mathfrak{X}$ is the set of points (z,u) where $\mathrm{Im}_\delta(z)=u\bar{u}$, together with a unique point "at infinity" c_∞ represented by the line ${}^t(1:0:0)$. The lines represented by $\partial\mathfrak{X}$ are the isotropic lines in $V_\mathbb{R}$. The set of *cusps* $\mathcal{C}\mathfrak{X}$ is the set of $\mathcal{K}$-rational isotropic lines, or, equivalently, the set of points on $\partial\mathfrak{X}$ with coordinates in $\mathcal{K}$. If $s\in\mathcal{K}$ and $r\in\mathbb{Q}$ we write

$$c_s^r=(r+\delta s\bar{s}/2,s). \tag{1.12}$$

Then $\mathcal{C}\mathfrak{X}=\{c_s^r|r\in\mathbb{Q},s\in\mathcal{K}\}\cup\{c_\infty\}$. The group $G=\mathbf{G}(\mathbb{Q})$ acts transitively on the cusps.

The stabilizer of a cusp is a Borel subgroup[1] in G_∞. Since G acts transitively on the cusps, we may assume that our cusp is c_∞. It is then easy to check that its stabilizer P_∞ has the form $P_\infty=M_\infty N_\infty$, where

$$M_\infty=\left\{tm(\alpha,\beta)=t\begin{pmatrix}\alpha&&\\&\beta&\\&&\bar{\alpha}^{-1}\end{pmatrix}|\,t\in\mathbb{R}_+^\times,\,\alpha\in\mathbb{C}^\times,\beta\in\mathbb{C}^1\right\}, \tag{1.13}$$

$$N_\infty=\left\{n(u,r)=\begin{pmatrix}1&\delta\bar{u}&r+\delta u\bar{u}/2\\&1&u\\&&1\end{pmatrix}|\,u\in\mathbb{C},r\in\mathbb{R}\right\}. \tag{1.14}$$

The matrix $tm(\alpha,\beta)$ belongs to U_∞ if and only if $t=1$, and to SU_∞ if furthermore $\beta=\bar{\alpha}/\alpha$. The group N_∞ is contained in SU_∞. The group $P=P_\infty\cap G$ consists of the same matrices with $\mathcal{K}$-rational entries. Since $N=N_\infty\cap G$ still acts transitively on the set of finite cusps c_s^r, we conclude that G acts *doubly transitively* on $\mathcal{C}\mathfrak{X}$.

Of particular interest to us will be the *geodesics* connecting an interior point (z,u) to a cusp $c\in\mathcal{C}\mathfrak{X}$. If $(z,u)=n(u,r)m(d,1)x_0$ (recall $x_0={}^t(\delta/2:0:1)$) where d is real and positive (i.e. $r=\mathrm{Re}\,z$ and $d=\sqrt{\lambda(z,u)}$) then the geodesic connecting (z,u) to c_∞ can be described by the formula

$$\begin{aligned}\gamma_u^r(t)&=n(u,r)m(t,1)x_0\\&=(r+\delta(u\bar{u}+t^2)/2,u)\quad(d\le t<\infty).\end{aligned} \tag{1.15}$$

The same geodesic extends in the opposite direction for $0<t\le d$, and if u and r lie in $\mathcal{K}$, it ends there in the cusp c_u^r. We shall call $\gamma_u^r(t)$ the *geodesic retraction* of $\mathfrak{X}$ to the cusp c_∞. As $0<t<\infty$ these parallel geodesics exhaust $\mathfrak{X}$, they converge to c_∞ as $t\to\infty$, and they pass through (z,u) precisely when $r=\mathrm{Re}\,z$ and $t=\sqrt{\lambda(z,u)}$. The points (z,u) and (z',u') lie on the same geodesic if and only if $u=u'$ and $\mathrm{Re}(z)=\mathrm{Re}(z')$.

[1] Note that any proper $\mathbb{R}$-parabolic subgroup of G_∞ is conjugate to P_∞, as $SU(2,1)$ has $\mathbb{R}$-rank 1.

1.2. **Picard modular surfaces over $\mathbb{C}$.**

1.2.1. *Lattices and their arithmetic groups.* Fix an $\mathcal{O}_\mathcal{K}$-invariant lattice $L \subset V$ which is *self-dual* in the sense that

$$L = \{u \in V | \langle u, v\rangle \in \mathbb{Z} \ \forall v \in L\}. \tag{1.16}$$

Equivalently, L is its own $\mathcal{O}_\mathcal{K}$-dual with respect to the hermitian pairing $(,)$. We assume also that the Steinitz class[2] of L as an $\mathcal{O}_\mathcal{K}$-module is $[\mathcal{O}_\mathcal{K}]$, or, what amounts to the same, that L is a free $\mathcal{O}_\mathcal{K}$-module. When we introduce the Shimura variety later on, we shall relax this last assumption, but the resulting scheme will be disconnected (over $\mathbb{C}$).

Fix an integer $N \geq 1$ and let

$$\Gamma = \{g \in G| \, gL = L \,\text{and}\; g(u) \equiv u \bmod NL \ \forall u \in L\}. \tag{1.17}$$

This Γ is a discrete subgroup of G_∞, contained in U_∞. It is easy to see that if $N \geq 3$ then Γ is torsion free, acts freely and faithfully on $\mathfrak{X}$, and is contained in SU_∞. From now on we assume that this is the case.

If $g \in G$ and $\mu(g) = 1$ (i.e. $g \in U$) the lattice gL is another lattice of the same sort and the discrete group corresponding to it is $g\Gamma g^{-1}$. Since U acts transitively on the cusps, this reduces the study of $\Gamma\backslash\mathfrak{X}$ near a cusp to the study of a neighborhood of the standard cusp c_∞ (at the price of changing L and Γ).

It is important to know the classification of lattices L as above (self-dual and $\mathcal{O}_\mathcal{K}$-free). Let e_1, e_2, e_3 be the standard basis of $\mathcal{K}^3$. Let

$$L_0 = Span_{\mathcal{O}_\mathcal{K}}\{\delta e_1, e_2, e_3\} \tag{1.18}$$

and

$$L_1 = Span_{\mathcal{O}_\mathcal{K}}\{\frac{\delta}{2}e_1 + e_3, e_2, \frac{\delta}{2}e_1 - e_3\}. \tag{1.19}$$

These two lattices are self-dual and of course, $\mathcal{O}_\mathcal{K}$-free. The following theorem is based on the local-global principle and a classification of lattices over $\mathbb{Q}_p$ by Shimura [Sh1].

Lemma 1.1. *([La1], p.25). For any lattice L as above there exists a $g \in U$ such that $gL = L_0$ or $gL = L_1$. If $D_\mathcal{K}$ is odd, L_0 and L_1 are equivalent. If $D_\mathcal{K}$ is even, they are inequivalent.*

Indeed, if $D_\mathcal{K}$ is even, $L_0 \otimes \mathbb{Q}_p$ and $L_1 \otimes \mathbb{Q}_p$ are U_p-equivalent for every $p \neq 2$, but not for $p = 2$.

1.2.2. *Picard modular surfaces and the Baily-Borel compactification.* We denote by X_Γ the complex surface $\Gamma\backslash\mathfrak{X}$. Since the action of Γ is free, X_Γ is smooth. We describe a topological compactification of X_Γ. A *standard neighborhood* of the cusp c_∞ is an open set of the form

$$\Omega_R = \{(z, u)|\lambda(z, u) > R\}. \tag{1.20}$$

The set $\mathcal{C}_\Gamma = \Gamma\backslash\mathcal{C}\mathfrak{X}$ is finite, and we write $c_\Gamma = \Gamma c$. We let X_Γ^* be the disjoint union of X_Γ and $\mathcal{C}_\Gamma$. We topologize it by taking $\Gamma\backslash\Omega_R \cup \{c_{\infty,\Gamma}\}$ as a basis of neighborhoods at $c_{\infty,\Gamma}$. If $c = g(c_\infty)$ where $g \in U$, we take $g(g^{-1}\Gamma g\backslash\Omega_R) \cup \{c_\Gamma\}$ instead. The following theorem is well-known.

[2] The Steinitz class of a finite projective $\mathcal{O}_\mathcal{K}$-module is the class of its top exterior power as an invertible module.

Theorem 1.2. *(Satake, Baily-Borel) X_Γ^* is projective and the singularities at the cusps are normal. In other words, there exists a normal complex projective surface S_Γ^* and a homeomorphism $\iota : S_\Gamma^*(\mathbb{C}) \simeq X_\Gamma^*$, which on $S_\Gamma(\mathbb{C}) = \iota^{-1}(X_\Gamma)$ is an isomorphism of complex manifolds. S_Γ^* is uniquely determined up to isomorphism.*

1.2.3. *The universal abelian variety over X_Γ.* With $x \in \mathfrak{X}$ and with our choice of L we shall now associate a PEL-structure $\underline{A}_x = (A_x, \lambda_x, \iota_x, \alpha_x)$ where

1. A_x is a 3-dimensional complex abelian variety,
2. λ_x is a principal polarization on A_x (i.e. an isomorphism $A_x \simeq A_x^t$ with its dual abelian variety induced by an ample line bundle),
3. $\iota_x : \mathcal{O}_\mathcal{K} \hookrightarrow End(A_x)$ is an embedding of CM type $(2,1)$ (i.e. the action of $\iota(a)$ on the tangent space of A_x at the origin induces the representation $2\Sigma + \bar{\Sigma}$) such that the Rosati involution induced by λ_x preserves $\iota(\mathcal{O}_\mathcal{K})$ and is given by $\iota(a) \mapsto \iota(\bar{a})$,
4. $\alpha_x : N^{-1}L/L \simeq A_x[N]$ is a full level N structure, compatible with the $\mathcal{O}_\mathcal{K}$-action and the polarization. The latter condition means that if we denote by $\langle , \rangle_\lambda$ the Weil "e_N-pairing" on $A_x[N]$ induced by λ_x, then for $l, l' \in N^{-1}L$

$$\langle \alpha_x(l), \alpha_x(l') \rangle_\lambda = e^{2\pi i N \langle l, l' \rangle}. \tag{1.21}$$

Let W_x be the negative definite complex line in $V_\mathbb{R} = \mathbb{C}^3$ defined by x, and $W_x^\perp$ its orthogonal complement, a positive definite plane. Let J_x be the complex structure which is multiplication by i on $W_x^\perp$ and by $-i$ on W_x. Let $A_x = (V_\mathbb{R}, J_x)/L$. Then the polarization form $\langle , \rangle$ is a Riemann form on L. One only has to verify that $\langle u, J_x v \rangle + i \langle u, v \rangle$ is a *positive definite* hermitian form. But up to the factor $|\delta|/2$ this is the same as (u, v) on $W_x^\perp$ and $-\overline{(u, v)}$ on W_x. The Riemann form determines a principal polarization on A_x as usual. The action of $\mathcal{O}_\mathcal{K}$ is derived from the underlying $\mathcal{K}$ structure of V. As we have changed the complex structure on W_x, the CM type is now (2,1). Finally the level N structure α_x is the identity map.

If $\gamma \in \Gamma$ then γ induces an isomorphism between $\underline{A}_x$ and $\underline{A}_{\gamma(x)}$. Conversely, if $\underline{A}_x$ and $\underline{A}_{x'}$ are isomorphic structures, it is easy to see that x' and x must belong to the same Γ-orbit. It follows that points of X_Γ are in a bijection with PEL structures of the above type for which the triple

$$(H_1(A_x, \mathbb{Z}), \iota_x, \langle , \rangle_{\lambda_x}) \tag{1.22}$$

is isomorphic to $(L, \iota, \langle , \rangle)$ (here ι refers to the $\mathcal{O}_\mathcal{K}$ action on L), with the further condition that α_x is compatible with the isomorphism between L and $H_1(A_x, \mathbb{Z})$ in the sense that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \to & N^{-1}L & \to & N^{-1}L/L & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \alpha_x & & \\ 0 & \to & H_1(A_x, \mathbb{Z}) & \to & N^{-1}H_1(A_x, \mathbb{Z}) & \to & A_x[N] & \to & 0 \end{array}. \tag{1.23}$$

1.2.4. *A "moving lattice" model for the universal abelian variety.* We want to assemble the individual A_x into an *abelian variety A over $\mathfrak{X}$*. In other words, we want to construct a 5-dimensional complex manifold A, together with a holomorphic map $A \to \mathfrak{X}$ whose fiber over x is identified with A_x. For that, as well as for the computation of the Gauss-Manin connection below, it is convenient to introduce another model, in which the complex structure on $\mathbb{C}^3$ is fixed, but the lattice varies.

For simplicity we assume from now on that $L = L_0$ is spanned over $\mathcal{O}_\mathcal{K}$ by $\delta e_1, e_2$ and e_3. The case of L_1 can be handled similarly.

Let $\mathbb{C}^3$ be given the usual complex structure, and let $a \in \mathcal{O}_\mathcal{K}$ act on it via the matrix

$$\iota'(a) = \begin{pmatrix} a & & \\ & a & \\ & & \bar{a} \end{pmatrix}. \tag{1.24}$$

Given $x = (z, u) \in \mathfrak{X}$ consider the lattice

$$L'_x = Span_{\iota'(\mathcal{O}_\mathcal{K})} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -u \end{pmatrix}, \begin{pmatrix} u \\ -z/\delta \\ z/\delta \end{pmatrix} \right\} \subset \mathbb{C}^3. \tag{1.25}$$

The map $T_x : \mathbb{C}^3 \to \mathbb{C}^3$ which sends $\zeta = {}^t(\zeta_1, \zeta_2, \zeta_3)$ to

$$T(\zeta) = \lambda(z,u)^{-1} \left\{ -\zeta_1 \begin{pmatrix} \bar{u}z \\ (z - \bar{z})/\delta \\ \bar{u} \end{pmatrix} - \zeta_2 \begin{pmatrix} \bar{z} + \delta u \bar{u} \\ u \\ 1 \end{pmatrix} + \bar{\zeta}_3 \begin{pmatrix} z \\ u \\ 1 \end{pmatrix} \right\} \tag{1.26}$$

is a complex linear isomorphism between $\mathbb{C}^3$ and $(V_\mathbb{R}, J_x)$. In fact, it sends $\mathbb{C}e_1 + \mathbb{C}e_2$ linearly to $W_x^\perp$ and $\mathbb{C}e_3$ conjugate-linearly to W_x. It intertwines the ι' action of $\mathcal{O}_\mathcal{K}$ on $\mathbb{C}^3$ with its ι-action on $(V_\mathbb{R}, J_x)$. It furthermore sends L'_x to L. In fact, an easy computation shows that it sends the three generating vectors of L'_x to $\delta e_1, e_2$ and e_3, respectively. We conclude that T_x induces an isomorphism

$$T_x : A'_x = \mathbb{C}^3 / L'_x \simeq A_x. \tag{1.27}$$

Consider the differential forms $d\zeta_1, d\zeta_2$ and $d\zeta_3$. As their periods along any $l \in L'_x$ vary holomorphically in z and u, the five coordinates $\zeta_1, \zeta_2, \zeta_3, z, u$ form a local system of coordinates on the family $A' \to \mathfrak{X}$. Identifying A' with A allows us to put the desired complex structure on the family A. Alternatively, we may define A' as the quotient of $\mathbb{C}^3 \times \mathfrak{X}$ by $\zeta \mapsto \zeta + l(z,u)$ where $l(z,u)$ varies over the holomorphic lattice-sections.

The model A' has another advantage, that will become clear when we examine the degeneration of the universal abelian variety at the cusp c_∞. It suffices to note at this point that the first two of the three generating vectors of L'_x depend only on u.

1.3. **The Picard moduli scheme.**

1.3.1. *The moduli problem.* Fix $N \geq 3$ as before. Fix the lattice $L = L_0 \subset V = \mathcal{K}^3$. Let R be an R_0-algebra (recall $R_0 = \mathcal{O}_\mathcal{K}[1/2d_\mathcal{K}N]$). Let $\mathcal{M}(R)$ be the collection of (isomorphism classes of) PEL structures $(A, \lambda, \iota, \alpha)$ where

1. A/R is an abelian scheme of relative dimension 3
2. $\lambda : A \simeq A^t$ is a principal polarization
3. $\iota : \mathcal{O}_\mathcal{K} \to End(A/R)$ is a homomorphism such that (1) ι makes $Lie(A/R)$ a locally free R-module of type $(2,1)$, (2) the Rosati involution induced on $\iota(\mathcal{O}_\mathcal{K})$ by λ is $\iota(a) \mapsto \iota(\bar{a})$.
4. $\alpha : N^{-1}L/L \simeq A[N]$ is an isomorphism of $\mathcal{O}_\mathcal{K}$-group schemes over R which is compatible with the polarization in the sense that there exists an isomorphism $\nu_N : \mathbb{Z}/N\mathbb{Z} \simeq \mu_N$ of group schemes over R such that

$$\left\langle \alpha(\frac{l}{N}), \alpha(\frac{l'}{N}) \right\rangle_\lambda = \nu_N(\langle l, l' \rangle \bmod N). \tag{1.28}$$

In addition we require that for every multiple N' of N, locally étale over $Spec(R)$, there exists a similar level N'-structure α', restricting to α on $N^{-1}L/L$. One says that α is *locally étale symplectic liftable* ([Lan], 1.3.6.2).

In view of Lemma 1.1, the last condition of symplectic liftability is void if $D_{\mathcal{K}}$ is odd, while if $D_{\mathcal{K}}$ is even it is equivalent to the following condition ([Bel], I.3.1):

- For any geometric point $\eta : R \to k$ (k algebraically closed field, necessarily of characteristic different from 2), the $\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_2$ polarized module $(T_2 A_\eta, \langle , \rangle_\lambda)$ is isomorphic to $(L \otimes \mathbb{Z}_2, \langle , \rangle)$ under a suitable identification of $\lim_{\leftarrow} \mu_{2^n}(k)$ with $\mathbb{Z}_2$.

The choice of L_0 was arbitrary. If we took L_1 as our basic lattice we would get a similar moduli problem.

A level N structure α can exist only if the group schemes $\mathbb{Z}/N\mathbb{Z}$ and μ_N become isomorphic over R, but the isomorphism ν_N is then determined by α.

$\mathcal{M}$ becomes a functor on the category of R_0-algebras (and more generally, on the category of R_0-schemes) in the obvious way. The following theorem is of fundamental importance ([Lan], I.4.1.11).

Theorem 1.3. *The functor $R \mapsto \mathcal{M}(R)$ is represented by a smooth quasi-projective scheme S over $Spec(R_0)$, of relative dimension 2.*

We call S the *(open) Picard modular surface of level N*. It comes equipped with a universal structure $(\mathcal{A}, \lambda, \iota, \alpha)$ of the above type over S. We call $\mathcal{A}$ the *universal abelian scheme* over S. For every R_0-algebra R and PEL structure in $\mathcal{M}(R)$, there exists a unique R-point of S such that the given PEL structure is obtained from the universal one by base-change.

1.3.2. *The Shimura variety Sh_K.* We briefly recall the interpretation of the Picard modular surface as a canonical model of a Shimura variety. The symmetric domain $\mathfrak{X}$ can be interpreted as a G_∞-conjugacy class of homomorphisms

$$h : \mathbb{S} = res_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_m \to \mathbf{G} \tag{1.29}$$

turning $(\mathbf{G}, \mathfrak{X})$ into a Shimura datum in the sense of Deligne [De]. The reflex field associated to this datum turns out to be $\mathcal{K}$. Let K_∞ be the stabilizer of x_0 in G_∞ and $K_f^0 \subset \mathbf{G}(\mathbb{A}_f)$ the subgroup stabilizing $\widehat{L} = L \otimes \widehat{\mathbb{Z}}$. Let K_f be the subgroup of K_f^0 inducing the identity on L/NL. Let $K = K_\infty K_f \subset \mathbf{G}(\mathbb{A})$. Then the Shimura variety Sh_K is a complex quasi-projective variety whose complex points are isomorphic, as a complex manifold, to the double coset space

$$\begin{aligned} Sh_K(\mathbb{C}) &\simeq \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K \\ &= \mathbf{G}(\mathbb{Q}) \backslash (\mathfrak{X} \times \mathbf{G}(\mathbb{A}_f) / K_f). \end{aligned} \tag{1.30}$$

The theory of Shimura varieties provides a *canonical model* for Sh_K over $\mathcal{K}$. The following important theorem complements the one on the representability of the functor $\mathcal{M}$.

Theorem 1.4. *The canonical model of Sh_K is the generic fiber $S_{\mathcal{K}}$ of S.*

Let us explain only how to associate to a point of $Sh_K(\mathbb{C})$ a point in $S(\mathbb{C})$. For that we have to associate an element of $\mathcal{M}(\mathbb{C})$ to $g \in \mathbf{G}(\mathbb{A})$, and show that the structures associated to g and to $\gamma g k$ ($\gamma \in G, k \in K$) are isomorphic. Let

$x = x_g = g_\infty(x_0) \in \mathfrak{X}$. Let $L_g = g_f(\widehat{L}) \cap V$ (the intersection taking place in $V_\mathbb{A} = \widehat{L} \otimes \mathbb{Q}$) and

$$A_g = (V_\mathbb{R}, J_x)/L_g. \tag{1.31}$$

Note that J_x depends only on $g_\infty K_\infty$ and L_g only on $g_f K_f^0$, so A_g depends only on gK^0.

Let $\tilde{\mu}(g)$ be the unique positive rational number such that for every prime p,

$$ord_p \tilde{\mu}(g) = ord_p \mu(g_p). \tag{1.32}$$

Such a rational number exists since $\mu(g_p)$ is a p-adic unit for almost all p and $\mathbb{Q}$ has class number 1. We claim that

$$\langle , \rangle_g = \tilde{\mu}(g)^{-1} \langle , \rangle : L_g \times L_g \to \mathbb{Q}$$

induces a principal polarization λ_g on A_g. That this is a (rational) Riemann form follows from the fact that $(u, v)_{J_x} = \langle u, J_x v \rangle + i \langle u, v \rangle$ is hermitian positive definite. That $\langle , \rangle_g$ is indeed $\mathbb{Z}$-valued and L_g is self-dual follows from the choice of $\tilde{\mu}(g)$ since locally at p the dual of $g_p L_p$ under $\langle , \rangle : V_p \times V_p \to \mathbb{Q}_p$ is $\mu(g_p)^{-1} g_p L_p$. We conclude that there exists a unique polarization $\lambda_g : A_g \to A_g^t$ such that

$$\langle u, v \rangle_{\lambda_g} = \exp(2\pi i l \langle u, v \rangle_g) \tag{1.33}$$

for every $u, v \in A_g[l] = l^{-1} L_g / L_g$ and every $l \geq 1$. This polarization is principal.

Since g_f commutes with the $\mathcal{K}$-structure on $V_\mathbb{A}$, L_g is still an $\mathcal{O}_\mathcal{K}$-lattice, hence ι_g is defined.

Finally α_g is derived from

$$N^{-1}L/L = N^{-1}\widehat{L}/\widehat{L} \xrightarrow{g_f} N^{-1}\widehat{L}_g/\widehat{L}_g = N^{-1}L_g/L_g = A_g[N]. \tag{1.34}$$

We note that α_g depends only on gK because $K_f \subset K_f^0$ is the principal level-N subgroup, and that it lifts to level N' structure for any multiple N' of N, by the same formula. The isomorphism $\nu_{N,g}$ between $\mathbb{Z}/N\mathbb{Z}$ and $\mu_N(\mathbb{C})$ that makes (1.28) work is self-evident (see (1.49)). Let $\underline{A}_g \in \mathcal{M}(\mathbb{C})$ be the structure just constructed.

Let now $\gamma \in \mathbf{G}(\mathbb{Q})$. Then the action of γ on V induces an isomorphism between the tuples $\underline{A}_g$ and $\underline{A}_{\gamma g}$. Indeed, $\gamma : V_\mathbb{R} \to V_\mathbb{R}$ intertwines the complex structures x_g and $x_{\gamma g}$, and carries L_g to $L_{\gamma g}$, so induces an isomorphism of the abelian varieties, which clearly commutes with the PEL structures.

This shows that $\underline{A}_g$ depends solely on the double coset of g in $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K$. One is left now with two tasks which we do not do in this survey: (i) Proving that if $\underline{A}_g \simeq \underline{A}_{g'}$ then g and g' belong to the same double coset, and that every $\underline{A} \in \mathcal{M}(\mathbb{C})$ is obtained in this way, (ii) Identifying the canonical model of Sh_K over $\mathcal{K}$ with $S_\mathcal{K}$.

1.3.3. *The connected components of Sh_K.* Recall that $\mathbf{G}' = \mathbf{SU} = \ker(\nu : \mathbf{G} \to \mathbf{T})$. Since $\mathbf{G}'$ is simple and simply connected, strong approximation holds and

$$\mathbf{G}'(\mathbb{A}) = \mathbf{G}'(\mathbb{Q}) G'_\infty K'_f. \tag{1.35}$$

Here $K' = K \cap \mathbf{G}'(\mathbb{A})$, $K'_f = K \cap \mathbf{G}'(\mathbb{A}_f)$. From the connectedness of G'_∞ we deduce that

$$\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A})/K' \tag{1.36}$$

is connected.

As $N \geq 3$, $\nu(K) \cap \mathcal{K}^\times = \{1\}$. Here $\mathcal{K}^\times = \nu(\mathbf{G}(\mathbb{Q}))$, and it follows that

$$\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A})/K' \hookrightarrow \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K \tag{1.37}$$

is injective. We now claim (see also Theorem 2.4 and 2.5 of [De]) that

$$\nu : \pi_0(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K) \simeq \pi_0(\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A})/\nu(K)) \tag{1.38}$$

is a bijection. For ν is surjective ([De] (0.2)) and continuous (on double coset spaces) so clearly induces a surjective map between the sets of connected components. On the other hand if $[g_1]$ and $[g_2]$ (double cosets of $g_i \in \mathbf{G}(\mathbb{A})$) are mapped by ν to the same connected component in $\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A})/\nu(K)$, then since G_∞ is mapped *onto* the connected component of the identity in $\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A})/\nu(K)$, modifying g_1 by an element of G_∞ we may assume that

$$\nu([g_1]) = \nu([g_2]) \in \mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A})/\nu(K), \tag{1.39}$$

without changing the connected component in which $[g_1]$ lies. Once this has been established, for appropriate representatives g_i of the double cosets, $g_1^{-1}g_2 \in \mathbf{G}'(\mathbb{A})$, so by the connectedness of $\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A})/K'$, $[g_1]$ and $[g_2]$ lie in the same connected component of $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K$.

The group $\pi_0(\mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A})/\nu(K))$ is the group

$$\mathcal{K}^\times\backslash\mathcal{K}^\times_{\mathbb{A}}/\mathbb{C}^\times\nu(K_f) = \mathcal{K}^\times\backslash\mathcal{K}^\times_f/\nu(K_f). \tag{1.40}$$

It sits in a short exact sequence

$$0 \to \mu_{\mathcal{K}}\backslash U_{\mathcal{K}}/\nu(K_f) \to \mathcal{K}^\times\backslash\mathcal{K}^\times_f/\nu(K_f) \xrightarrow{cl} Cl_{\mathcal{K}} \to 0, \tag{1.41}$$

where $U_{\mathcal{K}}$ is the product of local units at all the finite primes and $Cl_{\mathcal{K}}$ is the class group.

1.3.4. *The cl and ν_N invariants of a connected component.* The norm $N : \mathcal{K}^\times \to \mathbb{Q}^\times$ satisfies $N \circ \nu = \nu\nu^\rho = \mu$, hence induces a map

$$\mathcal{K}^\times\backslash\mathcal{K}^\times_f/\nu(K_f) \to \mathbb{Q}^\times_+\backslash\mathbb{Q}^\times_f/\mu(K_f). \tag{1.42}$$

Using the lattice L as an integral structure in V, we see that $\mathbf{G}$ comes from a group scheme $\mathbf{G}_\mathbb{Z}$ over $\mathbb{Z}$, whose points in any ring A are

$$\mathbf{G}_\mathbb{Z}(A) = \left\{(g,\mu) \in GL_{\mathcal{O}_\mathcal{K}\otimes A}(L_A) \times A^\times |\ \langle gu, gv\rangle = \mu\,\langle u,v\rangle\ \right\}. \tag{1.43}$$

We likewise get that μ is a homomorphism from $\mathbf{G}_\mathbb{Z}$ to $\mathbb{G}_m$. The diagram

$$\begin{array}{ccc} \mathbf{G}_\mathbb{Z}(\mathbb{Z}_p) & \xrightarrow{\mu} & \mathbb{Z}_p^\times \\ \downarrow & & \downarrow \\ \mathbf{G}_\mathbb{Z}(\mathbb{Z}_p/N\mathbb{Z}_p) & \xrightarrow{\mu} & (\mathbb{Z}_p/N\mathbb{Z}_p)^\times \end{array} \tag{1.44}$$

commutes, $\mathbf{G}_\mathbb{Z}(\mathbb{Z}_p) = K_p^0$ and the kernel of $\mathbf{G}_\mathbb{Z}(\mathbb{Z}_p) \to \mathbf{G}_\mathbb{Z}(\mathbb{Z}_p/N\mathbb{Z}_p)$ is K_p. This shows that $\mu(K_f) \subset \hat{\mathbb{Z}}^\times(N)$, the product of local units congruent to $1 \bmod N$. But

$$\mathbb{Q}^\times_+\backslash\mathbb{Q}^\times_f/\hat{\mathbb{Z}}^\times(N) = (\mathbb{Z}/N\mathbb{Z})^\times. \tag{1.45}$$

To conclude, we have shown the existence of two maps from the set of connected components:

$$cl : \pi_0(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K) \to Cl_\mathcal{K} \tag{1.46}$$

$$\nu_N : \pi_0(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K) \to (\mathbb{Z}/N\mathbb{Z})^\times. \tag{1.47}$$

These two maps are independent: together they map $\pi_0(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K)$ *onto* $Cl_{\mathcal{K}}\times(\mathbb{Z}/N\mathbb{Z})^\times$. On the other hand, they have a non-trivial common kernel, which grows with N, as is evident from the interpretation of $\mathcal{K}^\times\backslash\mathcal{K}_f^\times/\nu(K_f)$ as the Galois group of a certain class field extension of $\mathcal{K}$. The map cl gives the restriction to the Hilbert class field, while the map ν_N gives the restriction to the cyclotomic field $\mathbb{Q}(\mu_N)$. We have singled out cl and ν_N, because when $N\geq 3$, they have an interpretation in terms of the complex points of Sh_K.

Proposition 1.5. *Let* $[g]\in\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K=Sh_K(\mathbb{C})$. *Then*

(i) $cl([g])$ *is the* Steinitz class *of the lattice* $L_g=g_f(\hat{L})\cap V$ *in* $Cl_{\mathcal{K}}$.

(ii) $\nu_N([g])$ *is (essentially) the* $\nu_{N,g}$ *that appears in the definition of* α_g *(see 1.3.1).*

Proof. (i) $cl([g])$ is the class of the ideal $(\nu(g_f))$ associated to the idele $\nu(g_f)\in\mathcal{K}_f^\times$. This ideal is in the same class as $(\det(g_f))$, because $\mu(g_f)\in\mathbb{Q}_f^\times$, so $(\mu(g_f))$ is principal. But the class of $(\det(g_f))$ is the Steinitz class of L_g, since the Steinitz class of L is trivial.

(ii) To find $\nu_N([g])$ we first project the idele $\mu(g_f)$ to $\hat{\mathbb{Z}}^\times$ using $\mathbb{Q}_f^\times=\mathbb{Q}_+^\times\hat{\mathbb{Z}}^\times$. But this is just $\tilde{\mu}(g_f)^{-1}\mu(g_f)$. We then take the result modulo N, so

$$\nu_N([g])=\tilde{\mu}(g_f)^{-1}\mu(g_f)\bmod N. \tag{1.48}$$

Now the definition of the tuple $(A_g,\lambda_g,\iota_g,\alpha_g)$ is such that if $u,v\in N^{-1}L/L$ then

$$\begin{aligned}\langle\alpha_g(u),\alpha_g(v)\rangle_{\lambda_g} &= \exp\left(2\pi iN\left\langle g_fu,g_fv\right\rangle_g\right)\\ &= \exp\left(2\pi i\tilde{\mu}(g_f)^{-1}N\left\langle g_fu,g_fv\right\rangle\right)\\ &= \exp\left(2\pi i\tilde{\mu}(g_f)^{-1}\mu(g_f)N\left\langle u,v\right\rangle\right)\\ &= \exp\left(2\pi i\nu_N([g])N\left\langle u,v\right\rangle\right)\end{aligned} \tag{1.49}$$

Part (ii) follows if we identify $\nu_{N,g}\in Isom_{\mathbb{C}}(N^{-1}\mathbb{Z}/\mathbb{Z},\mu_N)$ with $\nu_N([g])\in(\mathbb{Z}/N\mathbb{Z})^\times$ using $\exp(2\pi i(\cdot))$. ∎

1.3.5. *The complex uniformization.* Recall that $\mathfrak{X}=G_\infty/K_\infty$ and that it was equipped with a base point x_0 (coresponding to $(z,u)=(\delta_{\mathcal{K}}/2,0)$ in the Siegel domain of the second kind). Let $1=g_1,\ldots,g_m\in\mathbf{G}(\mathbb{A}_f)$ ($m=\#(\mathcal{K}^\times\backslash\mathcal{K}_f^\times/\nu(K_f))$) be representatives of the connected components of $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K$, and define congruence groups

$$\Gamma_j=\mathbf{G}(\mathbb{Q})\cap g_jK_fg_j^{-1}. \tag{1.50}$$

We write $[x,g_j]$ for $\mathbf{G}(\mathbb{Q})(x,g_jK_f)\in\mathbf{G}(\mathbb{Q})\backslash(\mathfrak{X}\times\mathbf{G}(\mathbb{A}_f)/K_f)=\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K$. Then $[x',g_j]=[x,g_j]$ if and only if $x'=\gamma x$ for $\gamma\in\Gamma_j$. The map

$$\coprod_{j=1}^m X_{\Gamma_j}=\coprod_{j=1}^m\Gamma_j\backslash\mathfrak{X}\simeq Sh_K(\mathbb{C}) \tag{1.51}$$

sending Γ_jx to $[x,g_j]$ is an isomorphism.

Note that $\Gamma_1=\Gamma$ is the principal level-N congruence subgroup in $\mathbf{G}_{\mathbb{Z}}(\mathbb{Z})$, the stabilizer of L. Similarly, Γ_j is the principal level-N congruence subgroup in the stabilizer of L_{g_j}, and is thus a group of the type considered in 1.2.1, except that we have dropped the assumption on the Steinitz class of L_{g_j}. As $N\geq 3$, $\det(\gamma)=1$ and $\mu(\gamma)=1$ for all $\gamma\in\Gamma_j$, for every j. Indeed, on the one hand these are in $\mathcal{K}^\times$

and $\mathbb{Q}_+^\times$ respectively. On the other hand, they are local units which are congruent to $1 \bmod N$ everywhere. It follows that Γ_j are subgroups of $\mathbf{G}'(\mathbb{Q}) = \mathbf{SU}(\mathbb{Q})$.

We get a similar decomposition to connected components (as an algebraic surface)

$$S_{\mathbb{C}} = \coprod_{j=1}^{m} S_{\Gamma_j} \tag{1.52}$$

and we write $S_{\mathbb{C}}^* = \coprod_{j=1}^{m} S_{\Gamma_j}^*$ for the Baily-Borel compactification.

1.4. **Smooth compactifications.**

1.4.1. *The smooth compactification of* X_Γ. We begin by working in the complex analytic category and follow the exposition of [Cog]. The Baily-Borel compactification X_Γ^* is singular at the cusps, and does not admit a modular interpretation. For general unitary Shimura varieties, the theory of toroidal compactifications provides smooth compactifications that depend, in general, on extra data. It is a unique feature of Picard modular surfaces, stemming from the finiteness of $\mathcal{O}_{\mathcal{K}}^\times$, that this smooth compactification is canonical. As all cusps are equivalent (if we vary the lattice L or Γ), it is enough, as usual, to study the smooth compactification at c_∞. In [Cog] this is described for an arbitrary L (not even $\mathcal{O}_{\mathcal{K}}$-free), but for simplicity we write it down only for $L = L_0$.

As $N \geq 3$, elements of Γ stabilizing c_∞ lie in N_∞.[3] The computations, which we omit, are somewhat simpler if N is *even*, an assumption made for the rest of this section. Let

$$\Gamma_{cusp} = \Gamma \cap N_\infty. \tag{1.53}$$

Lemma 1.6. *Let $N \geq 3$ be even. The matrix $n(s,r) \in \Gamma_{cusp}$ if and only if: (i) ($d_{\mathcal{K}} \equiv 1 \bmod 4$) $s \in N\mathcal{O}_{\mathcal{K}}$, $r \in ND_{\mathcal{K}}\mathbb{Z}$, (ii) ($d_{\mathcal{K}} \equiv 2, 3 \bmod 4$) $s \in N\mathcal{O}_{\mathcal{K}}$ and $r \in 2^{-1}ND_{\mathcal{K}}\mathbb{Z}$.*

Let $M = N|D_{\mathcal{K}}|$ in case *(i)* and $M = 2^{-1}N|D_{\mathcal{K}}|$ in case *(ii)*. This is the *width of the cusp* c_∞. Let

$$q = q(z) = e^{2\pi i z/M}. \tag{1.54}$$

For $R > 0$, the domain $\Omega_R = \{(z,u) \in \mathfrak{X} |\, \lambda(z,u) > R\}$ is invariant under Γ_{cusp} and if R is large enough, two points of it are Γ-equivalent if and only if they are Γ_{cusp}-equivalent. A sufficiently small punctured neighborhood of c_∞ in X_Γ^* therefore looks like $\Gamma_{cusp} \backslash \Omega_R$. As

$$n(s,r)(z,u) = (z + \delta\bar{s}(u + s/2) + r, u + s) \tag{1.55}$$

we obtain the following description of $\Gamma_{cusp}\backslash\Omega_R$. Let $\Lambda = N\mathcal{O}_{\mathcal{K}}$ and $E = \mathbb{C}/\Lambda$, an elliptic curve with complex multiplication by $\mathcal{O}_{\mathcal{K}}$. Let $\mathcal{T}$ be the quotient

$$\mathcal{T} = (\mathbb{C} \times \mathbb{C})/\Lambda \tag{1.56}$$

where the action of $s \in \Lambda$ is via

$$[s] : (t,u) \mapsto (e^{2\pi i \delta\bar{s}(u+s/2)/M} t, u + s). \tag{1.57}$$

It is a line bundle over E via the second projection. We denote the class of (t,u) modulo the action of Λ by $[t,u]$.

[3] No confusion should arise from the use of the letter N to denote both the level and the unipotent radical of P.

Proposition 1.7. *Let $\mathcal{T}_R \subset \mathcal{T}$ be the disk bundle consisting of all the points $[t, u]$ where*

$$|t| < e^{-\pi|\delta|(R+u\bar{u})/M}. \tag{1.58}$$

(This condition is invariant under the action of Λ.) Let $\mathcal{T}'_R$ be the punctured disk bundle *obtained by removing the zero section from $\mathcal{T}_R$. Then the map $(z, u) \mapsto (q(z), u)$ induces an analytic isomorphism between $\Gamma_{cusp}\backslash\Omega_R$ and $\mathcal{T}'_R$.*

Proof. This follows from the discussion so far and the fact that $\lambda(z, u) > R$ is equivalent to the above condition on $t = q(z)$ ([Cog], Prop. 2.1). ∎

To get a smooth compactification $\bar{X}_\Gamma$ of X_Γ (as a complex surface), we glue the disk bundle $\mathcal{T}_R$ to X_Γ along $\mathcal{T}'_R$. In other words, we complete $\mathcal{T}'_R$ by adding the zero section, which is isomorphic to E. The same procedure should be carried out at any other cusp of $\mathcal{C}_\Gamma$.

Note that the geodesic (1.15) connecting $(z, u) \in \mathfrak{X}$ to the cusp c_∞ projects in $\bar{X}_\Gamma$ to a geodesic which meets E transversally at the point $u \bmod \Lambda$. We caution that this geodesic in X_Γ depends on (z, u) and c_∞ and not only on their images modulo Γ.

The line bundle $\mathcal{T}$ is the *inverse* of an ample line bundle on E. In fact, $\mathcal{T}^\vee$ is the N-th (resp. $2N$-th) power of one of the four basic *theta line bundles* if $d_\mathcal{K} \equiv 1 \bmod 4$ (resp. $d_\mathcal{K} \equiv 2, 3 \bmod 4$). A basic theta function of the lattice Λ satisfies, for $u \in \mathbb{C}$ and $s \in \Lambda$,

$$\theta(u + s) = \alpha(s) e^{2\pi\bar{s}(u+s/2)/|\delta|N^2} \theta(u) \tag{1.59}$$

where $\alpha : \Lambda \to \pm 1$ is a quasi-character (see [Mu], p.25). Recalling the relation between M and N, and the assumption that N was even, we easily get the relation between $\mathcal{T}$ and the theta line bundles.

1.4.2. *The smooth compactification of S.* The arithmetic compactification $\bar{S}$ of the Picard surface S (over R_0) is due to Larsen [La1] (see also [Bel] and [Lan]). We summarize the results in the following theorem. We mention first that as $S_\mathbb{C}$ has a canonical model S over R_0, its Baily-Borel compactification $S^*_\mathbb{C}$ has a similar model S^* over R_0, and S embeds in S^* as an open dense subscheme.

Theorem 1.8. *(i) There exists a projective scheme $\bar{S}$, smooth over R_0, of relative dimension 2, together with an open dense immersion of S in $\bar{S}$, and a proper morphism $p : \bar{S} \to S^*$, making the following diagram commutative*

$$\begin{array}{ccc} S & \rightarrow & \bar{S} \\ \downarrow & \overset{p}{\swarrow} & \\ S^* & & \end{array} . \tag{1.60}$$

(ii) As a complex manifold, there is an isomorphism

$$\bar{S}_\mathbb{C} \simeq \coprod_{j=1}^{m} \bar{X}_{\Gamma_j}, \tag{1.61}$$

extending the isomorphism of $S_\mathbb{C}$ with $\coprod_{j=1}^m X_{\Gamma_j}$.

(iii) Let $C = p^{-1}(S^* - S)$. Let R_N be the integral closure of R_0 in the ray class field $\mathcal{K}_N$ of conductor N over $\mathcal{K}$. Then the connected components of C_{R_N} are geometrically irreducible, and are indexed by the cusps of $S^*_{R_N}$ over which they

sit. Furthermore, each component $E \subset C_{R_N}$ is an elliptic curve with complex multiplication by $\mathcal{O}_{\mathcal{K}}$.

We call C the *cuspidal divisor.* If $c \in S^*_{\mathbb{C}} - S_{\mathbb{C}}$ is a cusp, we denote the complex elliptic curve $p^{-1}(c)$ by E_c. Bear in mind that while E_c is in principle definable over the Hilbert class field $\mathcal{K}_1$, no canonical model of it over that field is provided by $\bar{S}$. On the other hand, E_c does come with a canonical model over $\mathcal{K}_N$, and even over R_N.

We refer to [La1] and [Bel] for a moduli-theoretic interpretation of C as a moduli space for semi-abelian schemes with a suitable action of $\mathcal{O}_{\mathcal{K}}$ and a "level-N structure". Unfortunately these references do not give such a moduli interpretation to $\bar{S}$. While they do construct a universal semi-abelian scheme over $\bar{S}$ (see the next section), the level-N structure over S does not extend to a flat level-N structure over $\bar{S}$ in the ordinary sense, and the notion has to be modified over the boundary. What is evidently missing is the construction of a "Tate-Picard" object similar to the "generalized elliptic curve" which was constructed over the complete modular curve by Deligne and Rapoport in [De-Ra].

1.4.3. *Change of level.* Assume that $N \geq 3$ is even, and $N' = QN$. We then obtain a covering map $X_{\Gamma(N')} \to X_{\Gamma(N)}$ where by $\Gamma(N)$ we denote the group previously denoted by Γ. Near any of the cusps, the analytic model allows us to analyze this map locally. Let E' be an irreducible cuspidal component of $\bar{X}_{\Gamma(N')}$ mapping to the irreducible component E of $\bar{X}_{\Gamma(N)}$. The following is a consequence of the discussion in the previous sections.

Proposition 1.9. *The map $E' \to E$ is a multiplication-by-Q isogeny, hence étale of degree Q^2. When restricted to a neighborhood of E', the covering $\bar{X}_{\Gamma(N')} \to \bar{X}_{\Gamma(N)}$ is of degree Q^3, and has ramification index Q along E, in the normal direction to E.*

Corollary 1.10. *The pull-back to E' of the normal bundle $\mathcal{T}(N)$ of E is the Qth power of the normal bundle $\mathcal{T}(N')$ of E'.*

1.5. **The universal semi-abelian scheme $\mathcal{A}$.**

1.5.1. *The universal semi-abelian scheme over $\bar{S}$.* As Larsen and Bellaiche explain, the universal abelian scheme $\pi : \mathcal{A} \to S$ extends canonically to a semi-abelian scheme $\pi : \mathcal{A} \to \bar{S}$. The polarization λ extends over the boundary $C = \bar{S} - S$ to a principal polarization λ of the abelian part of $\mathcal{A}$. The action ι of $\mathcal{O}_{\mathcal{K}}$ extends to an action on the semi-abelian variety, which necessarily induces separate actions on the toric part and on the abelian part.

Let E be a connected component of C_{R_N}, mapping (over $\mathbb{C}$ and under the projection p) to the cusp $c \in S^*_{\mathbb{C}}$. Then there exist (1) a principally polarized elliptic curve B defined over R_N, with complex multiplication by $\mathcal{O}_{\mathcal{K}}$ and CM type Σ, and (2) an ideal $\mathfrak{a}$ of $\mathcal{O}_{\mathcal{K}}$, such that every fiber $\mathcal{A}_x$ of $\mathcal{A}$ over E is an $\mathcal{O}_{\mathcal{K}}$-group extension of B by the $\mathcal{O}_{\mathcal{K}}$-torus $\mathfrak{a} \otimes \mathbb{G}_m$. Both B (with its polarization) and the ideal class $[\mathfrak{a}] \in Cl_{\mathcal{K}}$ are uniquely determined by the cusp c. Only the extension class in the category of $\mathcal{O}_{\mathcal{K}}$-groups varies as we move along E. Note that since the Lie algebra of the torus is of type $(1,1)$, the Lie algebra of such an extension $\mathcal{A}_x$ is of type $(2,1)$, as is the case at an interior point $x \in S$. If we extend scalars to $\mathbb{C}$, the isomorphism type of B is given by another ideal class $[\mathfrak{b}]$ (i.e. $B(\mathbb{C}) \simeq \mathbb{C}/\mathfrak{b}$). In this case we say that the cusp c is of type $(\mathfrak{a}, \mathfrak{b})$.

The above discussion defines a homomorphism (of fppf sheaves over $Spec(R_N)$)

$$E \to Ext^1_{\mathcal{O}_\mathcal{K}}(B, \mathfrak{a} \otimes \mathbb{G}_m). \tag{1.62}$$

As we shall see soon, the Ext group is represented by an elliptic curve with CM by $\mathcal{O}_\mathcal{K}$, defined over R_N, and this map is an isogeny.

1.5.2. *$\mathcal{O}_\mathcal{K}$-semi-abelian schemes of type* $(\mathfrak{a}, \mathfrak{b})$. We digress to discuss the moduli space for semi-abelian schemes of the type found above points of E. Let R be an R_0-algebra, B an elliptic curve over R with complex multiplication by $\mathcal{O}_\mathcal{K}$ and CM type Σ, and $\mathfrak{a}$ an ideal of $\mathcal{O}_\mathcal{K}$. Consider a semi-abelian scheme $\mathcal{G}$ over R, endowed with an $\mathcal{O}_\mathcal{K}$ action $\iota : \mathcal{O}_\mathcal{K} \to End(\mathcal{G})$, and a short exact sequence

$$0 \to \mathfrak{a} \otimes \mathbb{G}_m \to \mathcal{G} \to B \to 0 \tag{1.63}$$

of $\mathcal{O}_\mathcal{K}$-group schemes over R. We call all this data a *semi-abelian scheme of type* $(\mathfrak{a}, B)$ (over R). The group classifying such structures is $Ext^1_{\mathcal{O}_\mathcal{K}}(B, \mathfrak{a}\otimes\mathbb{G}_m)$. Any $\chi \in \mathfrak{a}^* = Hom(\mathfrak{a}, \mathbb{Z})$ defines, by push-out, an extension $\mathcal{G}_\chi$ of B by $\mathbb{G}_m$, hence a point of $B^t = Ext^1(B, \mathbb{G}_m)$. We therefore get a homomorphism from $Ext^1_{\mathcal{O}_\mathcal{K}}(B, \mathfrak{a} \otimes \mathbb{G}_m)$ to $Hom(\mathfrak{a}^*, B^t)$. A simple check shows that its image is in $Hom_{\mathcal{O}_\mathcal{K}}(\mathfrak{a}^*, B^t) = \delta_\mathcal{K}\mathfrak{a} \otimes_{\mathcal{O}_\mathcal{K}} B^t$, and that this construction yields an isomorphism

$$Ext^1_{\mathcal{O}_\mathcal{K}}(B, \mathfrak{a} \otimes \mathbb{G}_m) \simeq \delta_\mathcal{K}\mathfrak{a} \otimes_{\mathcal{O}_\mathcal{K}} B^t. \tag{1.64}$$

Here we have used the canonical identification $\mathfrak{a}^* = \delta_\mathcal{K}^{-1}\mathfrak{a}^{-1}$ (via the trace pairing). Although $(\delta_\mathcal{K})$ is a principal ideal, so can be ignored, it is better to keep track of its presence. We emphasize that the CM type of B^t, with the natural action of $\mathcal{O}_\mathcal{K}$ derived from its action on B, is $\bar{\Sigma}$ rather than Σ.

Thus over $\delta_\mathcal{K}\mathfrak{a} \otimes_{\mathcal{O}_\mathcal{K}} B^t$ there is a universal semi-abelian scheme $\mathcal{G}(\mathfrak{a}, B)$ of type $(\mathfrak{a}, B)$, and any $\mathcal{G}$ as above, over any base R'/R, is obtained from $\mathcal{G}(\mathfrak{a}, B)$ by pull-back (specialization) with respect to a unique map $Spec(R') \to \delta_\mathcal{K}\mathfrak{a} \otimes_{\mathcal{O}_\mathcal{K}} B^t$.

When $R = \mathbb{C}$, $B \simeq \mathbb{C}/\mathfrak{b}$ for a unique ideal class $[\mathfrak{b}]$ (with $\mathcal{O}_\mathcal{K}$ acting via Σ). Then, canonically, $B^t = \mathbb{C}/\delta_\mathcal{K}^{-1}\bar{\mathfrak{b}}^{-1}$ (with $\mathcal{O}_\mathcal{K}$ acting via $\bar{\Sigma}$). The pairing between the lattices, $\mathfrak{b} \times \delta_\mathcal{K}^{-1}\bar{\mathfrak{b}}^{-1} \to \mathbb{Z}$ is $(x, y) \mapsto Tr_{\mathcal{K}/\mathbb{Q}}(x\bar{y})$. Since the $\mathcal{O}_\mathcal{K}$ action on B^t is via $\bar{\Sigma}$,

$$Ext^1_{\mathcal{O}_\mathcal{K}}(\mathbb{C}/\mathfrak{b}, \mathfrak{a} \otimes \mathbb{G}_m) \simeq \delta_\mathcal{K}\mathfrak{a} \otimes_{\mathcal{O}_\mathcal{K}} \mathbb{C}/\delta_\mathcal{K}^{-1}\bar{\mathfrak{b}}^{-1} = \mathbb{C}/\bar{\mathfrak{a}}\bar{\mathfrak{b}}^{-1}. \tag{1.65}$$

The universal semi-abelian variety $\mathcal{G}(\mathfrak{a}, B)$ will now be denoted $\mathcal{G}(\mathfrak{a}, \mathfrak{b})$. In 1.6.2 below we give a complex analytic model of this $\mathcal{G}(\mathfrak{a}, \mathfrak{b})$.

1.6. Degeneration of $\mathcal{A}$ along a geodesic connecting to a cusp.

1.6.1. *The degeneration to a semi-abelian variety.* It is instructive to use the "moving lattice model" to compute the degeneration of the universal abelian scheme along a geodesic, as we approach a cusp. To simplify the computations, assume for the rest of this section, as before, that $N \geq 3$ is even, and that the cusp is the standrad cusp at infinity $c = c_\infty$. In this case we have shown that $E_c = \mathbb{C}/\Lambda$, where $\Lambda = N\mathcal{O}_\mathcal{K}$, and we have given a neighborhood of E_c in $\bar{X}_\Gamma$ the structure of a disk bundle in a line bundle $\mathcal{T}$. See Proposition 1.7.

Consider the geodesic (1.15) connecting (z, u) to c_∞. Consider the universal abelian scheme in the moving lattice model (*cf* (1.27)). Of the three vectors used to span L'_x over $\mathcal{O}_\mathcal{K}$ in (1.25) the first two do not depend on z. As u is fixed along the geodesic, they are not changed. The third vector represents a cycle that vanishes at the cusp (together with all its $\mathcal{O}_\mathcal{K}$-multiples). We conclude that A'_x degenerates to

$$\mathbb{C}^3/Span_{\iota'(\mathcal{O}_\mathcal{K})}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\u\end{pmatrix}\right\}. \tag{1.66}$$

Making the change of variables $(\zeta_1',\zeta_2',\zeta_3')=(\zeta_1,\zeta_2+\bar{u}\zeta_1,\zeta_3)$ does not alter the $\mathcal{O}_\mathcal{K}$ action and gives the more symmetric model

$$\mathcal{G}_u=\mathbb{C}^3/Span_{\iota'(\mathcal{O}_\mathcal{K})}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\\bar{u}\\u\end{pmatrix}\right\} \tag{1.67}$$

(but note that ζ_2', unlike ζ_2, does not vary holomorphically in the *family* $\{\mathcal{G}_u\}$, only in each fiber individually).

Let $e(x)=e^{2\pi ix}:\mathbb{C}\to\mathbb{C}^\times$ be the exponential map, with kernel $\mathbb{Z}$. For any ideal $\mathfrak{a}$ of $\mathcal{O}_\mathcal{K}$ it induces a map

$$e_\mathfrak{a}:\mathfrak{a}\otimes\mathbb{C}\to\mathfrak{a}\otimes\mathbb{C}^\times \tag{1.68}$$

with kernel $\mathfrak{a}\otimes 1$. As usual we identify $\mathfrak{a}\otimes\mathbb{C}$ with $\mathbb{C}(\Sigma)\oplus\mathbb{C}(\bar{\Sigma})$, sending $a\otimes\zeta\mapsto(a\zeta,\bar{a}\zeta)$. We now note that if we use this identification to identify $\mathbb{C}^3$ with $\mathbb{C}\oplus(\mathcal{O}_\mathcal{K}\otimes\mathbb{C})$ (an identification which is compatible with the $\mathcal{O}_\mathcal{K}$ action) then the $\iota'(\mathcal{O}_\mathcal{K})$-span of the vector ${}^t(0,1,1)$ is just the kernel of $e_{\mathcal{O}_\mathcal{K}}$. We conclude that

$$\mathcal{G}_u\simeq\{\mathbb{C}\oplus(\mathcal{O}_\mathcal{K}\otimes\mathbb{C}^\times)\}/L_u \tag{1.69}$$

where L_u is the sub-$\mathcal{O}_\mathcal{K}$-module

$$L_u=\{(s,e_{\mathcal{O}_\mathcal{K}}(s\bar{u},\bar{s}u))|\,s\in\mathcal{O}_\mathcal{K}\}. \tag{1.70}$$

This clearly gives $\mathcal{G}_u$ the structure of an $\mathcal{O}_\mathcal{K}$-semi-abelian variety of type $(\mathcal{O}_\mathcal{K},\mathcal{O}_\mathcal{K})$, i.e. an extension

$$0\to\mathcal{O}_\mathcal{K}\otimes\mathbb{C}^\times\to\mathcal{G}_u\to\mathbb{C}/\mathcal{O}_\mathcal{K}\to 0. \tag{1.71}$$

1.6.2. *The analytic uniformization of the universal semi-abelian variety of type* $(\mathfrak{a},\mathfrak{b})$*.* We now compare the description that we have found for the degeneration of $\mathcal{A}$ along the geodesic connecting (z,u) to c_∞ with the analytic description of the universal semi-abelian variety of type $(\mathfrak{a},\mathfrak{b})$.

Proposition 1.11. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of* $\mathcal{O}_\mathcal{K}$*. For* $u\in\mathbb{C}$ *consider*

$$\mathcal{G}_u\simeq\{\mathbb{C}\oplus(\mathfrak{a}\otimes\mathbb{C}^\times)\}/L_u \tag{1.72}$$

where

$$L_u=\{(s,e_\mathfrak{a}(s\bar{u},\bar{s}u))|\,s\in\mathfrak{b}\}. \tag{1.73}$$

Then $\mathcal{G}_u$ *is a semi-abelian variety of type* $(\mathfrak{a},\mathfrak{b})$*, any complex semi-abelian variety of this type is a* $\mathcal{G}_u$*, and* $\mathcal{G}_u\simeq\mathcal{G}_v$ *if and only if* $u-v\in\bar{\mathfrak{a}}\bar{\mathfrak{b}}^{-1}$*.*

Proof. That $\mathcal{G}_u$ is a semi-abelian variety of type $(\mathfrak{a},\mathfrak{b})$ is obvious. That any abelian variety of this type is a $\mathcal{G}_u$ follows by passing to the universal cover $\mathbb{C}^2(\Sigma)\oplus\mathbb{C}(\bar{\Sigma})$, and noting that by a change of variables in the Σ- and $\bar{\Sigma}$-isotypical parts, we may assume that the lattice by which we divide is of the form

$$\mathfrak{a}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \oplus \mathfrak{b}\begin{pmatrix} 1 \\ \bar{u} \\ u \end{pmatrix}. \tag{1.74}$$

Finally, the map $u \mapsto [\mathcal{G}_u]$ is a homomorphism $\mathbb{C} \to Ext^1_{\mathcal{O}_\mathcal{K}}(\mathbb{C}/\mathfrak{b}, \mathfrak{a}\otimes\mathbb{C}^\times)$, so we only have to prove that $\mathcal{G}_u$ is split if and only if $u \in \overline{\mathfrak{a}\mathfrak{b}}^{-1}$. But one can check easily that $\mathcal{G}_u$ is trivial if and only if $(s\bar{u}, \bar{s}u) \in \ker e_\mathfrak{a} = \mathfrak{a}\otimes 1 = \{(a,\bar{a}) | a \in \mathfrak{a}\}$ for every $s \in \mathfrak{b}$, and this holds if and only if $u \in \overline{\mathfrak{a}\mathfrak{b}}^{-1}$. ∎

Corollary 1.12. *Let $N \geq 3$ be even. Let $c = c_\infty$ be the cusp at infinity. Then the map*

$$E_c \to Ext^1_{\mathcal{O}_\mathcal{K}}(\mathbb{C}/\mathcal{O}_\mathcal{K}, \mathcal{O}_\mathcal{K}\otimes\mathbb{C}^\times) \tag{1.75}$$

sending u to the isomorphism class of the semi-abelian variety above $u \bmod \Lambda$ is the isogeny of multiplication by N.

Proof. In view of the computations above, and the description of a neighborhood of E_c in $\bar{X}_\Gamma$ given in Proposition 1.7 this map is identified with the canonical map

$$\mathbb{C}/N\mathcal{O}_\mathcal{K} \to \mathbb{C}/\mathcal{O}_\mathcal{K}. \tag{1.76}$$

∎

The extra data carried by $u \in E_c$, which is forgotten by the map of the corollary, comes from the level N structure. As mentioned before, according to [La1] and [Bel] the cuspidal divisor C has a modular interpretation as the moduli space for semi-abelian schemes of the type considered above, together with level-N structure ($\mathcal{M}_{\infty,N}$ structures in the language of [Bel]). A level-N structure on a *semi-abelian* variety $\mathcal{G}$ of type $(\mathfrak{a},\mathfrak{b})$ consists of (i) a level-N structures $\alpha : N^{-1}\mathcal{O}_\mathcal{K}/\mathcal{O}_\mathcal{K} \simeq \mathfrak{a}\otimes\mu_N$ on the toric part (ii) a level-N structure $\beta : N^{-1}\mathcal{O}_\mathcal{K}/\mathcal{O}_\mathcal{K} \simeq N^{-1}\mathfrak{b}/\mathfrak{b} = B[N]$ on the abelian part (iii) an $\mathcal{O}_\mathcal{K}$-splitting γ of the map $\mathcal{G}[N] \to B[N]$.

Over $c = c_\infty$, when $\mathfrak{a} = \mathfrak{b} = \mathcal{O}_\mathcal{K}$, there are obvious natural choices for α and β (independent of u) but the splittings γ in (iii) form a torsor under $\mathcal{O}_\mathcal{K}/N\mathcal{O}_\mathcal{K}$. If we consider the splitting

$$\gamma_u : N^{-1}\mathcal{O}_\mathcal{K}/\mathcal{O}_\mathcal{K} \ni s \mapsto (s, e_{\mathcal{O}_\mathcal{K}}(s\bar{u}, \bar{s}u)) \bmod L_u \tag{1.77}$$

then the tuples $(\mathcal{G}_u, \alpha, \beta, \gamma_u)$ and $(\mathcal{G}_v, \alpha, \beta, \gamma_v)$ are isomorphic if and only if $u \equiv v \bmod N\mathcal{O}_\mathcal{K}$, i.e. if and only if u and v represent the same point of E_c.

2. The basic automorphic vector bundles

2.1. The vector bundles $\mathcal{P}$ and $\mathcal{L}$.

2.1.1. *Definition and first properties.* Recall our running assumptions and notation. The tame level $N \geq 3$, S is the Picard modular scheme over the base ring R_0, $\bar{S}$ is its smooth compactification, and $\mathcal{A}$ is the universal semi-abelian scheme over $\bar{S}$ (an abelian scheme over S) constructed by Larsen and Bellaiche.

Let $\omega_\mathcal{A}$ be the relative cotangent space at the origin of $\mathcal{A}$. If $e : \bar{S} \to \mathcal{A}$ is the zero section,

$$\omega_\mathcal{A} = e^*(\Omega^1_{\mathcal{A}/\bar{S}}). \tag{2.1}$$

This is a rank 3 vector bundle over $\bar{S}$ and the action of $\mathcal{O}_\mathcal{K}$ allows to decompose it according to types. We let

$$\mathcal{P} = \omega_\mathcal{A}(\Sigma), \;\; \mathcal{L} = \omega_\mathcal{A}(\bar{\Sigma}). \tag{2.2}$$

Then $\mathcal{P}$ is a plane bundle, and $\mathcal{L}$ a line bundle.

Over S (but not over the cuspidal divisor $C = \bar{S} - S$) we have the usual identification $\omega_\mathcal{A} = \pi_*\Omega^1_{\mathcal{A}/S}$. The relative de Rham cohomology of $\mathcal{A}/S$ is a rank 6 vector bundle sitting in an exact sequence (the Hodge filtration)

$$0 \to \omega_\mathcal{A} \to H^1_{dR}(\mathcal{A}/S) \to R^1\pi_*\mathcal{O}_\mathcal{A} \to 0. \tag{2.3}$$

Since, for any abelian scheme, $R^1\pi_*\mathcal{O}_\mathcal{A} = \omega^\vee_{\mathcal{A}^t}$ (canonical isomorphism, see [Mu]), and $\lambda : \mathcal{A} \to \mathcal{A}^t$ is an isomorphism which reverses CM types, we obtain an exact sequence

$$0 \to \omega_\mathcal{A} \to H^1_{dR}(\mathcal{A}/S) \to \omega^\vee_\mathcal{A}(\rho) \to 0. \tag{2.4}$$

The notation $\mathcal{M}(\rho)$ means that $\mathcal{M}$ is a vector bundle with an $\mathcal{O}_\mathcal{K}$ action and in $\mathcal{M}(\rho)$ the vector bundle structure is that of $\mathcal{M}$ but the $\mathcal{O}_\mathcal{K}$ action is conjugated. Decomposing according to types, we have two short exact sequences

$$\begin{array}{lll} 0 & \to & \mathcal{P} \to H^1_{dR}(\mathcal{A}/S)(\Sigma) \to \mathcal{L}^\vee(\rho) \to 0 \\ 0 & \to & \mathcal{L} \to H^1_{dR}(\mathcal{A}/S)(\bar{\Sigma}) \to \mathcal{P}^\vee(\rho) \to 0. \end{array} \tag{2.5}$$

The pairing $\langle , \rangle_\lambda$ on $H^1_{dR}(\mathcal{A}/S)$ induced by the polarization is $\mathcal{O}_S$-linear, alternating, perfect, and satisfies $\langle \iota(a)x, y\rangle_\lambda = \langle x, \iota(\bar{a})y\rangle_\lambda$. It follows that $H^1_{dR}(\mathcal{A}/S)(\Sigma)$ and $H^1_{dR}(\mathcal{A}/S)(\bar{\Sigma})$ are maximal isotropic subspaces, and are set in duality. As $\omega_\mathcal{A}$ is also isotropic, this pairing induces pairings

$$\mathcal{P} \times \mathcal{P}^\vee(\rho) \to \mathcal{O}_S, \;\; \mathcal{L} \times \mathcal{L}^\vee(\rho) \to \mathcal{O}_S. \tag{2.6}$$

These two pairings (in this order) are the tautological pairings between a vector bundle and its dual.

Another consequence of this discussion that we wish to record is the canonical isomorphism over S

$$\det \mathcal{P} = \mathcal{L}(\rho) \otimes \det\left(H^1_{dR}(\mathcal{A}/S)(\Sigma)\right). \tag{2.7}$$

2.1.2. *The factors of automorphy corresponding to $\mathcal{L}$ and $\mathcal{P}$.* The formulae below can be deduced also from the matrix calculations in the first few pages of [Sh2]. Let $\Gamma = \Gamma_j$ be one of the groups used in the complex uniformization of $S_\mathbb{C}$, *cf* Section 1.3.5. Via the analytic isomorphism $X_\Gamma \simeq S_\Gamma$ with the jth connected component, the vector bundles $\mathcal{P}$ and $\mathcal{L}$ are pulled back to X_Γ and then to the symmetric space $\mathfrak{X}$, where they can be trivialized, hence described by means of factors of automorphy. Let us denote by $\mathcal{P}_{an}$ and $\mathcal{L}_{an}$ the two vector bundles on X_Γ, in the complex analytic category, or their pull-backs to $\mathfrak{X}$.

To trivialize $\mathcal{L}_{an}$ we must choose a nowhere vanishing global section over $\mathfrak{X}$. As usual, we describe it only on the connected component containing the standard cusp, corresponding to $j = 1$ (where $L = L_{g_1} = L_0$). Recalling the "moving lattice model" (1.27) and the coordinates $\zeta_1, \zeta_2, \zeta_3$ introduced there, we note that $d\zeta_3$ is a generator of $\mathcal{L}_{an} = \omega_\mathcal{A}(\bar{\Sigma})$. For reasons that will become clear later (*cf* Section

2.6) we use $2\pi i \cdot d\zeta_3$ to trivialize $\mathcal{L}_{an}$ over $\mathfrak{X}$. Suppose

$$\gamma = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in \Gamma \subset SU_\infty. \tag{2.8}$$

If $\gamma(z,u) = (z', u')$ then

$$z' = \frac{a_1 z + b_1 u + c_1}{a_3 z + b_3 u + c_3}, \quad u' = \frac{a_2 z + b_2 u + c_2}{a_3 z + b_3 u + c_3} \tag{2.9}$$

and

$$\gamma \begin{pmatrix} z \\ u \\ 1 \end{pmatrix} = j(\gamma; z, u) \begin{pmatrix} z' \\ u' \\ 1 \end{pmatrix}, \quad j(\gamma; z, u) = a_3 z + b_3 u + c_3. \tag{2.10}$$

Lemma 2.1. *The following relation holds for every $\gamma \in U_\infty$*

$$\lambda(z,u) = \lambda(\gamma(z,u)) \cdot |j(\gamma; z, u)|^2. \tag{2.11}$$

Proof. Let $v = v(z,u) = {}^t(z,u,1)$. Then

$$\lambda(z,u) = -(v,v). \tag{2.12}$$

As $v(\gamma(z,u)) = j(\gamma; z,u)^{-1} \cdot \gamma(v(z,u))$ the lemma follows from $(\gamma v, \gamma v) = (v,v)$. ∎

Let $\mathcal{V} = Lie(\mathcal{A}/\mathfrak{X}) = \omega^\vee_{\mathcal{A}/\mathfrak{X}}$ and $\mathcal{W} = \mathcal{V}(\bar{\Sigma}) = \mathcal{L}^\vee_{an}$ (a line bundle). At a point $x = (z,u) \in \mathfrak{X}$ the fiber $\mathcal{V}_x$ is identified canonically with $(V_\mathbb{R}, J_x)$ and then $\mathcal{W}_x = W_x = \mathbb{C} \cdot {}^t(z,u,1)$.

Proposition 2.2. *For $x = (z,u) \in \mathfrak{X}$ let*

$$v_3(z,u) = \lambda(z,u)^{-1} \begin{pmatrix} z \\ u \\ 1 \end{pmatrix} \in \mathcal{W}_x. \tag{2.13}$$

Then (i) $v_3(z,u)$ is a nowhere vanishing holomorphic section of $\mathcal{W}$, (ii) $\langle d\zeta_3, v_3 \rangle \equiv 1$, (iii) the automorphy factor corresponding to $d\zeta_3$ is the function $j(\gamma; z,u)$.

Proof. Since, by construction, $d\zeta_3$ is a nowhere vanishing holomorphic section of $\mathcal{L}$ (over $\mathfrak{X}$), (i) follows from (ii). To prove (ii) we transfer $v_3(z,u)$ to the moving lattice model and get ${}^t(0,0,1)$, which is the dual vector to $d\zeta_3$. To prove (iii) we compute in $V_\mathbb{R}$ (with the original complex structure!)

$$\frac{\gamma_* v_3(z,u)}{v_3(\gamma(z,u))} = \frac{\lambda(\gamma(z,u))}{\lambda(z,u)} j(\gamma; z,u) = \overline{j(\gamma; z,u)}^{-1}, \tag{2.14}$$

and recall that since $W_{\gamma(z,u)}$ is precisely the line where the complex structure in $(V_\mathbb{R}, J_{\gamma(z,u)})$ has been reversed, in $(V_\mathbb{R}, J_{\gamma(z,u)})$ we have

$$\frac{\gamma_* v_3(z,u)}{v_3(\gamma(z,u))} = j(\gamma; z,u)^{-1}. \tag{2.15}$$

Dualizing, we get $(x = (z,u))$

$$\frac{(\gamma^{-1})^* d\zeta_3|_x}{d\zeta_3|_{\gamma(x)}} = j(\gamma, x). \tag{2.16}$$

This concludes the proof. ∎

Consider next the plane bundle $\mathcal{P}_{an}$. As we will only be interested in scalar-valued modular forms, we do not compute its matrix-valued factor of automorphy (but see [Sh2]). It is important to know, however, that the line bundle $\det \mathcal{P}_{an}$ gives nothing new.

Proposition 2.3. *There is an isomorphism of analytic line bundles over X_Γ,*

$$\det \mathcal{P}_{an} \simeq \mathcal{L}_{an}. \tag{2.17}$$

Moreover, $d\zeta_1 \wedge d\zeta_2$ is a nowhere vanishing holomorphic section of $\det \mathcal{P}_{an}$ over $\mathfrak{X}$, and the factor of automorphy corresponding to it is $j(\gamma; z, u)$.

Proof. Since a holomorphic line bundle on $X_\Gamma = \Gamma \backslash \mathfrak{X}$ is determined, up to an isomorphism, by its factor of automorphy, and $j(\gamma; z, u)$ is the factor of automorphy of $\mathcal{L}_{an}$ corresponding to $d\zeta_3$, it is enough to prove the second statement. Let $\mathcal{U} = \mathcal{V}(\Sigma)$ be the plane bundle dual to $\mathcal{P}_{an}$. Let

$$v_1(z,u) = -\lambda(z,u)^{-1} \begin{pmatrix} \bar{u}z \\ (z-\bar{z})/\delta \\ \bar{u} \end{pmatrix} \tag{2.18}$$

and

$$v_2(z,u) = -\lambda(z,u)^{-1} \begin{pmatrix} \bar{z} + \delta u \bar{u} \\ u \\ 1 \end{pmatrix} \tag{2.19}$$

(considered as vectors in $(V_\mathbb{R}, J_x) = \mathcal{V}_x$). As we have seen in (1.27), these two vector fields are sections of $\mathcal{U}$ and at each point $x \in \mathfrak{X}$ form a basis dual to $d\zeta_1$ and $d\zeta_2$. It follows that they are holomorphic sections, and that $v_1 \wedge v_2$ is the basis dual to $d\zeta_1 \wedge d\zeta_2$. We must show that the factor of automorphy corresponding to $v_1 \wedge v_2$ is $j(\gamma; z, u)^{-1}$, i.e. that

$$\frac{\gamma_*(v_1 \wedge v_2(z,u))}{v_1 \wedge v_2(\gamma(z,u))} = j(\gamma; z, u)^{-1}. \tag{2.20}$$

Working in $V_\mathbb{R} = \mathbb{C}^3$ (with the original complex structure)

$$\begin{aligned} \frac{\gamma_*(v_1 \wedge v_2(z,u))}{v_1 \wedge v_2(\gamma(z,u))} \cdot \frac{1}{\overline{j(\gamma; z, u)}} &= \\ \frac{\gamma_*(v_1 \wedge v_2(z,u))}{v_1 \wedge v_2(\gamma(z,u))} \cdot \frac{\gamma_* v_3(z,u)}{v_3(\gamma(z,u))} &= \frac{\gamma_*(v_1 \wedge v_2 \wedge v_3(z,u))}{v_1 \wedge v_2 \wedge v_3(\gamma(z,u))}. \end{aligned} \tag{2.21}$$

But

$$v_1 \wedge v_2 \wedge v_3(z,u) = \delta \lambda(z,u)^{-1} e_1 \wedge e_2 \wedge e_3, \tag{2.22}$$

because

$$\det \begin{pmatrix} \bar{u}z & \bar{z} + \delta u \bar{u} & z \\ (z-\bar{z})/\delta & u & u \\ \bar{u} & 1 & 1 \end{pmatrix} = \delta \lambda(z,u)^2. \tag{2.23}$$

As $\det(\gamma) = 1$, this gives

$$\frac{\gamma_*(v_1 \wedge v_2(z,u))}{v_1 \wedge v_2(\gamma(z,u))} \cdot \frac{1}{\overline{j(\gamma; z, u)}} = \frac{\lambda(\gamma(z,u))}{\lambda(z,u)} = \frac{1}{j(\gamma; z, u)\overline{j(\gamma; z, u)}}, \tag{2.24}$$

and the proof is complete. ∎

2.1.3. *The relation* $\det \mathcal{P} \simeq \mathcal{L}$ *over* $\bar{S}_{\mathcal{K}}$. The isomorphism between $\det \mathcal{P}$ and $\mathcal{L}$ is in fact algebraic, and even extends to the generic fiber $\bar{S}_{\mathcal{K}}$ of the smooth compactification.

Proposition 2.4. *One has* $\det \mathcal{P} \simeq \mathcal{L}$ *over* $\bar{S}_{\mathcal{K}}$.

Proof. Since $Pic(\bar{S}_{\mathcal{K}}) \subset Pic(\bar{S}_{\mathbb{C}})$ it is enough to prove the proposition over $\mathbb{C}$. By GAGA, it is enough to establish the triviality of $\det \mathcal{P} \otimes \mathcal{L}^{-1}$ in the analytic category. For each connected component X_Γ of $S_{\mathbb{C}}$, the section $(d\zeta_1 \wedge d\zeta_2) \otimes d\zeta_3^{-1}$ descends from $\mathfrak{X}$ to X_Γ, because $d\zeta_1 \wedge d\zeta_2$ and $d\zeta_3$ have the same factor of automorphy $j(\gamma, x)$ ($\gamma \in \Gamma$, $x \in \mathfrak{X}$). This section is nowhere vanishing on X_Γ, and extends to a nowhere vanishing section on $\bar{X}_\Gamma$, trivializing $\det \mathcal{P} \otimes \mathcal{L}^{-1}$. In fact, if c is the standard cusp, $d\zeta_1 \wedge d\zeta_2$ and $d\zeta_3$ are already well-defined and nowhere vanishing sections of $\det \mathcal{P}$ and $\mathcal{L}$ in the neighborhood

$$\overline{\Gamma_{cusp} \backslash \Omega_R} = (\Gamma_{cusp} \backslash \Omega_R) \cup E_c \tag{2.25}$$

of E_c (see 1.4.1). This is a consequence of the fact that $j(\gamma, x) = 1$ for $\gamma \in \Gamma_{cusp}$.

An alternative proof is to use Theorem 4.8 of [Ha]. In our case it gives a *functor* $\mathcal{V} \mapsto [\mathcal{V}]$ from the category of $\mathbf{G}(\mathbb{C})$-equivariant vector bundles on the compact dual $\mathbb{P}^2_{\mathbb{C}}$ of Sh_K to the category of vector bundles with $\mathbf{G}(\mathbb{A}_f)$-action on the inverse system of Shimura varieties Sh_K. Here $\mathbb{P}^2_{\mathbb{C}} = \mathbf{G}(\mathbb{C})/\mathbf{H}(\mathbb{C})$, where $\mathbf{H}(\mathbb{C})$ is the parabolic group stabilizing the line $\mathbb{C} \cdot {}^t(\delta/2, 0, 1)$ in $\mathbf{G}(\mathbb{C}) = GL_3(\mathbb{C}) \times \mathbb{C}^\times$, and the irreducible $\mathcal{V}$ are associated with highest weight representations of the Levi factor $\mathbf{L}(\mathbb{C})$ of $\mathbf{H}(\mathbb{C})$. It is strightforward to check that $\det \mathcal{P}$ and $\mathcal{L}$ are associated with the same character of $\mathbf{L}(\mathbb{C})$, up to a twist by a character of $\mathbf{G}(\mathbb{C})$, which affects the $\mathbf{G}(\mathbb{A}_f)$-action (hence the normalization of Hecke operators), but not the structure of the line bundles themselves. The functoriality of Harris' construction implies that $\det \mathcal{P}$ and $\mathcal{L}$ are isomorphic also algebraically. ∎

We de not know if $\det \mathcal{P}$ and $\mathcal{L}$ are isomorphic as algebraic line bundles over S. This would be equivalent, by (2.7), to the statement that for every PEL structure $(A, \lambda, \iota, \alpha) \in \mathcal{M}(R)$, for any R_0-algebra R, $\det(H^1_{dR}(A/R)(\Sigma))$ is the trivial line bundle on $Spec(R)$. To our regret, we have not been able to establish this, although a similar statement in the "Siegel case", namely that for any principally polarized abelian scheme (A, λ) over R, $\det H^1_{dR}(A/R)$ is trivial, follows at once from the Hodge filtration (2.4). Our result, however, suffices to guarantee the following corollary, which is all that we will be using in the sequel.

Corollary 2.5. *For any characteristic* p *geometric point* $Spec(k) \to Spec(R_0)$, *we have* $\det \mathcal{P} \simeq \mathcal{L}$ *on* $\bar{S}_k$. *A similar statement holds for morphisms* $SpecW(k) \to Spec(R_0)$.

Proof. Since $\bar{S}$ is a regular scheme, $\det \mathcal{P} \otimes \mathcal{L}^{-1} \simeq \mathcal{O}(D)$ for a Weil divisor D supported on vertical fibers over R_0. Since any connected component Z of $\bar{S}_k$ is irreducible, we can modify D so that D and Z are disjoint, showing that $\det \mathcal{P} \otimes \mathcal{L}^{-1}|_Z$ is trivial. The second claim is proved similarly. ∎

2.1.4. *Modular forms.* Let R be an R_0-algebra. A *modular form* of weight $k \geq 0$ and level $N \geq 3$ defined over R is an element of the finite R-module

$$M_k(N, R) = H^0(\bar{S}_R, \mathcal{L}^k). \tag{2.26}$$

We usually omit the subscript R, remembering that $\bar{S}$ is now to be considered over R. The well-known Koecher principle says that $H^0(\bar{S},\mathcal{L}^k)=H^0(S,\mathcal{L}^k)$. See [Bel], Section 2.2, for an arithmetic proof that works integrally over any R_0-algebra R. A *cusp form* is an element of the space

$$M_k^0(N,R)=H^0(\bar{S},\mathcal{L}^k\otimes\mathcal{O}(C)^\vee). \tag{2.27}$$

As we shall see below (*cf* Corollary 2.10), if $k\geq 3$, there is an isomorphism $\mathcal{L}^k\otimes\mathcal{O}(C)^\vee\simeq\Omega^2_{\bar{S}}\otimes\mathcal{L}^{k-3}$. In particular, cusp forms of weight 3 are "the same" as holomorphic 2-forms on $\bar{S}$.

An alternative definition (à la Katz) of a modular form of weight k and level N defined over R, is as a "rule" f which assigns to every R-scheme T, and every $\underline{A}=(A,\lambda,\iota,\alpha)\in\mathcal{M}(T)$, together with a nowhere vanishing section $\omega\in H^0(T,\omega_{A/T}(\bar{\Sigma}))$, an element $f(\underline{A},\omega)\in H^0(T,\mathcal{O}_T)$ satisfying

- $f(\underline{A},\lambda\omega)=\lambda^{-k}f(\underline{A},\omega)$ for every $\lambda\in H^0(T,\mathcal{O}_T)^\times$
- The "rule" f is compatible with base change T'/T.

Indeed, if f is an element of $M_k(N,R)$, then given such an $\underline{A}$ and ω, the universal property of S produces a unique morphism $\varphi:T\to S$ over R, $\varphi^*\mathcal{A}=A$, and we may let $f(\underline{A},\omega)=\varphi^*f/\omega^k$. Conversely, given such a rule f we may cover S by Zariski open sets T where $\mathcal{L}$ is trivialized, and then the sections $f(\mathcal{A}_T,\omega_T)\omega_T^k$ (ω_T a trivializing section over T) glue to give $f\in M_k(N,R)$. While viewing f as a "rule" rather than a section is a matter of language, it is sometimes more convenient to use this language.

Let $R\to R'$ be a homomorphism of R_0-algebras. Then Bellaiche proved the following theorem ([Bel], 1.1.5).

Theorem 2.6. *If $k\geq 3$ (resp. $k\geq 6$) then $M_k^0(N,R)$ (resp. $M_k(N,R)$) is a locally free finite R-module, and the base-change homomorphism*

$$R'\otimes M_k^0(N,R)\simeq M_k^0(N,R') \tag{2.28}$$

is an isomorphism (resp. base change for $M_k(N,R)$).

Bellaiche considers only weights divisible by 3, but his proofs generalize to all k (*cf* remark on the bottom of p.43 in [Bel]).

Over $\mathbb{C}$, pulling back to $\mathfrak{X}$ and using the trivialization of $\mathcal{L}$ given by the nowhere vanishing section $2\pi i\cdot d\zeta_3$, a modular form of weight k is a collection $(f_j)_{1\leq j\leq m}$ of holomorphic functions on $\mathfrak{X}$ satisfying

$$f_j(\gamma(z,u))=j(\gamma;z,u)^kf_j(z,u)\quad\forall\gamma\in\Gamma_j \tag{2.29}$$

(the Koecher principle means that no condition has to be imposed at the cusps).

2.2. **The Kodaira Spencer isomorphism.** Let $\pi:A\to S$ be an abelian scheme of relative dimension 3, as in the Picard moduli problem. The Gauss-Manin connection

$$\nabla:H^1_{dR}(A/S)\to H^1_{dR}(A/S)\otimes_{\mathcal{O}_S}\Omega^1_S \tag{2.30}$$

defines the Kodaira-Spencer map

$$KS\in Hom_{\mathcal{O}_S}(\omega_A\otimes_{\mathcal{O}_S}\omega_{A^t},\Omega^1_S) \tag{2.31}$$

as the composition of the maps

$$\begin{aligned} \omega_A &= H^0(A, \Omega^1_{A/S}) \hookrightarrow H^1_{dR}(A/S) \xrightarrow{\nabla} H^1_{dR}(A/S) \otimes_{\mathcal{O}_S} \Omega^1_S \\ &\twoheadrightarrow R^1\pi_*\mathcal{O}_A \otimes_{\mathcal{O}_S} \Omega^1_S \simeq \omega^\vee_{A^t} \otimes_{\mathcal{O}_S} \Omega^1_S, \end{aligned} \tag{2.32}$$

and finally using $Hom(L, M^\vee \otimes N) = Hom(L \otimes M, N)$. Recall that if A is endowed with an $\mathcal{O}_\mathcal{K}$ action via ι then the induced action of $a \in \mathcal{O}_K$ on A^t is induced from the action on $Pic(A)$, taking a line bundle $\mathcal{M}$ to $\iota(a)^*\mathcal{M}$. As the polarization $\lambda : A \to A^t$ is $\mathcal{O}_S$-linear but satisfies $\lambda \circ \iota(a) = \iota(a^\rho) \circ \lambda$, it follows that the induced $\mathcal{O}_\mathcal{K}$ action on A^t is of type $(1, 2)$, hence $\omega^\vee_{A^t}$ is of type $(1, 2)$.

Lemma 2.7. *The map KS induces maps*

$$\begin{aligned} KS(\Sigma) &: \omega_A(\Sigma) \to \omega^\vee_{A^t}(\Sigma) \otimes_{\mathcal{O}_S} \Omega^1_S \\ KS(\bar\Sigma) &: \omega_A(\bar\Sigma) \to \omega^\vee_{A^t}(\bar\Sigma) \otimes_{\mathcal{O}_S} \Omega^1_S \end{aligned} \tag{2.33}$$

hence maps, denoted by the same symbols,

$$\begin{aligned} KS(\Sigma) &: \omega_A(\Sigma) \otimes_{\mathcal{O}_S} \omega_{A^t}(\Sigma) \to \Omega^1_S \\ KS(\bar\Sigma) &: \omega_A(\bar\Sigma) \otimes_{\mathcal{O}_S} \omega_{A^t}(\bar\Sigma) \to \Omega^1_S. \end{aligned} \tag{2.34}$$

The CM-type-reversing isomorphism $\lambda^* : \omega_{A^t} \to \omega_A$ induced by the principal polarization satisfies

$$KS(\Sigma)(\lambda^* x \otimes y) = KS(\bar\Sigma)(\lambda^* y \otimes x) \tag{2.35}$$

for all $x \in \omega_{A^t}(\bar\Sigma)$ and $y \in \omega_{A^t}(\Sigma)$.

Proof. The first claim follows from the fact that the Gauss-Manin connection commutes with the endomorphisms, hence preserves CM types. The second claim is a consequence of the symmetry of the polarization, see [Fa-Ch], Prop. 9.1 on p.81 (in the Siegel modular case). ∎

Observe that $\omega_A(\Sigma) \otimes_{\mathcal{O}_S} \omega_{A^t}(\Sigma)$, as well as $\omega_A(\bar\Sigma) \otimes_{\mathcal{O}_S} \omega_{A^t}(\bar\Sigma)$, are vector bundles of rank 2.

Lemma 2.8. *If S is the Picard modular surface and $A = \mathcal{A}$ is the universal abelian variety, then*

$$KS(\Sigma) : \omega_\mathcal{A}(\Sigma) \otimes_{\mathcal{O}_S} \omega_{\mathcal{A}^t}(\Sigma) \to \Omega^1_S \tag{2.36}$$

is an isomorphism, and so is $KS(\bar\Sigma)$.

Proof. This is well-known and follows from deformation theory. For a self-contained proof, see [Bel], Prop. II.2.1.5. ∎

Proposition 2.9. *The Kodaira-Spencer map induces a canonical isomorphism of vector bundles over S*

$$\mathcal{P} \otimes \mathcal{L} \simeq \Omega^1_S. \tag{2.37}$$

Proof. We need only use λ^* to identify $\omega_{\mathcal{A}^t}(\Sigma)$ with $\omega_\mathcal{A}(\bar\Sigma)$. ∎

We refer to Corollary 2.16 for an extension of this result to $\bar S$.

Corollary 2.10. *There is an isomorphism of line bundles $\mathcal{L}^3 \simeq \Omega^2_S$.*

Proof. Take determinants and use $\det \mathcal{P} \simeq \mathcal{L}$. We emphasize that while $KS(\Sigma)$ is canonical, the identification of $\det \mathcal{P}$ with $\mathcal{L}$ depends on a choice, which we shall fix later on once and for all. ∎

The last corollary should be compared to the case of the open modular curve $Y(N)$, where the *square* of the Hodge bundle $\omega_{\mathcal{E}}$ of the universal elliptic curve, becomes isomorphic to $\Omega^1_{Y(N)}$. Over $\mathbb{C}$, as the isomorphism between $\mathcal{L}^3$ and Ω^2_S takes $d\zeta_3^{\otimes 3}$ to a constant multiple of $dz \wedge du$ (see Corollary 2.18), the differential form corresponding to a modular form $(f_j)_{1\le j\le m}$ of weight 3, is (up to a constant) $(f_j(z,u)dz\wedge du)_{1\le j\le m}$.

2.3. **Extensions to the boundary of S.**

2.3.1. *The vector bundles $\mathcal{P}$ and $\mathcal{L}$ over C.* Let $E \subset C_{R_N}$ be a connected component of the cuspidal divisor (over the integral closure R_N of R_0 in the ray class field $\mathcal{K}_N$). As we have seen, E is an elliptic curve with CM by $\mathcal{O}_{\mathcal{K}}$. If the cusp at which E sits is of type $(\mathfrak{a}, B)$ ($\mathfrak{a}$ an ideal of $\mathcal{O}_{\mathcal{K}}$, B an elliptic curve with CM by $\mathcal{O}_{\mathcal{K}}$ defined over R_N) then E maps via an isogeny to $\delta_{\mathcal{K}}\mathfrak{a}\otimes_{\mathcal{O}_{\mathcal{K}}} B^t = Ext^1_{\mathcal{O}_{\mathcal{K}}}(B, \mathfrak{a}\otimes\mathbb{G}_m)$. In particular, E and B are isogenous over $\mathcal{K}_N$.

Consider $\mathcal{G}$, the universal semi-abelian $\mathcal{O}_{\mathcal{K}}$-abelian three-fold of type $(\mathfrak{a}, B)$, over $\delta_{\mathcal{K}}\mathfrak{a}\otimes_{\mathcal{O}_{\mathcal{K}}} B^t$. The semi-abelian scheme $\mathcal{A}$ over E is the pull-back of this $\mathcal{G}$. Clearly, $\omega_{\mathcal{A}/E} = \mathcal{P}\oplus\mathcal{L}$ and $\mathcal{P} = \omega_{\mathcal{A}/E}(\Sigma)$ admits over E a canonical rank 1 sub-bundle $\mathcal{P}_0 = \omega_B$. Since the toric part and the abelian part of $\mathcal{G}$ are constant, $\mathcal{L}, \mathcal{P}_0$ and $\mathcal{P}_\mu = \mathcal{P}/\mathcal{P}_0$ are all trivial line bundles when restricted to E. It can be shown that $\mathcal{P}$ itself is not trivial over E.

2.3.2. *More identities over $\bar{S}$.* We have seen that $\Omega^2_S \simeq \mathcal{L}^3$. For the following proposition, compare [Bel], Lemme II.2.1.7.

Proposition 2.11. *Working over $\mathcal{K}_N$, let E_j ($1\le j\le h$) be the connected components of C. Then*

$$\Omega^2_{\bar{S}} \simeq \mathcal{L}^3 \otimes \bigotimes_{j=1}^{h} \mathcal{O}(E_j)^\vee. \tag{2.38}$$

Proof. By [Hart] II.6.5, $\Omega^2_{\bar{S}} \simeq \mathcal{L}^3 \otimes \bigotimes_{j=1}^h \mathcal{O}(E_j)^{n_j}$ for some integers n_j and we want to show that $n_j = -1$ for all j. By the adjunction formula on the smooth surface $\bar{S}$, if we denote by $K_{\bar{S}}$ a canonical divisor, $\mathcal{O}(K_{\bar{S}}) = \Omega^2_{\bar{S}}$, then

$$0 = 2g_{E_j} - 2 = E_j.(E_j + K_{\bar{S}}). \tag{2.39}$$

We conclude that

$$\deg(\Omega^2_{\bar{S}}|_{E_j}) = E_j.K_{\bar{S}} = -E_j.E_j > 0. \tag{2.40}$$

Here $E_j.E_j < 0$ because E_j can be contracted to a point (Grauert's theorem). As $\mathcal{L}|_{E_j}$ and $\mathcal{O}(E_i)|_{E_j}$ $(i \ne j)$ are trivial we get

$$-E_j.E_j = n_j E_j.E_j, \tag{2.41}$$

hence $n_j = -1$ as desired. ∎

2.4. **Fourier-Jacobi expansions.**

2.4.1. *The infinitesimal retraction.* We follow the arithmetic theory of Fourier-Jacobi expansions as developed in [Bel]. Let $\widehat{S}$ be the formal completion of $\bar{S}$ along the cuspidal divisor $C = \bar{S} - S$. We work over R_0, and denote by $C^{(n)}$ the n-th infinitesimal neighborhood of C in $\bar{S}$. The closed immersion $i : C \hookrightarrow \widehat{S}$ admits a canonical left inverse $r : \widehat{S} \to C$, a *retraction* satisfying $r \circ i = Id_C$. This is not automatic, but rather a consequence of the rigidity of tori, as explained in [Bel], Proposition II.2.4.2. As a corollary, the universal semi-abelian scheme $\mathcal{A}_{/C^{(n)}}$ is the pull-back of $\mathcal{A}_{/C}$ via r. The same therefore holds for $\mathcal{P}$ and $\mathcal{L}$, namely there are natural isomorphisms $r^*(\mathcal{P}|_C) \simeq \mathcal{P}|_{C^{(n)}}$ and $r^*(\mathcal{L}|_C) \simeq \mathcal{L}|_{C^{(n)}}$. As a consequence, the filtration

$$0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0 \tag{2.42}$$

extends canonically to $C^{(n)}$. Since $\mathcal{L}, \mathcal{P}_0$ and $\mathcal{P}_\mu$ are trivial on C, they are trivial over $C^{(n)}$ as well.

2.4.2. *Arithmetic Fourier-Jacobi expansions.* We fix an arbitrary noetherian R_0-algebra R and consider all our schemes over R, without a change in notation. As usual, we let $\mathcal{O}_{\widehat{S}} = \lim_{\leftarrow} \mathcal{O}_{C^{(n)}}$ (a sheaf in the Zariski topology on C). Via r^*, this is a sheaf of $\mathcal{O}_C$-modules. Choose a global nowhere vanishing section $s \in H^0(C, \mathcal{L})$ trivializing $\mathcal{L}$. Such a section is unique up to a unit of R on each connected component of C. This s determines an isomorphism

$$\mathcal{L}^k|_{\widehat{S}} \simeq \mathcal{O}_{\widehat{S}}, \quad f \mapsto f/s^k \tag{2.43}$$

for each k, hence a ring homomorphism

$$FJ : \oplus_{k=0}^{\infty} M_k(N, R) \to H^0(C, \mathcal{O}_{\widehat{S}}). \tag{2.44}$$

We call $FJ(f)$ the (arithmetic) Fourier-Jacobi expansion of f. It depends on s in an obvious way.

To understand the structure of $H^0(C, \mathcal{O}_{\widehat{S}})$ let $\mathcal{I} \subset \mathcal{O}_{\bar{S}}$ be the sheaf of ideals defining C, so that $C^{(n)}$ is defined by $\mathcal{I}^n$. The conormal sheaf $\mathcal{N} = \mathcal{I}/\mathcal{I}^2$ is the restriction $i^*\mathcal{O}_{\bar{S}}(-C)$ of $\mathcal{O}_{\bar{S}}(-C)$ to C. It is an ample invertible sheaf on C, since (over R_N) its degree on each component E_j is $-E_j^2 > 0$.

Now r^* supplies, for every $n \geq 2$, a canonical splitting of

$$0 \to \mathcal{I}/\mathcal{I}^n \to \mathcal{O}_{\bar{S}}/\mathcal{I}^n \mathrel{\overset{\curvearrowleft}{\to}} \mathcal{O}_{\bar{S}}/\mathcal{I} \to 0. \tag{2.45}$$

Inductively, we get a direct sum decomposition

$$\mathcal{O}_{\bar{S}}/\mathcal{I}^n \simeq \bigoplus_{m=0}^{n-1} \mathcal{I}^m/\mathcal{I}^{m+1} \tag{2.46}$$

as $\mathcal{O}_C$-modules, hence, since $\mathcal{I}^m/\mathcal{I}^{m+1} \simeq \mathcal{N}^m$, an isomorphism

$$H^0(C, \mathcal{O}_{C^{(n)}}) \simeq \bigoplus_{m=0}^{n-1} H^0(C, \mathcal{N}^m), \quad f \mapsto \sum_{m=0}^{n} c_m(f). \tag{2.47}$$

This isomorphism respects the multiplicative structure, so is a ring isomorphism. Going to the projective limit, and noting that the $c_m(f)$ are independent of n, we get

$$FJ(f) = \sum_{m=0}^{\infty} c_m(f) \in \prod_{m=0}^{\infty} H^0(C, \mathcal{N}^m). \tag{2.48}$$

2.4.3. *Fourier-Jacobi expansions over* $\mathbb{C}$. Working over $\mathbb{C}$, we shall now relate the infinitesimal retraction r to the geodesic retraction, and the powers of the conormal bundle $\mathcal{N}$ to theta functions. Recall the analytic compactification of X_Γ described in Proposition 1.7. Let E be the connected component of $\bar{X}_\Gamma - X_\Gamma$ corresponding to the standrad cusp c_∞. As before, we denote by $E^{(n)}$ its nth infinitesimal neighborhood. The line bundle $\mathcal{T}|_E$ is just the analytic normal bundle to E, hence we have an isomorphism

$$\mathcal{N}_{an} \simeq \mathcal{T}^\vee \tag{2.49}$$

between the analytification of $\mathcal{N} = \mathcal{I}/\mathcal{I}^2$ and the dual of $\mathcal{T}$.

Lemma 2.12. *The infinitesimal retraction $r : E^{(n)} \to E$ coincides with the map induced by the geodesic retraction (1.15).*

Proof. The meaning of the lemma is this. The infinitesimal retraction induces a map of ringed spaces

$$r_{an} : E^{(n)}_{an} \to E_{an} \tag{2.50}$$

where E_{an} is the analytic space associated to E with its sheaf of analytic functions $\mathcal{O}^{hol}_E$, and $E^{(n)}_{an}$ is the same topological space with the sheaf $\mathcal{O}^{hol}_{\bar{S}}/\mathcal{I}^n_{an}$. The geodesic retraction (sending (z, u) to $u \bmod \Lambda$) is an analytic map $r_{geo} : E_{an}(\varepsilon) \to E_{an}$, where $E_{an}(\varepsilon)$ is our notation for some tubular neighborhood of E_{an} in $\bar{S}_{an}$. On the other hand, there is a canonical map *can* of ringed spaces from $E^{(n)}_{an}$ to $E_{an}(\varepsilon)$. We claim that these three maps satisfy $r_{geo} \circ can = r_{an}$.

To prove the lemma, note that the infinitesimal retraction $r : E^{(n)} \to E$ is uniquely characterized by the fact that the $\mathcal{O}_\mathcal{K}$-semi-abelian variety $\mathcal{A}_x$ at a point x of $E^{(n)}$, obtained as a specialization of the universal family $\mathcal{A}$, is the pull-back of the universal semi-abelian variety at $r(x)$. See [Bel], II.2.4.2. The computations of section 1.6 show that the same is true for the infinitesimal retraction obtained from the geodesic retraction. We conclude that the two retractions agree on the level of "truncated Taylor expansions". ∎

Consider now a modular form of weight k and level N over $\mathbb{C}$, $f \in M_k(N, \mathbb{C})$. Using the trivialization of $\mathcal{L}_{an}$ over the symmetric space $\mathfrak{X}$ given by $2\pi i \cdot d\zeta_3$ as discussed in Section 2.1.2, we identify f with a collection of *functions* f_j on $\mathfrak{X}$, transforming under Γ_j according to the automorphy factor $j(\gamma; z, u)^k$. As usual we look at $\Gamma = \Gamma_1$ only, and at the expansion of $f = f_1$ at the standard cusp c_∞, the other cusps being in principle similar. On the arithmetic FJ expansion side this means that we concentrate on one connected component E of C, which lies on the connected component of $S_\mathbb{C}$ corresponding to $g_1 = 1$. It also means that as the section s used to trivialize $\mathcal{L}$ along E, we must use a section that, analytically, coincides with $2\pi i \cdot d\zeta_3$.

Pulling back the sheaf $\mathcal{N}_{an}$ from $E = \mathbb{C}/\Lambda$ to $\mathbb{C}$, it is clear that $q = q(z) = e^{2\pi i z/M}$ maps, at each $u \in \mathbb{C}$, to a generator of $\mathcal{T}^\vee = \mathcal{N}_{an} = \mathcal{I}_{an}/\mathcal{I}^2_{an}$, and we denote by q^m the corresponding generator of $\mathcal{N}^m_{an} = \mathcal{I}^m_{an}/\mathcal{I}^{m+1}_{an}$. If

$$f(z, u) = \sum_{m=0}^{\infty} \theta_m(u) e^{2\pi i m z/M} = \sum_{m=0}^{\infty} \theta_m(u) q^m \tag{2.51}$$

is the complex analytic Fourier expansion of f at a neighborhood of c_∞, then $c_m(z, u) = \theta_m(u) q^m \in H^0(E, \mathcal{N}^m_{an})$ is just the restriction of the section denoted above by $c_m(f)$ to E. The functions θ_m are classical elliptic theta functions (for the lattice Λ).

2.5. The Gauss-Manin connection in a neighborhood of a cusp.

2.5.1. *A computation of* ∇ *in the complex model.* We shall now compute the Gauss-Manin connection in the complex ball model near the standard cusp c_∞. Recall that we use the coordinates $(z, u, \zeta_1, \zeta_2, \zeta_3)$ as in Section 1.2.4. Here $d\zeta_1$ and $d\zeta_2$ form a basis for $\mathcal{P}$ and $d\zeta_3$ for $\mathcal{L}$. The same coordinates served to define also the semi-abelian variety $\mathcal{G}_u$ (denoted also $\mathcal{A}_u$) over the cuspidal component E at c_∞, *cf* Section 1.6. As explained there (1.69), the projection to the abelian part is given by the coordinate ζ_1 (modulo $\mathcal{O}_\mathcal{K}$), so $d\zeta_1$ is a basis for the sub-line-bundle of $\omega_{\mathcal{A}/E}$ coming from the abelian part, which was denoted $\mathcal{P}_0$. In section 2.4.1 above it was explained how to extend the filtration $\mathcal{P}_0 \subset \mathcal{P}$ canonically to the formal neighborhood $\widehat{S}$ of E using the retraction r, by pulling back from the boundary. It was also noted that complex analytically, the retraction r is the germ of the geodesic retraction introduced earlier. From the analytic description of the degeneration of $\mathcal{A}_{(z,u)}$ along a geodesic, it becomes clear that $\mathcal{P}_0 = r^*(\mathcal{P}_0|_E)$ is just the line bundle $\mathcal{O}_{\widehat{S}} \cdot d\zeta_1 \subset \omega_{\mathcal{A}/\widehat{S}}$. It follows that $\mathcal{P}_\mu = \mathcal{O}_{\widehat{S}} \cdot d\zeta_2 \bmod \mathcal{P}_0$.

We shall now pull back these vector bundles to the ball $\mathfrak{X}$, and compute the Gauss Manin connection ∇ complex analytically on $\omega_{\mathcal{A}/\mathfrak{X}}$. We write $\mathcal{P}_0 = \mathcal{O}_\mathfrak{X} \cdot d\zeta_1$ for $\mathcal{P}_{0,an}$ etc. dropping the decoration *an.* Let

$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \\ u \end{pmatrix}, \alpha_3 = \begin{pmatrix} u \\ -z/\delta \\ z/\delta \end{pmatrix} \tag{2.52}$$

and

$$\alpha_1' = \iota'(\omega_\mathcal{K})\alpha_1 = \begin{pmatrix} 0 \\ \omega_\mathcal{K} \\ \bar{\omega}_\mathcal{K} \end{pmatrix}, \alpha_2' = \iota'(\omega_\mathcal{K})\alpha_2 = \begin{pmatrix} \omega_\mathcal{K} \\ 0 \\ \bar{\omega}_\mathcal{K} u \end{pmatrix}, \tag{2.53}$$

$$\alpha_3' = \iota'(\omega_\mathcal{K})\alpha_3 = \begin{pmatrix} \omega_\mathcal{K} u \\ -\omega_\mathcal{K} z/\delta \\ \bar{\omega}_\mathcal{K} z/\delta \end{pmatrix}.$$

These 6 vectors span $L'_{(z,u)}$ over $\mathbb{Z}$. Let $\beta_1, \ldots, \beta_3'$ be the dual basis to $\{\alpha_1, \ldots, \alpha_3'\}$ in $H^1_{dR}(\mathcal{A}/\mathcal{O}_\mathfrak{X})$, i.e. $\int_{\alpha_1} \beta_1 = 1$ etc. As the periods of the β_i's along the integral homology are constant, the β-basis is horizontal for the Gauss-Manin connection. The first coordinate of the α_i and α_i' gives us

$$d\zeta_1 = 0 \cdot \beta_1 + 1 \cdot \beta_2 + u \cdot \beta_3 + 0 \cdot \beta_1' + \omega_\mathcal{K} \cdot \beta_2' + \omega_\mathcal{K} u \cdot \beta_3', \tag{2.54}$$

and we find that

$$\nabla(d\zeta_1) = (\beta_3 + \omega_\mathcal{K}\beta_3') \otimes du. \tag{2.55}$$

Similarly, we find

$$\begin{aligned} \nabla(d\zeta_2) &= -\delta^{-1}(\beta_3 + \omega_\mathcal{K}\beta_3') \otimes dz \\ \nabla(d\zeta_3) &= (\beta_2 + \bar{\omega}_\mathcal{K}\beta_2') \otimes du + \delta^{-1}(\beta_3 + \bar{\omega}_\mathcal{K}\beta_3') \otimes dz. \end{aligned} \tag{2.56}$$

2.5.2. *A computation of KS in the complex model.* We go on to compute the Kodaira-Spencer map on $\mathcal{P}$, i.e. the map denoted $KS(\Sigma)$. For that we have to take $\nabla(d\zeta_1)$ and $\nabla(d\zeta_2)$ and project them to $R^1\pi_*\mathcal{O}_{\mathcal{A}}(\Sigma)\otimes\Omega^1_{\mathfrak{X}}$. We then pair the result, using the polarization form $\langle,\rangle_\lambda$ on $H^1_{dR}(\mathcal{A})$ (reflecting the isomorphism

$$R^1\pi_*\mathcal{O}_{\mathcal{A}}(\Sigma) = Lie(\mathcal{A}^t)(\Sigma) = \omega^\vee_{\mathcal{A}^t}(\Sigma) \simeq \mathcal{L}^\vee(\rho) \tag{2.57}$$

coming from λ), with $d\zeta_3$.

To perform the computation we need two lemmas

Lemma 2.13. *The Riemann form on L'_x, associated to the polarization λ, is given in the basis $\alpha_1,\alpha_2,\alpha_3,\alpha'_1,\alpha'_2,\alpha'_3$ by the matrix*

$$J = \begin{pmatrix} &&&&&1\\ &&&&-1&\\ &&&1&&\\ &&-1&&&\\ &1&&&&\\ -1&&&&& \end{pmatrix}. \tag{2.58}$$

Proof. This is an easy computation using the transition map T between L and L'_x and the fact that on L the Riemann form is the alternating form $\langle,\rangle = \mathrm{Im}_\delta(,)$. ∎

For the formulation of the next lemma recall that if A is a complex abelian variety, a polarization $\lambda : A\to A^t$ induces an alternating form $\langle,\rangle_\lambda$ on $H^1_{dR}(A)$ as well as a Riemann form on the integral homology $H_1(A,\mathbb{Z})$. We compare the two.

Lemma 2.14. *Let (A,λ) be a principally polarized complex abelian variety. If $\alpha_1,\dots,\alpha_{2g}$ is a symplectic basis for $H_1(A,\mathbb{Z})$ in which the associated Riemann form is given by a matrix J, and $\beta_1,\dots,\beta_{2g}$ is the dual basis of $H^1_{dR}(A)$, then the matrix of the bilinear form $\langle,\rangle_\lambda$ on $H^1_{dR}(A)$ in the basis $\beta_1,\dots,\beta_{2g}$ is $(2\pi i)^{-1}J$.*

Proof. These are essentially Riemann's bilinear relations. For example, if A is the Jacobian of a curve $\mathcal{C}$ and the basis $\alpha_1,\dots,\alpha_{2g}$ has the standard intersection matrix

$$J = \begin{pmatrix} 0 & I\\ -I & 0\end{pmatrix} \tag{2.59}$$

then the lemma follows from the well-known formula for the cup product (ξ,η being differentials of the second kind on $\mathcal{C}$)

$$\xi\cup\eta = \frac{1}{2\pi i}\sum_{i=1}^{g}\left(\int_{\alpha_i}\xi\int_{\alpha_{i+g}}\eta - \int_{\alpha_i}\eta\int_{\alpha_{i+g}}\xi\right). \tag{2.60}$$

∎

Using the two lemmas we get

$$\begin{aligned} KS(d\zeta_1\otimes d\zeta_3) &= \left\langle \beta_3+\omega_{\mathcal{K}}\beta'_3, d\zeta_3\right\rangle_\lambda\cdot du\\ &= -\delta(2\pi i)^{-1}du. \end{aligned} \tag{2.61}$$

Similarly,

$$\begin{aligned} KS(d\zeta_2\otimes d\zeta_3) &= \left\langle -\delta^{-1}(\beta_3+\omega_{\mathcal{K}}\beta'_3), d\zeta_3\right\rangle_\lambda\cdot dz\\ &= (2\pi i)^{-1}dz. \end{aligned} \tag{2.62}$$

We summarize.

Proposition 2.15. *Let $z, u, \zeta_1, \zeta_2, \zeta_3$ be the standard coordinates in a neighborhood of the cusp c_∞. Then, complex analytically, the Kodaira-Spencer isomorphism*

$$KS(\Sigma) : \mathcal{P} \otimes \mathcal{L} \simeq \Omega^1_{\mathfrak{X}} \tag{2.63}$$

is given by the formulae

$$KS(d\zeta_1 \otimes d\zeta_3) = -\delta(2\pi i)^{-1} du, \;\; KS(d\zeta_2 \otimes d\zeta_3) = (2\pi i)^{-1} dz. \tag{2.64}$$

Corollary 2.16. *The Kodaira-Spencer isomorphism $\mathcal{P} \otimes \mathcal{L} \simeq \Omega^1_S$ extends meromorphically over $\bar{S}$. Moreover, in a formal neighborhood $\widehat{S}$ of C, its restriction to the line sub-bundle $\mathcal{P}_0 \otimes \mathcal{L}$ is holomorphic, and on any direct complement of $\mathcal{P}_0 \otimes \mathcal{L}$ in $\mathcal{P} \otimes \mathcal{L}$ it has a simple pole along C.*

Proof. As we have seen, $d\zeta_1 \otimes d\zeta_3$ and $d\zeta_2 \otimes d\zeta_3$ define a basis of $\mathcal{P} \otimes \mathcal{L}$ at the boundary, with $d\zeta_1 \otimes d\zeta_3$ spanning the line sub-bundle $\mathcal{P}_0 \otimes \mathcal{L}$. On the other hand du is holomorphic there, while dz has a simple pole along the boundary. ∎

Corollary 2.17. *The induced map*

$$\kappa : \Omega^1_{\mathfrak{X}} \to \mathcal{P}_\mu \otimes \mathcal{L} \tag{2.65}$$

($\mathcal{P}_\mu = \mathcal{P}/\mathcal{P}_0$) obtained by inverting the isomorphism $KS(\Sigma)$ and dividing $\mathcal{P}$ by $\mathcal{P}_0$ kills du and maps dz to $2\pi i \cdot d\zeta_2 \otimes d\zeta_3$.

Proof. As we have seen, $d\zeta_1$ is a basis for $\mathcal{P}_0$. ∎

Corollary 2.18. *The isomorphism $\mathcal{L}^3 \simeq \Omega^2_S$ maps $d\zeta_3^{\otimes 3}$ to a constant multiple of $dz \wedge du$.*

Proof. The isomorphism $\det \mathcal{P} \simeq \mathcal{L}$ carries $d\zeta_1 \wedge d\zeta_2$ to a constant multiple of $d\zeta_3$, so the corollary follows from (2.64). ∎

2.5.3. *Transferring the results to the algebraic category.* The computations in the analytic category over $\mathfrak{X}$ of course descend (still in the analytic category) to $S_\mathbb{C}$, because they are local in nature. They then hold *a fortiori* in the formal completion $\widehat{S}_\mathbb{C}$ along the cuspidal component E. But the Gauss-Manin and Kodaira-Spencer maps are defined algebraically on S, and both $\Omega^1_{\widehat{S}}$ and $\omega_{\mathcal{A}/\widehat{S}}$ are flat over R_0, so from the validity of the formulae over $\mathbb{C}$ we deduce their validity in $\widehat{S}$ over R_0, provided we identify the differential forms figuring in them (suitably normalized) with elements of $\Omega^1_{\widehat{S}}$ and $\omega_{\mathcal{A}/\widehat{S}}$ defined over R_0. In particular, they hold in the characteristic p fiber as well.

From the relation

$$\frac{dq}{q} = \frac{2\pi i}{M} dz \tag{2.66}$$

we deduce that the map κ has a simple zero along the cuspidal divisor.

Finally, although we have done all the computations at one specific cusp, it is clear that similar computations hold at any other cusp.

2.6. **Fields of rationality.**

2.6.1. *Rationality of local sections of $\mathcal{P}$ and $\mathcal{L}$.* We have compared the arithmetic surface S with the complex analytic surfaces $\Gamma_j\backslash\mathfrak{X}$ $(1 \le j \le m)$, and the compactifications of these two models. We have also compared the universal semi-abelian scheme $\mathcal{A}$ and the automorphic vector bundles $\mathcal{P}$ and $\mathcal{L}$ in both models. In this section we want to compare the local parameters obtained from the two presentations, and settle the question of rationality. To avoid issues of class numbers, we shall work rationally and not integrally.

We shall need to look at local parameters at the cusps, and as the cusps are defined only over $\mathcal{K}_N$, we shall work with $S_{\mathcal{K}_N}$ instead of $S_{\mathcal{K}}$. With a little more care, working with Galois orbits of cusps, we could probably prove rationality over $\mathcal{K}$, but for our purpose $\mathcal{K}_N$ is good enough.

If ξ and η belong to a $\mathcal{K}_N$-module, we write $\xi \sim \eta$ to mean that $\eta = c\xi$ for some $c \in \mathcal{K}_N^\times$.

We begin with the vector bundles $\mathcal{P}$ and $\mathcal{L}$. Over $\mathbb{C}$ they yield analytic vector bundles $\mathcal{P}_{an}$ and $\mathcal{L}_{an}$ on each X_{Γ_j} $(1 \le j \le m)$. Assume for the rest of this section that $j = 1$ and write $\Gamma = \Gamma_1$. Similar results will hold for every j. The vector bundles $\mathcal{P}$ and $\mathcal{L}$ are trivialized over the unit ball $\mathfrak{X}$ by means of the nowhere vanishing sections $d\zeta_3 \in H^0(\mathfrak{X}, \mathcal{L}_{an})$ and $d\zeta_1, d\zeta_2 \in H^0(\mathfrak{X}, \mathcal{P}_{an})$. These sections do not descend to X_Γ, but

$$\sigma_{an} = (d\zeta_1 \wedge d\zeta_2) \otimes d\zeta_3^{-1} \in H^0(X_\Gamma, \det \mathcal{P} \otimes \mathcal{L}^{-1}) \tag{2.67}$$

does, as the factors of automorphy of $d\zeta_1 \wedge d\zeta_2$ and $d\zeta_3$ are the same (*cf* Section 2.1.2). Furthermore, this factor of automorphy (i.e. $j(\gamma; z, u)$) is trivial on Γ_{cusp}, the stabilizer of c_∞ in Γ, so $d\zeta_1 \wedge d\zeta_2$ and $d\zeta_3$ define sections of $\det \mathcal{P}$ and $\mathcal{L}$ on $\widehat{S}_{\mathbb{C}}$, the formal completion of $\bar{S}_{\mathbb{C}}$ along the cuspidal divisor $E_c = p^{-1}(c_\infty) \subset \bar{S}_{\mathbb{C}}$. We have noted already that along $E_c, \mathcal{P}$ has a canonical filtration

$$0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0 \tag{2.68}$$

and that $d\zeta_1$ is a generator of $\mathcal{P}_0$. (Compare (1.63) and (1.71) and note that the projection to $\mathbb{C}/\mathcal{O}_{\mathcal{K}} = B(\mathbb{C})$ is via the coordinate ζ_1, so $d\zeta_1$ is a generator of $\mathcal{P}_0|_{E_c} = \omega_B$). As we have shown in Section 2.4.1, this filtration extends to the formal neighborhood $\widehat{S}_{\mathbb{C}}$ of E_c. The vector bundles $\mathcal{P}$ and $\mathcal{L}$, as well as the filtration on $\mathcal{P}$ over $\widehat{S}_{\mathbb{C}}$, are defined over $\mathcal{K}_N$. It makes sense therefore to ask if certain sections are $\mathcal{K}_N$-rational. Recall that the cusp c_∞ is of type $(\mathcal{O}_{\mathcal{K}}, \mathcal{O}_{\mathcal{K}})$.

Proposition 2.19. *(i)* $2\pi i \cdot d\zeta_3 \in H^0(\widehat{S}_{\mathcal{K}_N}, \mathcal{L})$. *In other words, this section is* $\mathcal{K}_N$*-rational.*

(ii) Similarly $2\pi i \cdot d\zeta_2$ *projects (modulo* $\mathcal{P}_0$*) to a* $\mathcal{K}_N$*-rational section of* $\mathcal{P}_\mu$.

(iii) Let B *be the elliptic curve over* $\mathcal{K}_N$ *associated with the cusp* c_∞ *as in Section 1.5.1. Let* $\Omega_B \in \mathbb{C}^\times$ *be a period of a basis* ω *of* $\omega_B = H^0(B, \Omega^1_{B/\mathcal{K}_N})$ *(i.e. the lattice of periods of* ω *is* $\Omega_B \cdot \mathcal{O}_{\mathcal{K}}$*). This* Ω_B *is well-defined up to an element of* $\mathcal{K}_N^\times$*. Then* $\Omega_B \cdot d\zeta_1 \in H^0(\widehat{S}_{\mathcal{K}_N}, \mathcal{P}_0)$ *is* $\mathcal{K}_N$*-rational.*

Proof. Let E be the component of $C_{\mathcal{K}_N}$ which over $\mathbb{C}$ becomes E_c. Let $\mathcal{G}$ be the universal semi-abelian scheme over E. Then $\mathcal{G}$ is a semi-abelian scheme which is an extension of $B \times_{\mathcal{K}_N} E$ by the torus $(\mathcal{O}_{\mathcal{K}} \otimes \mathbb{G}_{m,\mathcal{K}_N}) \times_{\mathcal{K}_N} E$. At any point $u \in E(\mathbb{C})$ we have the analytic model $\mathcal{G}_u$ (1.69) for the fiber of $\mathcal{G}$ at u, but the abelian part and the toric part are constant. Over E the line bundle $\mathcal{P}_0$ is (by definition) $\omega_{B \times E/E}$. As the lattice of periods of a suitable $\mathcal{K}_N$-rational differential is $\Omega_B \cdot \mathcal{O}_{\mathcal{K}}$, while the lattice of periods of $d\zeta_1$ is $\mathcal{O}_{\mathcal{K}}$, part (iii) follows. For parts (i) and (ii) observe that the toric part of $\mathcal{G}$ is in fact defined over $\mathcal{K}$, and that $e^*_{\mathcal{O}_\mathcal{K}}$ maps the cotangent space of $\mathcal{O}_\mathcal{K} \otimes \mathbb{G}_{m,\mathcal{K}}$ isomorphically to the $\mathcal{K}$-span of $2\pi i d\zeta_2$ and $2\pi i d\zeta_3$. ∎

Corollary 2.20. *$\Omega_B \cdot \sigma_{an}$ is a nowhere vanishing global section of* $\det \mathcal{P} \otimes \mathcal{L}^{-1}$ *over S_Γ, rational over $\mathcal{K}_N$.*

Proof. Recall that we denote by S_Γ the connected component of $S_{\mathcal{K}_N}$ whose associated analytic space is the complex manifold X_Γ. We have seen that as an analytic section $\Omega_B \cdot \sigma_{an}$ descends to X_Γ and extends to the smooth compactification $\bar{X}_\Gamma$. By GAGA, it is algebraic. Since $\bar{X}_\Gamma$ is connected, to check its field of definition, it is enough to consider it at one of the cusps. By the Proposition, its restriction to the formal neighborhood of E_c ($c = c_\infty$) is defined over $\mathcal{K}_N$. ∎

The complex periods Ω_B (and their powers) appear as the transcendental parts of special values of L-functions associated with Grossencharacters of $\mathcal{K}$. They are therefore instrumental in the construction of p-adic L functions on $\mathcal{K}$. We expect them to appear in the p-adic interpolation of holomorphic Eisenstein series on the group $\mathbf{G}$, much as powers of $2\pi i$ (values of $\zeta(2k)$) appear in the p-adic interpolation of Eisenstein series on $GL_2(\mathbb{Q})$.

2.6.2. *Rationality of local parameters at the cusps.* We keep the assumptions and the notation of the previous section. Analytically, neighborhoods of E_{c_∞} were described in Section 1.4.1 with the aid of the parameters (z, u). Let $\widehat{S}$ denote the formal completion of $\bar{S}_{\mathcal{K}_N}$ along E. Let $r : \widehat{S} \to E$ be the infinitesimal retraction discussed in Section 2.4.1. If $i : E \hookrightarrow \widehat{S}$ is the closed embedding then $r \circ i = Id_E$. If $\mathcal{I}$ is the sheaf of definition of E, then $\mathcal{N} = \mathcal{I}/\mathcal{I}^2$ is the conormal bundle to E, hence its analytification is the dual of the line bundle $\mathcal{T}$,

$$\mathcal{N}_{an} = \mathcal{T}^\vee. \tag{2.69}$$

Consider $r^*\mathcal{N}$ on $\widehat{S}$. The retraction allows us to split the exact sequence

$$0 \to \mathcal{N} \to i^*\Omega^1_{\widehat{S}} \to \Omega^1_E \to 0 \tag{2.70}$$

using $\Omega^1_E = i^*r^*\Omega^1_E \subset i^*\Omega^1_{\widehat{S}}$. Thus $i^*\Omega^1_{\widehat{S}} = \mathcal{N} \times \Omega^1_E$. The map $i \circ r : \widehat{S} \to \widehat{S}$ induces a sheaf homomorphism $r^*i^*\Omega^1_{\widehat{S}} \to \Omega^1_{\widehat{S}}$, which becomes the identity if we restrict it to E (i.e. follow it with i^*). By Nakayama's lemma, it is an isomorphism. It follows that

$$\Omega^1_{\widehat{S}} = r^*i^*\Omega^1_{\widehat{S}} = r^*\mathcal{N} \times r^*\Omega^1_E. \tag{2.71}$$

Let $x \in E$ and represent it by $u \in \mathbb{C}$ (modulo Λ). Then $q = e^{2\pi i z/M}$, where M is the width of the cusp (1.54), is a local analytic parameter on a classical neighborhood U_x of x which vanishes to first order along E. Note that q depends on the choice of u (see Remark below). It follows that dq, the image of q in $\mathcal{I}_{an}/\mathcal{I}^2_{an}$, is a basis of $\mathcal{N}_{an}$ (on $U_x \cap E$). But

$$2\pi i \cdot dz = M\frac{dq}{q} \tag{2.72}$$

is independent of u (see (1.55)), so represents a global meromorphic section of $r^*\mathcal{N}_{an}$, with a simple pole along $E \subset \widehat{S}_\mathbb{C}$. By GAGA, this section is (meromorphic) algebraic.

Proposition 2.21. *(i) The section $2\pi i\cdot dz$ is $\mathcal{K}_N$-rational, i.e. it is the analytification of a section of $r^*\mathcal{N}$. (ii) The section $\Omega_B\cdot du$ is $\mathcal{K}_N$-rational, i.e. belongs to $H^0(E,\Omega^1_{E/\mathcal{K}_N})$.*

Proof. The proof relies on the Kodaira-Spencer isomorphism $KS(\Sigma)$ (2.64), which is a $\mathcal{K}_N$-rational (even $\mathcal{K}$-rational) algebraic isomorphism between $\mathcal{P}\otimes\mathcal{L}$ and Ω^1_S. As we have shown, it extends to a meromorphic homomorphism from $\mathcal{P}\otimes\mathcal{L}$ to $\Omega^1_{\bar{S}}$ over $\bar{S}$. Over $\widehat{S}$ it induces an isomorphism of $\mathcal{P}_0\otimes\mathcal{L}$ onto $r^*\Omega^1_E\subset\Omega^1_{\widehat{S}}$ carrying the $\mathcal{K}_N$-rational section $\Omega_B d\zeta_1\otimes 2\pi i d\zeta_3$ to $-\Omega_B\delta\cdot du$, proving part (ii) of the proposition. It also carries $2\pi i d\zeta_2\otimes 2\pi i d\zeta_3$ to $2\pi i dz$, but the latter is only meromorphic. We may summarize the situation over $\widehat{S}$ by the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \to & \widehat{\mathcal{I}}\otimes\mathcal{P}_0\otimes\mathcal{L} & \to & \widehat{\mathcal{I}}\otimes\mathcal{P}\otimes\mathcal{L} & \to & \widehat{\mathcal{I}}\otimes\mathcal{P}_\mu\otimes\mathcal{L} & \to & 0 \\
 & & \downarrow & & \downarrow KS(\Sigma) & & \downarrow & & \\
0 & \to & r^*\Omega^1_E & \to & \Omega^1_{\widehat{S}} & \to & r^*\mathcal{N} & \to & 0
\end{array}
.\tag{2.73}
$$

Let h be a $\mathcal{K}_N$-rational local equation of E, i.e. a $\mathcal{K}_N$-rational section of $\mathcal{I}$ in some Zariski open U intersecting E non-trivially, vanishing to first order along $E\cap U$. The differential $\eta=h\cdot(2\pi i dz)$ is regular on U, and to prove that it is $\mathcal{K}_N$-rational we may restrict it to $\widehat{S}$ and check rationality there. But in $\widehat{S}$ we have a $\mathcal{K}_N$-rational product decomposition $\Omega^1_{\widehat{S}}=r^*\mathcal{N}\times r^*\Omega^1_E$ and the projection of η to the second factor is 0, so it is enough to prove rationality of its projection to $r^*\mathcal{N}$. This projection is the image, under $KS(\Sigma)$, of $h\cdot(2\pi i d\zeta_2\otimes 2\pi i d\zeta_3 \bmod \mathcal{P}_0\otimes\mathcal{L})$, so our assertion follows from parts (i) and (ii) of the previous proposition. This proves that η, hence $h^{-1}\eta=2\pi i dz$ is a $\mathcal{K}_N$-rational differential. An alternative proof of part (ii) is to note that E is isogenous over $\mathcal{K}_N$ to B, so up to a $\mathcal{K}_N$-multiple has the same period. ∎

Remark 2.1. *Unlike $2\pi i dz=Mdq/q$, the parameter q is not a well-defined parameter at x, and depends not only on x, but also on the point u used to uniformize it. If we change u to $u+s$ $(s\in\Lambda)$ then q is multiplied by the factor $e^{2\pi i\delta\bar{s}(u+s/2)M}$, so although $\mathcal{O}^{hol}_{\bar{S}_{\mathbb{C}},x}\subset\widehat{\mathcal{O}}_{\bar{S}_{\mathbb{C}},x}$ and analytic parameters may be considered as formal parameters, the question whether q itself is $\mathcal{K}_N$-rational is not well-defined (in sharp contrast to the case of modular curves!).*

2.6.3. *Normalizing the isomorphism* $\det\mathcal{P}\simeq\mathcal{L}$. Let us fix a nowhere vanishing section

$$\sigma\in H^0(S_{\mathcal{K}},\det\mathcal{P}\otimes\mathcal{L}^{-1}).\tag{2.74}$$

This section is determined up to $\mathcal{K}^\times$. From now on we shall use this section to identify $\det\mathcal{P}$ with $\mathcal{L}$ whenever such an identification is needed. From Corollary 2.20 we deduce that when we base change to $\mathbb{C}$, on each connected component X_Γ

$$\sigma\sim\Omega_B\cdot\sigma_{an}.\tag{2.75}$$

3. Picard moduli schemes modulo an inert prime p

3.1. **The stratification.**

3.1.1. *The three strata.* Fix a prime $p > 2$ which is inert in $\mathcal{K}$ and relatively prime to N. Then $R_0/pR_0 = \mathbb{F}_{p^2}$ and we consider the characteristic p fiber of the smooth compactification $\bar{S}$ of S,

$$\bar{S} \times_{Spec(R_0)} Spec(R_0/pR_0). \tag{3.1}$$

From now on we shall write $\bar{S}$ to denote this scheme, rather than the original one over R_0. We let $\mathcal{A}$, as before, stand for the universal semi-abelian variety over $\bar{S}$. The structure of $\bar{S}$ has been worked out by Vollaard [V]. We record her main results. Recall that an abelian variety over a field of characteristic p is called supersingular if the Newton polygon of its p-divisible group is of constant slope $1/2$. It is called superspecial if it is isomorphic to a product of supersingular elliptic curves.

Theorem 3.1. *(i) There exists a closed reduced* 1*-dimensional subscheme* $S_{ss} \subset \bar{S}$ *(the supersingular locus), disjoint from the cuspidal divisor (i.e. contained in* S*), which is uniquely characterized by the fact that for every geometric point* x *of* S*, the abelian variety* $\mathcal{A}_x$ *is supersingular if and only if* $x \in S_{ss}$*.*

(ii) Let S_{ssp} *be the singular locus on* S_{ss}*. Then* x *lies in* S_{ssp} *if and only if* $\mathcal{A}_x$ *is superspecial. If* $s_0 \in S_{ssp}$ *then* s_0 *is rational over* $\mathbb{F}_{p^2}$ *and*

$$\widehat{\mathcal{O}}_{S,s_0} \simeq \mathbb{F}_{p^2}[[u,v]]/(u^{p+1}+v^{p+1}). \tag{3.2}$$

(iii) Assume that N *is large enough (depending on* p*). Then the irreducible components of* S_{ss} *are rational over* $\mathbb{F}_{p^2}$*, nonsingular, and in fact are all isomorphic to the Fermat curve*

$$x^{p+1}+y^{p+1}+z^{p+1}=0. \tag{3.3}$$

There are p^3+1 *points of* S_{ssp} *on each irreducible component, and through each such point pass* $p+1$ *irreducible components. Any two irreducible components are either disjoint or intersect transversally at a unique point.*

We call $\bar{S}_\mu = \bar{S} - S_{ss}$ (or $S_\mu = \bar{S}_\mu \cap S$) the *$\mu$-ordinary* or *generic* locus, $S_{gss} = S_{ss} - S_{ssp}$ the *general supersingular locus*, and S_{ssp} the *superspecial locus*. Then $\bar{S} = \bar{S}_\mu \cup S_{gss} \cup S_{ssp}$ is a stratification. The three strata are of dimensions 2,1, and 0 respectively, the closure of each stratum contains the lower dimensional ones, and each of the three is open in its closure.

3.1.2. *The p-divisible groups.* Bültel and Wedhorn [Bu-We] and Vollaard describe the p-divisible group $\mathcal{A}_x(p)$ of the abelian variety $\mathcal{A}_x$ for x in the various strata. Let $x : Spec(k) \to S_\mu$ be a geometric point (k an algebraically closed field of characteristic p) and $A = \mathcal{A}_x$ (an abelian variety over k). Then the p-divisible group $A(p)$ has a three-step $\mathcal{O}_\mathcal{K}$-invariant filtration

$$0 \subset Fil^2A(p) \subset Fil^1A(p) \subset Fil^0A(p) = A(p) \tag{3.4}$$

such that $gr^2 = Fil^2$ is a group of multiplicative type (connected, with étale dual), $gr^1 = Fil^1/Fil^2$ is connected with a connected dual, and $gr^0 = Fil^0/Fil^1$ is étale. Each of the three graded pieces is a p-divisible group of height 2 ($\mathcal{O}_\mathcal{K}$-height 1). As every p-divisible group over an algebraically closed field of characteristic p splits canonically into a product of a group of multiplicative type, a group which is connected with a connected dual, and an étale group, the above filtration splits.

We have

$$\begin{aligned} gr^2A(p) &\simeq \mathfrak{a}\otimes\mu_{p^\infty} \\ gr^1A(p) &\simeq \mathfrak{G} \\ gr^0A(p) &\simeq \mathfrak{c}\otimes\mathbb{Q}_p/\mathbb{Z}_p \end{aligned} \tag{3.5}$$

where $\mathfrak{a}$ and $\mathfrak{c}$ are ideals of $\mathcal{O}_{\mathcal{K}}$ and $\mathfrak{G}$ is the p-divisible group of a supersingular elliptic curve over $\bar{\mathbb{F}}_p$ (the group denoted by $G_{1,1}$ in the Manin-Dieudonné classification [Dem]). Although, up to isomorphism, it is possible to substitute $\mathcal{O}_{\mathcal{K}}$ for $\mathfrak{a}$ or $\mathfrak{c}$, it is sometimes more natural to allow this extra freedom in notation.

Geometric points of S_{ss} classify abelian varieties $A = \mathcal{A}_x$ for which $A(p)$ is isogenous to $\mathfrak{G}^3$, and such a point x is superspecial if and only if $A(p)$ is isomorphic to $\mathfrak{G}^3$.

The a-number of A is the dimension of the k-linear space $Hom(\alpha_p, A[p])$. It is 1 if $x \in S_\mu \cup S_{gss}$ and 3 if $x \in S_{ssp.}$ The stratification which we have described coincides, in our simple case, with the Ekedahl-Oort stratification [Oo],[Mo],[We].

3.2. **New relations between automorphic vector bundles in characteristic p.**

3.2.1. *The line bundles $\mathcal{P}_0$ and $\mathcal{P}_\mu$ over $\bar{S}_\mu$.* Consider the universal semi-abelian variety $\mathcal{A}$ over the Zariski open set $\bar{S}_\mu$. As we have seen in Section 2.3.1, over the cuspidal divisor C, $\mathcal{P} = \omega_{\mathcal{A}}(\Sigma)$ admits a canonical filtration

$$0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0 \tag{3.6}$$

where $\mathcal{P}_0$ is the cotangent space to the abelian part of $\mathcal{A}$, and $\mathcal{P}_\mu$ the Σ-component of the cotangent space to the toric part of $\mathcal{A}$. This filtration exists already in characteristic 0, but when we reduce modulo p it extends, as we now show, to the whole of $\bar{S}_\mu$.

Let $\mathcal{A}[p]^0$ be the connected part of the subgroup scheme $\mathcal{A}[p]$. Then $\mathcal{A}[p]^0$ is finite flat over $\bar{S}_\mu$ of rank p^4. (It is clearly flat and quasi-finite, and the fiber rank can be computed separately on C and on S_μ. Since the rank is constant, the morphism to $\bar{S}_\mu$ is actually finite, *cf.* [De-Ra], Lemme 1.19.) Let

$$0 \subset \mathcal{A}[p]^\mu \subset \mathcal{A}[p]^0 \tag{3.7}$$

be the maximal subgroup-scheme of multiplicative type. Since at every geometric point of $\bar{S}_\mu$, $\mathcal{A}[p]^\mu$ is of rank p^2, this subgroup is also finite flat over $\bar{S}_\mu$. It is also $\mathcal{O}_{\mathcal{K}}$-invariant. Over the cuspidal divisor C, $\mathcal{A}[p]^\mu$ is the p-torsion in the toric part of $\mathcal{A}$, and over S_μ

$$\mathcal{A}[p]^\mu = \mathcal{A}[p] \cap Fil^2\mathcal{A}(p). \tag{3.8}$$

As $\omega_{\mathcal{A}}$ is killed by p, we have $\omega_{\mathcal{A}} = \omega_{\mathcal{A}[p]} = \omega_{\mathcal{A}[p]^0}$. Let $\omega_{\mathcal{A}}^\mu = \omega_{\mathcal{A}[p]^\mu}$, a rank-2 $\mathcal{O}_{\mathcal{K}}$-vector bundle of type $(1,1)$. The kernel of $\omega_{\mathcal{A}[p]^0} \to \omega_{\mathcal{A}[p]^\mu}$ is then a line bundle $\mathcal{P}_0$ of type $(1,0)$ and we get the short exact sequence

$$0 \to \mathcal{P}_0 \to \omega_{\mathcal{A}} \to \omega_{\mathcal{A}}^\mu \to 0 \tag{3.9}$$

over the whole of $\bar{S}_\mu$. Decomposing according to types and setting $\mathcal{P}_\mu = \omega_{\mathcal{A}}^\mu(\Sigma)$, we get the desired filtration.

3.2.2. *Frobenius and Verschiebung.* Write $\mathcal{G}$ for $\mathcal{A}[p]^0$ (a finite flat $\mathcal{O}_\mathcal{K}$-group scheme over $\bar{S}$) and $\mathcal{G}^{(p)}$ for the base-change of $\mathcal{G}$ with respect to the absolute Frobenius morphism of degree p. In other words, if we denote by ϕ the homomorphism $x \mapsto x^p$ (of any $\mathbb{F}_p$-algebra), and by $\Phi : \bar{S} \to \bar{S}$ the corresponding map of schemes, then

$$\mathcal{G}^{(p)} = \bar{S} \times_{\Phi, \bar{S}} \mathcal{G}. \tag{3.10}$$

The *relative Frobenius* is an $\mathcal{O}_{\bar{S}}$-linear homomorphism $Frob_\mathcal{G} : \mathcal{G} \to \mathcal{G}^{(p)}$, characterized by the fact that $pr_2 \circ Frob_\mathcal{G}$ is the absolute Frobenius morphism of $\mathcal{G}$. The *relative Verschiebung* is an $\mathcal{O}_S$-linear homomorphism $Ver_\mathcal{G} : \mathcal{G}^{(p)} \to \mathcal{G}$. Under Cartier duality $Ver_\mathcal{G}$ is dual to $Frob_{\mathcal{G}^D}$, where we denote by $\mathcal{G}^D$ the Cartier dual of $\mathcal{G}$.

Recall that $\omega_\mathcal{G} = \omega_\mathcal{A}$, and similarly $\omega_{\mathcal{G}^{(p)}} = \omega_{\mathcal{A}^{(p)}} = \omega_\mathcal{A}^{(p)} = \mathcal{O}_{\bar{S}} \otimes_{\phi, \mathcal{O}_{\bar{S}}} \omega_\mathcal{A}$. The morphism $Ver_\mathcal{G} : \mathcal{G}^{(p)} \to \mathcal{G}$ induces a homomorphism of vector bundles $V : \omega_\mathcal{A} \to \omega_\mathcal{A}^{(p)}$, and taking Σ-components we get

$$V : \mathcal{P} = \omega_\mathcal{A}(\Sigma) \to \omega_\mathcal{A}^{(p)}(\Sigma) = \omega_\mathcal{A}(\bar{\Sigma})^{(p)} = \mathcal{L}^{(p)}. \tag{3.11}$$

Over $\bar{S}_\mu$ this map fits in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \leftarrow & \omega_\mathcal{A}^\mu(\Sigma) = \mathcal{P}_\mu & \leftarrow & \mathcal{P} & \leftarrow & \mathcal{P}_0 & \leftarrow & 0 \\ & & \downarrow \simeq & & \downarrow V & & \downarrow & & \\ 0 & \leftarrow & (\omega_\mathcal{A}^\mu(\bar{\Sigma}))^{(p)} & \leftarrow & \mathcal{L}^{(p)} & \leftarrow & 0 & \leftarrow & 0 \end{array}. \tag{3.12}$$

The right vertical arrow is 0 since V kills $\mathcal{P}_0$, as $\mathfrak{G}$ is of local-local type. The left vertical map is an isomorphism, since Ver is an isomorphism on p-divisible groups of multiplicative type. We conclude that

$$\mathcal{P}_0 = \ker(V : \mathcal{P} \to \mathcal{L}^{(p)}). \tag{3.13}$$

3.2.3. *Relations between $\mathcal{P}_0, \mathcal{P}_\mu$ and $\mathcal{L}$ over $\bar{S}_\mu$.* We first recall a general lemma.

Lemma 3.2. *Let $\mathcal{M}$ be a line bundle over a scheme S in characteristic p. Let $\Phi : S \to S$ be the absolute Frobenius and $\mathcal{M}^{(p)} = \Phi^*\mathcal{M}$. Then the map $\mathcal{M}^{(p)} \to \mathcal{M}^p$*

$$a \otimes m \mapsto a \cdot m \otimes \cdots \otimes m \tag{3.14}$$

is an isomorphism of line bundles over S.

Since $\mathcal{L}^{(p)} \simeq \mathcal{L}^p$ by the lemma, we have

$$\mathcal{L}^p \simeq \mathcal{P}/\mathcal{P}_0 = \mathcal{P}_\mu. \tag{3.15}$$

Finally, from $\mathcal{P}_0 \otimes \mathcal{P}_\mu \simeq \det \mathcal{P} \simeq \mathcal{L}$ we get

$$\mathcal{P}_0 \simeq \mathcal{L}^{1-p}. \tag{3.16}$$

We have proved:

Proposition 3.3. *Over $\bar{S}_\mu$, $\mathcal{P}_\mu \simeq \mathcal{L}^p$ and $\mathcal{P}_0 \simeq \mathcal{L}^{1-p}$.*

In the same vein we get a commutative diagram for the $\bar{\Sigma}$ parts

$$\begin{array}{ccccccccc} 0 & \leftarrow & \omega_\mathcal{A}^\mu(\bar{\Sigma}) & \leftarrow & \mathcal{L} & \leftarrow & 0 & \leftarrow & 0 \\ & & \downarrow \simeq & & \downarrow V & & \downarrow & & \\ 0 & \leftarrow & (\omega_\mathcal{A}^\mu(\Sigma))^{(p)} & \leftarrow & \mathcal{P}^{(p)} & \leftarrow & \mathcal{P}_0^{(p)} & \leftarrow & 0 \end{array} \tag{3.17}$$

and deduce that V is injective on $\mathcal{L}$ and

$$\mathcal{P}^{(p)} = \mathcal{P}_0^{(p)} \oplus V(\mathcal{L}). \tag{3.18}$$

Thus over $\bar{S}_\mu$, $\mathcal{P}$ has a canonical filtration by $\mathcal{P}_0$, but the induced filtration on $\mathcal{P}^{(p)}$ already splits as a direct sum.

Remark 3.1. *Working over* $\bar{\mathbb{F}}_p$, *and restricting attention to a connected component* E *of* C, $\mathcal{P}|_E$ *is a* non-split *extension of* $\mathcal{P}_\mu$ *by* $\mathcal{P}_0$. *However, both* $\mathcal{P}_\mu$ *and* $\mathcal{P}_0$ *are trivial on* E, *so the extension is described by a non-zero class* $\xi \in H^1(E, \mathcal{O}_E)$. *The extension* $\mathcal{P}^{(p)}|_E$ *is then described by* $\xi^{(p)}$. *The semilinear map* $\xi \mapsto \xi^{(p)}$ *is the Cartier-Manin operator, and since* E *is a supersingular elliptic curve,* $\xi^{(p)} = 0$ *and* $\mathcal{P}^{(p)}|_E$ *splits. Thus at least over* C, *the splitting of* $\mathcal{P}^{(p)}$ *is consistent with what we know so far.*

Since V induces an isomorphism of $\mathcal{L}$ onto $\mathcal{P}^{(p)}/\mathcal{P}_0^{(p)} \simeq (\mathcal{P}/\mathcal{P}_0)^p$ and $\mathcal{P}/\mathcal{P}_0 \simeq \mathcal{L}^p$ we conclude that over $\bar{S}_\mu$, $\mathcal{L} \simeq \mathcal{L}^{p^2}$. In the next section we realize this isomorphism via the Hasse invariant. Combining what was proved so far we easily get the following.

Proposition 3.4. *Over* $\bar{S}_\mu$, $\mathcal{L}^{p^2} \simeq \mathcal{L}$. *For* $k \geq 1$ *odd,* $\mathcal{P}^{(p^k)} \simeq \mathcal{L}^{p-1} \oplus \mathcal{L}$. *For* $k \geq 2$ *even,* $\mathcal{P}^{(p^k)} \simeq \mathcal{L}^{1-p} \oplus \mathcal{L}^p$, *but for* $k = 0$ *we only have an exact sequence*

$$0 \to \mathcal{L}^{1-p} \to \mathcal{P} \to \mathcal{L}^p \to 0. \tag{3.19}$$

Corollary 3.5. *Over* $\bar{S}_\mu$, $\mathcal{L}^{p^2-1}, \mathcal{P}_\mu^{p^2-1}$ *and* $\mathcal{P}_0^{p+1}$ *are trivial line bundles.*

3.2.4. *Extending the filtration on* $\mathcal{P}$ *over* S_{gss}. In order to determine to what extent the filtration on $\mathcal{P}$ and the relation between $\mathcal{L}$ and the two graded pieces of the filtration extend into the supersingular locus, we have to employ Dieudonné theory.

Proposition 3.6. *Let* $\mathcal{P}_0 = \ker(V : \mathcal{P} \to \mathcal{L}^{(p)})$. *Then over the whole of* $\bar{S} - S_{ssp}$, $V(\mathcal{P}) = \mathcal{L}^{(p)}$ *and* $\mathcal{P}_0$ *is a rank 1 submodule. Let* $\mathcal{P}_\mu = \mathcal{P}/\mathcal{P}_0$. *Then* $\mathcal{P}_\mu \simeq \mathcal{L}^p$, $\mathcal{P}_0 \simeq \mathcal{L}^{1-p}$ *and the filtration (3.19) is valid there.*

Proof. Everything is a formal consequence of the fact that V maps $\mathcal{P}$ *onto* $\mathcal{L}^{(p)}$, and the relation $\det \mathcal{P} \simeq \mathcal{L}$. Over $\bar{S}_\mu$ the proposition was verified in the previous subsection, so it is enough to prove that $V(\mathcal{P}) = \mathcal{L}^{(p)}$ in the fiber of any geometric point $x \in S_{gss}(k)$ (k algebraically closed). We use the description of $H^1_{dR}(\mathcal{A}_x/k)$ given in Lemma 4.9 below, due to Bültel and Wedhorn. In the notation of that lemma, $\mathcal{P}_x$ is spanned over k by e_1 and e_2 and $\mathcal{L}_x$ by f_3, while $V(e_1) = 0$, $V(e_2) = f_3^{(p)}$. This concludes the proof. ∎

Proposition 3.7. *Over the whole of* $\bar{S} - S_{ssp}$, V *maps* $\mathcal{L}$ *injectively onto a sub-line-bundle of* $\mathcal{P}^{(p)}$.

Proof. Once again, we know it already over $\bar{S}_\mu$, and it remains to check the assertion fiber-wise on S_{gss}. We refer again to Lemma 4.9, and find that $V(f_3) = e_1^{(p)}$, which proves our claim. ∎

The emerging picture is this: Outside the superspecial points, V maps $\mathcal{L}$ injectively onto a sub-line-bundle of $\mathcal{P}^{(p)}$, and $V^{(p)}$ maps $\mathcal{P}^{(p)}$ surjectively onto $\mathcal{L}^{(p^2)}$. However, the line $V(\mathcal{L})$ coincides with the line $\mathcal{P}_0^{(p)} = \ker(V^{(p)})$ only on the general supersingular locus, while on its complement $\bar{S}_\mu$ the two lines make up a frame for $\mathcal{P}^{(p)}$ (3.18). One can be a little more precise. The equation

$$V(\mathcal{L}) = \mathcal{P}_0^{(p)} \tag{3.20}$$

(i.e. $V^{(p)} \circ V(\mathcal{L}) = 0$) is the *defining equation* of S_{ss} in the sense that when expressed in local coordinates it defines S_{ss} with its reduced subscheme structure. See Proposition 3.9 below.

For a superspecial x, $\mathcal{A}_x$ is isomorphic to a product of three supersingular elliptic curves, so V vanishes on the whole of $\omega_{\mathcal{A}_x}$. The analysis of the last paragraph breaks up. To complete the picture, we shall now prove that there does not exist any way to extend the filtration (3.19) across such an x.

Proposition 3.8. *It is impossible to extend the filtration* $0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0$ *along* S_{ss} *in a neighborhood of a superspecial point* x.

Proof. To simplify the computations we change the base field to $\bar{\mathbb{F}}_p$. At any superspecial point there are $p+1$ branches of S_{ss} meeting transversally. We shall prove the proposition by showing that along any one of these branches (labelled by ζ, a $p+1$-st root of -1) $\mathcal{P}_0$ approaches a line $\mathcal{P}_x[\zeta] \subset \mathcal{P}_x$, but these $p+1$ lines are distinct. In other words, on the normalization of S_{ss} we can extend the filtration uniquely, but the extension does not come from S_{ss}.

Before we go into the proof a word of explanation is needed. In Section 4.3 below we study the deformation of the action of V on $\omega_{\mathcal{A}}$ near a *general* supersingular point $x \in S_{gss}$. For that purpose the first infinitesimal neighborhood of x suffices, and we end up using Grothendieck's crystalline deformation theory, even though we cast it in the language of de Rham cohomology. At a point $x \in S_{ssp}$, in contrast, we need to work in the full formal neighborhood of x in S, or at least in an Artinian neighborhood which no longer admits a divided power structure. The reason is that the singularity of S_{ss} at x is formally of the type $Spec(\bar{\mathbb{F}}_p[[u,v]]/(u^{p+1}+v^{p+1}))$, as we shall see in (3.27). Crystalline deformation theory is inadequate, and we need to use Zink's "displays". As the theory of displays is covariant, we start with the covariant Cartier module of $A = \mathcal{A}_x$ rather than the contravariant Dieudonné module, and look for its universal deformation.

Let us review the (confusing) functoriality of these two modules. For the moment, let A be any abelian variety over $\bar{\mathbb{F}}_p$. If D is the (contravariant) Dieudonné module of A and M is its (covariant) Cartier module, then $D/pD = H^1_{dR}(A)$ and $M/pM = H^1_{dR}(A^t)$ are set in duality. The dual of

$$V : D/pD \to (D/pD)^{(p)} \tag{3.21}$$

($V = Ver_A^*$, Ver_A being the Verschiebung isogeny from $A^{(p)}$ to A) is the map

$$F : (M/pM)^{(p)} \to M/pM \tag{3.22}$$

($F = Frob^*_{A^t}$, $Frob_{A^t}$ being the Frobenius isogeny from A^t to $A^{t(p)}$). As usual, since $\bar{\mathbb{F}}_p$ is perfect, we may view V as a ϕ^{-1}-linear map of D/pD, and F as a ϕ-linear map of M/pM. Replacing A by A^t we then also have a map F on D/pD and V on M/pM. The Hodge filtration $\omega_A \subset H^1_{dR}(A)$ is $(D/pD)[F]$. Its dual is the quotient $Lie(A) = H^1(A^t, \mathcal{O})$ of $H^1_{dR}(A^t)$, identified with M/VM. Compare [Oda], Corollary 5.11.

This reminder tells us that when we pass from the contravariant theory to the covariant one, instead of looking for the deformation of V on ω_A we should look for the deformation of F on $Lie(A) = M/VM$. At a superspecial point F annihilates $Lie(A)$, but at nearby points in S it need not annihilate it anymore.

Now let $x \in S_{ssp}$ and $A = \mathcal{A}_x$. The Cartier module (modulo p) M/pM of A admits a symplectic basis $f_3, e_1, e_2, e_3, f_1, f_2$ where $\mathcal{O}_\mathcal{K}$ acts on the e_i via Σ and on the f_i via $\bar{\Sigma}$, where the polarization pairing is $\langle e_i, f_j\rangle = -\langle f_j, e_i\rangle = \delta_{ij}$ and $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$, and where f_3, e_1, e_2 project to a basis of $Lie(A) = M/VM$. With an appropriate choice of the basis, the Frobenius F on M/pM is the ϕ-linear map whose matrix with respect to the basis $f_3, e_1, e_2, e_3, f_1, f_2$ is

$$\begin{pmatrix} 0 & & & 0 & & \\ & 0 & & & 0 & \\ & & 0 & & & 0 \\ 1 & & & 0 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \end{pmatrix}. \tag{3.23}$$

All this can be deduced from [Bu-We], 3.2.

To construct the universal display we follow the method of [Go-O]. See also [An-Go]. With local coordinates u and v we write $\widehat{S} = Spf\ \bar{\mathbb{F}}_p[[u, v]]$ for the formal completion of S at x. We study the deformation of F to

$$F : H^1_{dR}(\mathcal{A}^t/\widehat{S})^{(p)} \to H^1_{dR}(\mathcal{A}^t/\widehat{S}). \tag{3.24}$$

We use a basis $f_3, e_1, \dots, f_2$ satisfying the same assumptions as above with respect to the $\mathcal{O}_\mathcal{K}$-type and the polarization pairing. Then one can choose u and v and the basis of $H^1_{dR}(\mathcal{A}^t/\widehat{S})$ so that the universal Frobenius is given by the matrix

$$F = \begin{pmatrix} 0 & u & v & 0 & & \\ u & 0 & & & 0 & \\ v & & 0 & & & 0 \\ 1 & & & 0 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \end{pmatrix}. \tag{3.25}$$

Since the first three vectors project onto a basis of $Lie(\mathcal{A})$, the matrix of $F : Lie(\mathcal{A})^{(p)} \to Lie(\mathcal{A})$ is the 3×3 upper left corner, and the matrix of $F^2 (= F \circ F^{(p)})$ is (note the semilinearity)

$$\begin{pmatrix} u^{p+1} + v^{p+1} & & \\ & u^{p+1} & uv^p \\ & vu^p & v^{p+1} \end{pmatrix}. \tag{3.26}$$

This matrix is sometimes called the Hasse-Witt matrix. Thus on $Lie(\mathcal{A})(\bar{\Sigma})^{(p^2)} = \mathcal{L}^{(p^2)\vee}$ the action of F^2 is given by multiplication by $u^{p+1} + v^{p+1}$. As the supersingular locus is the locus where the action of F on the Lie algebra is nilpotent, it follows that the local (formal) equation of S_{ss} at x is

$$u^{p+1} + v^{p+1} = 0. \tag{3.27}$$

Note that this equation guarantees also that the lower 2×2 block, representing the action of F^2 on the Σ-part of the Lie algebras is (semi-linearly) nilpotent, i.e.

$$\begin{pmatrix} u^{p+1} & uv^p \\ vu^p & v^{p+1} \end{pmatrix} \begin{pmatrix} u^{p^2(p+1)} & u^{p^2}v^{p^3} \\ v^{p^2}u^{p^3} & v^{p^2(p+1)} \end{pmatrix} = 0. \tag{3.28}$$

We write $\widehat{S}_{ss} = Spf(\bar{\mathbb{F}}_p[[u, v]]/(u^{p+1} + v^{p+1}))$ for the formal completion of S_{ss} at x. Letting ζ run over the $p+1$ roots of -1 we recover the $p+1$ formal branches through x as the "lines"

$$u = \zeta v. \tag{3.29}$$

We write $\widehat{S}_{ss}[\zeta] = Spf(\bar{\mathbb{F}}_p[[u,v]]/(u-\zeta v))$ for this branch. When we restrict (pull back) the vector bundle $Lie(\mathcal{A})$ to $\widehat{S}_{ss}[\zeta]$, $\ker(F) \cap Lie(\mathcal{A})^{(p)}$ (the dual of $\mathcal{P}_\mu$) becomes

$$\ker \begin{pmatrix} 0 & \zeta v & v \\ \zeta v & 0 & 0 \\ v & 0 & 0 \end{pmatrix}. \tag{3.30}$$

When $v \neq 0$ (i.e. outside the point x) this is the line

$$\bar{\mathbb{F}}_p \begin{pmatrix} 0 \\ 1 \\ -\zeta \end{pmatrix} \tag{3.31}$$

As these lines are distinct, the filtration of $\mathcal{P}$ can not be extended across x. ∎

3.3. The Hasse invariant $h_{\bar{\Sigma}}$.

3.3.1. *Definition of $h_{\bar{\Sigma}}$.* The construction and main properties of the Hasse invariant that we are about to describe, have been given (for any unitary Shimura variety) by Goldring and Nicole in [Go-Ni]. Let R be an $R_0/pR_0 = \mathbb{F}_{p^2}$-algebra. Let A be an abelian scheme over R and

$$A^{(p)} = A \times_{SpecR,\phi} SpecR \tag{3.32}$$

where ϕ is the p-power map. If α is an endomorphism of A then $\alpha^{(p)} = \alpha \times 1$ is an endomorphism of $A^{(p)}$ and

$$\alpha^{(p)} \circ Frob_{A/R} = Frob_{A/R} \circ \alpha. \tag{3.33}$$

If $\iota : \mathcal{O}_\mathcal{K} \to End_R(A)$ is a ring homomorphism we define $\iota^{(p)} : \mathcal{O}_\mathcal{K} \to End_R(A^{(p)})$ by

$$\iota^{(p)}(a) = \iota(a)^{(p)}. \tag{3.34}$$

If, via ι, A has CM by $\mathcal{O}_\mathcal{K}$ and type $(2,1)$, then $A^{(p)}$ will have, via $\iota^{(p)}$, type $(1,2)$. The isogenies $Frob_{A/R}$ and $Ver_{A/R}$ induce R-linear maps $(F_{A/R})_*$ and $(V_{A/R})_*$ on Lie algebras. Since

$$\iota^{(p)}(a)_* \circ (F_{A/R})_* = (F_{A/R})_* \circ \iota(a)_* \tag{3.35}$$

(and similarly for $V_{A/R}$), $(F_{A/R})_*$ and $(V_{A/R})_*$ *preserve* types. As $Lie(A^{(p)})(\bar{\Sigma})$ is 2-dimensional but $Lie(A)(\bar{\Sigma})$ is 1-dimensional,

$$(V_{A/R})_* : Lie(A^{(p)}) \to Lie(A) \tag{3.36}$$

must have a kernel in $Lie(A^{(p)})(\bar{\Sigma})$ which is at least one-dimensional. Consider

$$V_*^2 := (V_{A/R})_* \circ (V_{A^{(p)}/R})_* : Lie(A)^{(p^2)} \to Lie(A) \tag{3.37}$$

which again preserves types. We remark that under the identification of $Lie(A)$ with $H^1(A^t, \mathcal{O})$, $(V_{A/R})_* = (F_{A^t/R})^*$.

Definition 3.1. *The Hasse invariant is*

$$h_{\bar{\Sigma}} = V_*^2(\bar{\Sigma}) : Lie(A)(\bar{\Sigma})^{(p^2)} \to Lie(A)(\bar{\Sigma}). \tag{3.38}$$

Recall (Lemma 3.2) that if $\mathcal{M}$ is a line bundle on a scheme S in characteristic p, then $\mathcal{M}^{(p)} = \mathcal{O}_S \otimes_{\phi,\mathcal{O}_S} \mathcal{M}$ is another line bundle over S and the map

$$\mathcal{M}^{(p)} \ni 1 \otimes m \mapsto m \otimes \cdots \otimes m \in \mathcal{M}^p \tag{3.39}$$

is an isomorphism of line bundles over S. This applies in particular to the line bundle $Lie(\mathcal{A})(\bar{\Sigma}) = \mathcal{L}^\vee$ on the Picard modular surface S (over R_0/pR_0).

We conclude that

$$\begin{aligned} h_{\bar{\Sigma}} &\in Hom_S(Lie(\mathcal{A})(\bar{\Sigma})^{p^2}, Lie(\mathcal{A})(\bar{\Sigma})) \\ &= H^0(S, \mathcal{L}^{p^2-1}). \end{aligned} \tag{3.40}$$

This means that *the Hasse invariant is a modular form of weight p^2-1 in characteristic p.*

Theorem 3.9. *The Hasse invariant is invertible on S_μ and vanishes on $S_{ss} = S - S_\mu$ to order one. More precisely, S_μ is an open subscheme whose complement S_{ss} is a divisor, and when we endow this divisor with its induced reduced subscheme structure, it becomes the Cartier divisor $div(h_{\bar{\Sigma}})$.*

Proof. Dualizing the definition, the Hasse invariant vanishes precisely where $V^*_{\mathcal{A}}(\mathcal{L})$ is contained in $\ker(V^*_{\mathcal{A}^{(p)}} : \mathcal{P}^{(p)} \to \mathcal{L}^{(p^2)})$. We have already seen that over S_μ the latter is the line bundle $\mathcal{P}_0^{(p)}$ and that V^* sends $\mathcal{L}$ isomorphically onto a direct complement of $\mathcal{P}_0^{(p)}$, *cf* (3.18). Thus the Hasse invariant does not vanish on S_μ. To prove that $h_{\bar{\Sigma}}$ vanishes on S_{ss} to order 1 we must study the Dieudonné module at an infinitesimal neighborhood of a point $x \in S_{gss}$ and compute $V^{*(p)} \circ V^*$ using local coordinates there. This can be extracted from [Bu-We], but since later in our work we explain it in deatil (for the purpose of studying the theta operator), we shall now refer to Section 4.3. In Lemma 4.9 we describe the (contravariant) Dieudonné module at x. In subsection 4.3.3 we describe its infinitesimal deformation. Using the local coordinates u and v, and the notation used there, $f_3 - uf_1 - vf_2$ becomes a basis for $\mathcal{L}$ over the first infinitesimal neighborhood of x. We then compute

$$\begin{aligned} V^*(f_3 - uf_1 - vf_2) &= e_1^{(p)} - ue_2^{(p)} \\ V^{*(p)}(e_1^{(p)} - ue_2^{(p)}) &= -uf_3^{(p^2)} = -u \cdot (f_3 - uf_1 - vf_2)^{(p^2)}. \end{aligned} \tag{3.41}$$

Apart from working in the cotangent space instead of the tangent space, $V^{*(p)} \circ V^*$ is the Hasse invariant (use $Hom(M,N) = Hom(N^\vee, M^\vee)$). It follows that after $\mathcal{L}$ has been locally trivialized, the equation $h_{\bar{\Sigma}} = 0$ becomes $u = 0$, which is the local equation for S_{gss}. ∎

3.3.2. *Nonvanishing of $h_{\bar{\Sigma}}$ at the cusps.*

Proposition 3.10. *The Hasse invariant extends to a holomorphic section of $\mathcal{L}^{p^2-1}$ over $\bar{S}$, which is nowhere vanishing on $\bar{S}_\mu$. If we trivialize $\mathcal{L}|_C$ then the restriction of $h_{\bar{\Sigma}}$ to the cuspidal divisor C becomes a nowhere vanishing locally constant function.*

Proof. Extendibility holds by the Koecher principle for any modular form. One can even deduce non-vanishing at the cusps, but here we may argue directly. The same definition as the one given over S, with the abelian variety $\mathcal{A}$ replaced by $\mathcal{G} = \mathcal{A}[p]^0$ ($\mathcal{A}$ signifying now the semi-abelian variety over $\bar{S}$), defines $h_{\bar{\Sigma}}$ over the complete Picard surface:

$$h_{\bar{\Sigma}} = (V_{\mathcal{G}})_* \circ (V_{\mathcal{G}^{(p)}})_* : Lie(\mathcal{G})(\bar{\Sigma})^{(p^2)} \to Lie(\mathcal{G})(\bar{\Sigma}). \tag{3.42}$$

The same argument as above shows that $h_{\bar{\Sigma}}$ is nowhere vanishing on the whole of $\bar{S}_\mu$. Since $\mathcal{L}|_C$ is trivial, the last statement is obvious. ∎

Corollary 3.11. *The scheme $S^*_\mu = S^* - S_{ss}$ is affine and over $\bar{\mathbb{F}}_p$, the intersection of S_{ss} with every connected component of S is connected.*

Proof. The line bundle $\mathcal{L}$ is ample on S, even over R_0.[4] Hence for large enough m, which we can take to be a multiple of p^2-1, $\mathcal{L}^m$ is very ample, and by [La1] the Baily-Borel compactification S^* is the closure of S in the projective embedding supplied by the linear system $H^0(S,\mathcal{L}^m)$. It follows that $\mathcal{L}^m$ has an extension to a line bundle on S^* which we denote $\mathcal{O}_{S^*}(1)$, since it comes from the restriction of the $\mathcal{O}(1)$ of the projective space to S^*. Moreover, Larsen proves that on the smooth compactification $\bar{S}$, $\mathcal{L}^m = \pi^*\mathcal{O}_{S^*}(1)$ where $\pi : \bar{S} \to S^*$.[5]

Replacing $h_{\bar{\Sigma}}$ by its power $h_{\bar{\Sigma}}^{m/(p^2-1)}$, this power becomes a global section of $\mathcal{L}^m$, hence its zero locus S_{ss} a hyperplane section of S^* in the projective embeding supplied by $H^0(S,\mathcal{L}^m)$. Its complement is therefore affine. The last claim follows from the fact [Hart], III, 7.9, that a positive dimensional hyperplane section of a smooth (or more generally, normal) projective variety is connected. ∎

The scheme $\bar{S}_\mu$ is of course far from affine, as it contains the complete curves E as cuspidal divisors.

3.4. **A secondary Hasse invariant on the supersingular locus.** In his forthcoming Ph.D. thesis [Bo], Boxer develops a general theory of secondary Hasse invariants defined on lower strata of Shimura varieties of Hodge type. See also [Kos]. In this section we provide an independent approach, in the case of Picard modular surfaces, affording a detailed study of its properties. As an application we relate the number of irreducible components of S_{ss} to the Euler number of $S_{\mathbb{C}}$, and through it to the value of the function $L(s,(\frac{D_K}{\cdot}))$ at $s=3$.

3.4.1. *Definition of h_{ssp}.* As we have seen, along the general supersingular locus S_{gss}, Verschiebung induces isomorphisms

$$V_{\mathcal{L}} : \mathcal{L} \simeq \mathcal{P}_0^{(p)}, \;\; V_{\mathcal{P}} : \mathcal{P}_\mu \simeq \mathcal{L}^{(p)}. \tag{3.43}$$

(The first is unique to S_{gss}, the second holds also on the μ-ordinary stratum.) Consider the isomorphism

$$V_{\mathcal{P}}^{(p)} \otimes V_{\mathcal{L}}^{-1} : \mathcal{P}_\mu^{(p)} \otimes \mathcal{P}_0^{(p)} \simeq \mathcal{L}^{(p^2)} \otimes \mathcal{L} \simeq \mathcal{L}^{p^2+1}. \tag{3.44}$$

Its source is the line bundle $\det \mathcal{P}^{(p)}$ which is identified with $\mathcal{L}^{(p)} \simeq \mathcal{L}^p$. We therefore get a nowhere vanishing section

$$\tilde{h}_{ssp} \in H^0(S_{gss}, \mathcal{L}^{p^2-p+1}). \tag{3.45}$$

Our "secondary" Hasse invariant is the nowhere vanishing section

$$h_{ssp} = \tilde{h}_{ssp}^{p+1} \in H^0(S_{gss}, \mathcal{L}^{p^3+1}). \tag{3.46}$$

We shall show that h_{ssp} extends to a holomorphic section on S_{ss}, and vanishes at the superspecial points (to a high order).

---

[4] One way to see it is to use the ampleness of the Hodge bundle $\det \omega_{\mathcal{A}} \simeq \mathcal{L}^2$ (pull back from Siegel space, where it is known to be ample by [Fa-Ch]).

[5] It is not clear that $\mathcal{L}$ itself has an extension to a line bundle on S^*, or that $\pi_*\mathcal{L}$, which is a coherent sheaf extending $\mathcal{L}|_S$, is a line bundle (the problem lying of course only at the cusps). In other words, it is not clear that we can extract an mth root of $\mathcal{O}_{S^*}(1)$ as a line bundle.

3.4.2. *Computations at the superspecial points.* We refer again to the computations of Proposition 3.8, and work over $\bar{\mathbb{F}}_p$. Dualizing the display described there, and using the letters e_i, f_j to denote *the dual basis* to the basis used there we get the following.

Lemma 3.12. *Let $x \in S_{ssp}$ be a superspecial point. There exist formal coordinates u and v so that the formal completion of S at x is $\widehat{S} = Spf(\bar{\mathbb{F}}_p[[u,v]])$, and $D = H^1_{dR}(\mathcal{A}/\widehat{S})$ has a basis $f_3, e_1, e_2, e_3, f_1, f_2$ over $\bar{\mathbb{F}}_p[[u,v]]$ with the following properties:*

(i) f_3, e_1, e_2 is a basis for $\omega_{\mathcal{A}}$

(ii) The basis is symplectic, i.e. the polarization form is $\langle e_i, f_j\rangle = -\langle f_j, e_i\rangle = \delta_{ij}$, $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$.

(iii) $\mathcal{O}_{\mathcal{K}}$ acts on the e_i via Σ and on the f_j via $\bar{\Sigma}$.

(iv) $V : D \to D^{(p)}$ is given by $Vf_3 = ue_1^{(p)} + ve_2^{(p)}$, $Ve_1 = uf_3^{(p)}$, $Ve_2 = vf_3^{(p)}$, $Ve_3 = f_3^{(p)}$, $Vf_1 = e_1^{(p)}$, $Vf_2 = e_2^{(p)}$.

Using the lemma, we compute along $\widehat{S}_{ss}[\zeta]$, where $u = \zeta v$ ($\zeta^{p+1} = -1$). Denote by $\mathcal{P}[\zeta]$, $\mathcal{P}_0[\zeta]$ and $\mathcal{L}[\zeta]$ the pull-backs of the corresponding vector bundles to the branch $\widehat{S}_{ss}[\zeta]$. The map $V_{\mathcal{L}}$ is given by

$$f_3 \mapsto ue_1^{(p)} + ve_2^{(p)} = v\cdot(\zeta e_1^{(p)} + e_2^{(p)}) = v\cdot(\zeta^p e_1 + e_2)^{(p)} \in \mathcal{P}_0^{(p)}[\zeta]. \tag{3.47}$$

Use $e_1 \wedge e_2 = e_1 \wedge (\zeta^p e_1 + e_2)$ as a basis for $\det \mathcal{P}[\zeta]$. Since $V_{\mathcal{P}}^{(p)}$ maps $e_1^{(p)}$ to $(\zeta v)^p f_3^{(p^2)}$, $\tilde{h}_{ssp}$ maps $e_1^{(p)} \wedge e_2^{(p)} = e_1^{(p)} \wedge (\zeta^p e_1 + e_2)^{(p)}$ to

$$\begin{aligned} \tilde{h}_{ssp}(e_1^{(p)} \wedge e_2^{(p)}) &= \zeta^p v^{p-1} f_3^{(p^2)} \otimes f_3 \\ &= \zeta^p v^{p-1} f_3^{p^2+1} = \zeta u^{p-1} f_3^{p^2+1}. \end{aligned} \tag{3.48}$$

Lemma 3.13. *There does not exist a function $g \in \bar{\mathbb{F}}_p[[u,v]]/(u^{p+1}+v^{p+1})$ on $\widehat{S}_{ss}$ whose restriction to the branch $\widehat{S}_{ss}[\zeta]$ is ζu^{p-1}.*

Proof. Had there been such a function, represented by a power series $G \in \bar{\mathbb{F}}_p[[u,v]]$, then we would get $vg = u^p$ on $\widehat{S}_{ss}[\zeta]$ for every ζ, hence

$$vG - u^p \in (u^{p+1} + v^{p+1}) \subset \bar{\mathbb{F}}_p[[u,v]]. \tag{3.49}$$

But any power series in the ideal $(u^{p+1}+v^{p+1})$ contains only terms of degree $\geq p+1$, while in $vG - u^p$ we can not cancel the term u^p. ∎

The lemma means that $\tilde{h}_{ssp}$ can not be extended over S_{ss} to a section of $Hom(\det \mathcal{P}^{(p)}, \mathcal{L}^{p^2+1}) \simeq \mathcal{L}^{p^2-p+1}$. However, when we raise it to a $p+1$ power the dependence on ζ disappears. It then extends to a section h_{ssp} of $\mathcal{L}^{p^3+1}$ over S_{ss}, given over $\widehat{S}$ by the equation

$$h_{ssp} = \varepsilon u^{p^2-1} f_3^{p^3+1}, \tag{3.50}$$

where $\varepsilon \in \bar{\mathbb{F}}_p[[u,v]]^\times$ depends on the isomorphism between $\det \mathcal{P}$ and $\mathcal{L}$.

Theorem 3.14. *The secondary Hasse invariant h_{ssp} belongs to $H^0(S_{ss}, \mathcal{L}^{p^3+1})$. It vanishes precisely at the points of S_{ssp}. The subscheme "$h_{ssp} = 0$" of S_{ss} is not reduced. At $x \in S_{ssp}$, with u and v as above, it is the spectrum of*

$$\bar{\mathbb{F}}_p[[u,v]]/(u^{p+1} + v^{p+1}, u^{p^2-1}, v^{p^2-1}). \tag{3.51}$$

From now on we assume that N is large enough (depending on p) so that Theorem 3.1(iii) holds. Each irreducible component of S_{ss} is non-singular, and h_{ssp} has a zero of order p^2-1 at each superspecial point on such a component. Each component contains p^3+1 superspecial points. It follows that if Z is such a component,

$$\deg(\mathcal{L}^{p^3+1}|_Z) = \deg(div_Z(h_{ssp})) = (p^3+1)(p^2-1). \tag{3.52}$$

We get the following corollary.

Corollary 3.15. *Let Z be an irreducible component of S_{ss}, and assume that N is large enough. Then* $\deg(\mathcal{L}|_Z) = p^2-1$.

3.4.3. *Application to the number of irreducible components of S_{ss}.* We want to give an interesting application of the last corollary to the computation of the number of irreducible components of S_{ss}. Let

$$Z = \bigcup_{i=1}^{n} Z_i \tag{3.53}$$

be the decomposition into irreducible components of a single connected component Z of S_{ss} (recall that Z is the intersection of S_{ss} with a connected component of $\bar{S}$). The Z_i are smooth, and their genus is $p(p-1)/2$.

Theorem 3.16. *Let c_2 be the Euler characteristic of the connected component of $\bar{S}$ containing Z, i.e. if over $\mathbb{C}$ this connected component is $\bar{X}_\Gamma$ then*

$$c_2 = \sum_{i=0}^{4} (-1)^i \dim_{\mathbb{C}} H^i(\bar{X}_\Gamma, \mathbb{C}). \tag{3.54}$$

Then

$$3n = c_2. \tag{3.55}$$

Proof. Computing intersection numbers,

$$(Z.Z) = n(p^3+1)p + \sum_{i=1}^{n} (Z_i.Z_i). \tag{3.56}$$

Denote by $K_{\bar{S}}$ a canonical divisor on the given connected component of $\bar{S}$. From the adjunction formula,

$$p(p-1) - 2 = 2g(Z_i) - 2 = Z_i.(Z_i + K_{\bar{S}}). \tag{3.57}$$

As we have seen in Proposition 2.11, $\mathcal{O}(K_{\bar{S}} + C) \simeq \mathcal{L}^3$ where C is the cuspidal divisor. Hence

$$(Z_i.K_{\bar{S}}) = Z_i.(K_{\bar{S}} + C) = \deg(\mathcal{L}^3|_{Z_i}) = 3(p^2-1) \tag{3.58}$$

by the last Corollary. We get

$$(Z_i.Z_i) = -2p^2 - p + 1. \tag{3.59}$$

Plugging this into the expression for $(Z.Z)$ we get

$$(Z.Z) = n(p^2-1)^2. \tag{3.60}$$

On the other hand, since Z is the divisor of the Hasse invariant on $\bar{S}$, $div(h_{\bar{\Sigma}}) = Z$, we get $\mathcal{O}(Z) = \mathcal{L}^{p^2-1}$ so

$$n = c_1(\mathcal{L})^2. \tag{3.61}$$

From this we get $9n = (K_{\bar{S}}+C).(K_{\bar{S}}+C)$. Holzapfel [Ho] (4.3.11') on p.184, implies

$$9n = 3c_2(X_\Gamma) = 3c_2(\bar{X}_\Gamma) \tag{3.62}$$

proving the theorem. ∎

In [Ho] (5A.4.3), p.325 Holzapfel proves that for Γ the stabilizer of L_0

$$c_2(X_\Gamma) = \varepsilon \frac{3|D_\mathcal{K}|^{5/2}}{32\pi^3} L\left(3, \left(\frac{D_\mathcal{K}}{\cdot}\right)\right) = -\frac{3\varepsilon}{16} L\left(-2, \left(\frac{D_\mathcal{K}}{\cdot}\right)\right), \tag{3.63}$$

where $\varepsilon = 1$, unless $D_\mathcal{K} = -3$, in which case $\varepsilon = 3$. For a congruence subgroup Γ' we have $c_2(X_{\Gamma'}) = [\Gamma : \Gamma']c_2(X_\Gamma)$. Formulae of this form have been previously obtained, for split reductive groups, by Harder.

3.4.4. *The classes of Ekedahl-Oort strata in the Chow ring of $\bar{S}$.* Let us denote by $CH = CH(\bar{S})$ the Chow ring with $\mathbb{Q}$ coefficients of $\bar{S}$ over $\bar{\mathbb{F}}_p$, and let $CH_\mathcal{L}$ the $\mathbb{Q}$-subalgebra generated by $c_1(\mathcal{L})$ (note that we now use c_i to denote Chern classes of vector bundles in CH and not in cohomology). Recall that the Ekedahl-Oort strata of $\bar{S}$ consist of $\bar{S}$ itself, $Z = S_{ss}$ in codimension 1, and $W = S_{ssp}$ in codimension 2.

Proposition 3.17. *The classes $[\bar{S}], [Z]$ and $[W]$ in CH belong to $CH_\mathcal{L}$.*

A similar result for Siegel modular varieties has been proved by van der Geer in [vdG] and for Hilbert modular varieties in [Go-O].

Proof. Without loss of generality we may assume that N is large enough, so that Theorem 3.1(iii) applies. As the divisor of the Hasse invariant is $Z, [Z] = (p^2 - 1)c_1(\mathcal{L})$. It remains to deal with $[W]$. Let $f : \tilde{Z} \to \bar{S}$ be the map from the normalization of Z to $\bar{S}$ with image Z. Then the projection formula implies

$$[Z].c_1(\mathcal{L}) = f_*(f^*(c_1(\mathcal{L})) = f_*(c_1(f^*\mathcal{L})). \tag{3.64}$$

Write $\tilde{Z} = \coprod_{i=1}^n \tilde{Z}_i$ and note that $(p^3+1)c_1(f^*\mathcal{L})$ is the class of $div(h_{ssp})$ on $\tilde{Z}$, which is represented by the 0-cycle consisting of the superspecial points on each $\tilde{Z}_i$ with multiplicities $p^2 - 1$. It follows that

$$(p^3+1)[Z].c_1(\mathcal{L}) = (p^2-1)(p+1)[W]. \tag{3.65}$$

Substituting $[Z] = (p^2-1)c_1(\mathcal{L})$ we get

$$[W] = (p^2 - p + 1)c_1(\mathcal{L})^2. \tag{3.66}$$

∎

3.5. **The open Igusa surfaces.**

3.5.1. *The Igusa scheme.* Let $N \geq 3$ as always, and let $\mathcal{M}$ be the moduli problem of Section 1.3.1. Let $n \geq 1$ and consider the following moduli problem on R_0/pR_0-algebras:

- $\mathcal{M}_{Ig(p^n)}(R)$ is the set of isomorphism classes of triples $(\underline{A}, A[p^n]^\mu, \varepsilon)$ where $\underline{A} \in \mathcal{M}(R)$, $A[p^n]^\mu \subset A[p^n]$ is a finite flat $\mathcal{O}_\mathcal{K}$-subgroup scheme of rank p^{2n} of multiplicative type, and

$$\varepsilon : \delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes \mu_{p^n} \simeq A[p^n]^\mu \tag{3.67}$$

is an isomorphism of $\mathcal{O}_\mathcal{K}$-group schemes over R.

It is clear that if $(\underline{A}, A[p^n]^\mu, \varepsilon) \in \mathcal{M}_{Ig(p^n)}(R)$ then A is fiber-by-fiber μ-ordinary and therefore $\underline{A} \in \mathcal{M}(R)$ defines an R-point of S_μ. It is also clear that the functor $R \rightsquigarrow \mathcal{M}_{Ig(p^n)}(R)$ is relatively representable over $\mathcal{M}$, and therefore as $N \geq 3$ and $\mathcal{M}$ is representable, this functor is also representable by a scheme $Ig_\mu(p^n)$ which maps to S_μ. See [Ka-Ma] for the notion of relative representability. We call $Ig_\mu(p^n)$ the *Igusa scheme of level* p^n.

Proposition 3.18. *The morphism* $\tau : Ig_\mu(p^n) \to S_\mu$ *is finite and étale, with the Galois group* $\Delta(p^n) = (\mathcal{O}_\mathcal{K}/p^n\mathcal{O}_\mathcal{K})^\times$ *acting as a group of deck transformations.*

Proof. Every μ-ordinary abelian variety has a *unique* finite flat $\mathcal{O}_\mathcal{K}$-subgroup scheme of multiplicative type $A[p^n]^\mu$ of rank p^{2n}. Such a subgroup scheme is, locally in the étale topology, isomorphic to $\delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes \mu_{p^n}$, and any two isomorphisms differ by a unique automorphism of $\delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes \mu_{p^n}$. But $\Delta(p^n) = Aut_{\mathcal{O}_\mathcal{K}}(\delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes \mu_{p^n})$. If we let $\gamma \in \Delta(p^n)$ act on the triple $(\underline{A}, A[p^n]^\mu, \varepsilon)$ via

$$\gamma((\underline{A}, A[p^n]^\mu, \varepsilon)) = (\underline{A}, A[p^n]^\mu, \varepsilon \circ \gamma^{-1}) \tag{3.68}$$

$\Delta(p^n)$ becomes a group of deck transformation and the proof is complete. ∎

3.5.2. *A compactification over the cusps.* The proof of the following proposition mimics the construction of $\bar{S}$. We omit it.

Proposition 3.19. *Let* $\overline{Ig}_\mu(p^n)$ *be the normalization of* $\bar{S}_\mu = \bar{S} - S_{ss}$ *in* $Ig_\mu(p^n)$. *Then* $\overline{Ig}_\mu(p^n) \to \bar{S}_\mu$ *is finite étale and the action of* $\Delta(p^n)$ *extends to it. The boundary* $\overline{Ig}_\mu(p^n) - Ig_\mu(p^n)$ *is non-canonically identified with* $\Delta(p^n) \times C$.

We define similarly Ig_μ^*, and note that it is finite étale over S_μ^*.

Proposition 3.20. *Let* $\mathcal{A}$ *denote the pull-back of the universal semi-abelian variety from* $\bar{S}_\mu$ *to* $\overline{Ig}_\mu(p^n)$. *Then* $\mathcal{A}$ *is equipped with a canonical Igusa level structure*

$$\varepsilon : \delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes \mu_{p^n} \simeq \mathcal{A}[p^n]^\mu. \tag{3.69}$$

Over C *and after base change to* R_N/pR_N *the toric part of* $\mathcal{A}$ *is locally Zariski of the form* $\mathfrak{a} \otimes \mathbb{G}_m$ *and* ε *is then an* $\mathcal{O}_\mathcal{K}$*-linear isomorphism between* $\delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes \mu_{p^n}$ *and* $\mathfrak{a} \otimes \mu_{p^n}$.

3.5.3. *A trivialization of* $\mathcal{L}$ *over the Igusa surface.* From now on we focus on $\overline{Ig}_\mu = \overline{Ig}_\mu(p)$ although similar results hold when $n > 1$, and would be instrumental in the study of p-adic modular forms. The vector bundle $\omega_\mathcal{A}$ pulls back to a similar vector bundle over $\overline{Ig}_\mu$. But there

$$\omega_\mathcal{A}^\mu := \omega_{\mathcal{A}[p]^\mu} \tag{3.70}$$

is a rank 2 quotient bundle stable under $\mathcal{O}_\mathcal{K}$ (of type $(1,1)$), and the isomorphism ε induces an isomorphism

$$\varepsilon^* : \omega_\mathcal{A}^\mu \simeq \omega_{\delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes \mu_p}. \tag{3.71}$$

Now $Lie(\delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes \mu_p) = \delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes Lie(\mu_p) = \delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes Lie(\mathbb{G}_m)$ and by duality

$$\omega_{\delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K} \otimes \mu_p} = \mathcal{O}_\mathcal{K} \otimes \omega_{\mathbb{G}_m}, \tag{3.72}$$

with $1 \otimes dT/T$ as a generator (if T is the parameter of $\mathbb{G}_m$). Here we have used the fact that the $\mathbb{Z}$-dual of $\delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K}$ is $\mathcal{O}_\mathcal{K}$ via the trace pairing. This is the constant vector bundle $\mathcal{O}_\mathcal{K} \otimes R = R(\Sigma) \oplus R(\bar{\Sigma})$.

Proposition 3.21. *The line bundles $\mathcal{L}$, $\mathcal{P}_0$ and $\mathcal{P}_\mu$ are trivial over $\overline{Ig}_\mu$.*

Proof. Use ε^* as an isomorphism between vector bundles and note that $\mathcal{L} = \omega^\mu_{\mathcal{A}}(\bar{\Sigma})$ and $\mathcal{P}_\mu = \omega^\mu_{\mathcal{A}}(\Sigma)$. The relation $\mathcal{P}_0 \otimes \mathcal{P}_\mu = \det \mathcal{P} \simeq \mathcal{L}$ implies the triviality of $\mathcal{P}_0$ as well. ∎

Note that the trivialization of $\mathcal{L}$ and $\mathcal{P}_\mu$ is canonical, because it uses only the tautological map ε which exists over the Igusa scheme. The trivialization of $\mathcal{P}_0$ on the other hand depends on how we realize the isomorphism $\det \mathcal{P} \simeq \mathcal{L}$.

We can now give an alternative proof to the fact that $\mathcal{L}^{p^2-1}$ and $\mathcal{P}_\mu^{p^2-1}$ are trivial on $\bar{S}_\mu$. Since $\overline{Ig}_\mu$ is an étale cover of $\bar{S}_\mu$ of order p^2-1, $\det \tau_*(\tau^*\mathcal{L}) \simeq \mathcal{L}^{p^2-1}$. As $\tau^*\mathcal{L}$ is already trivial, so is $\mathcal{L}^{p^2-1}$ on the base. The same argument works for $\mathcal{P}_\mu$ and for $\mathcal{P}_0$. The fact that $\mathcal{P}_0^{p+1}$ is already trivial could be deduced by a similar argument had we worked out an analogue of $Ig(p)$ classifying *symplectic isomorphisms* of $\mathfrak{G}[p]$ with $gr^1 A[p]$. The role of $\Delta(p)$ for such a moduli space would be assumed by

$$\Delta^1(p) = \ker(N : (\mathcal{O}_\mathcal{K}/p\mathcal{O}_\mathcal{K})^\times \to \mathbb{F}_p^\times), \tag{3.73}$$

which is a group of order $p+1$. We do not go any further in this direction here.

3.6. **Compactification of the Igusa surface along the supersingular locus.**

3.6.1. *Extracting a p^2-1 root from $h_{\bar{\Sigma}}$ over $\overline{Ig}_\mu$.* Let $a(1)$ be the canonical nowhere vanishing section of $\mathcal{L}$ over $\overline{Ig}_\mu$ which is sent to $e_{\bar{\Sigma}} \cdot (1 \otimes dT/T)$ under the trivialization

$$\varepsilon^* : \mathcal{L} = \omega^\mu_{\mathcal{A}}(\bar{\Sigma}) \simeq (\mathcal{O}_\mathcal{K} \otimes \omega_{\mathbb{G}_m})(\bar{\Sigma}) = R(\bar{\Sigma}). \tag{3.74}$$

Here R is any R_0/pR_0-algebra over which we choose to work. In other words, $a(1) = (\varepsilon^*)^{-1}(e_{\bar{\Sigma}} \cdot 1 \otimes dT/T)$. Dually, $a(1)$ is the homomorphism from $Lie(\mathcal{A})(\bar{\Sigma})$ to $\delta_\mathcal{K}^{-1} \otimes Lie(\mathbb{G}_m)(\bar{\Sigma})$ arising from ε^{-1}. Let $a(k) = a(1)^{\otimes k} \in H^0(\overline{Ig}_\mu, \mathcal{L}^k)$.

Proposition 3.22. *(i) Let $\gamma \in \Delta(p) = (\mathcal{O}_\mathcal{K}/p\mathcal{O}_\mathcal{K})^\times$. Then $\Delta(p)$ acts on $H^0(\overline{Ig}_\mu, \mathcal{L})$ and*

$$\gamma^* a(1) = \bar{\Sigma}(\gamma)^{-1} \cdot a(1). \tag{3.75}$$

(ii) The section $a(1)$ is a p^2-1 root of the Hasse invariant over $\overline{Ig}_\mu$, i.e.

$$a(p^2-1) = h_{\bar{\Sigma}}. \tag{3.76}$$

Proof. (i) This part is a restatement of the action of $\Delta(p)$. At two points of $Ig_\mu(R)$ lying over the same point of $S_\mu(R)$ and differing by the action of $\gamma \in \Delta(p)$, the canonical embeddings

$$\delta_\mathcal{K}^{-1} \otimes \mu_p \hookrightarrow A[p] \tag{3.77}$$

differ by $\iota(\gamma)$ (3.68). The induced trivializations of $Lie(A)(\bar{\Sigma})$ differ by $\bar{\Sigma}(\gamma)$ and by duality we get (i).

(ii) Since over any $\mathbb{F}_p$-base, $Ver_{\mathbb{G}_m} = 1$, we have a commutative diagram

$$\begin{array}{ccc} Lie(\mathcal{A})(\bar{\Sigma})^{(p^2)} & \stackrel{V_*^2}{\to} & Lie(\mathcal{A})(\bar{\Sigma}) \\ \downarrow a(1)^{(p^2)} & & \downarrow a(1) \\ \delta_\mathcal{K}^{-1} \otimes Lie(\mathbb{G}_m)(\bar{\Sigma}) & = & \delta_\mathcal{K}^{-1} \otimes Lie(\mathbb{G}_m)(\bar{\Sigma}) \end{array}. \tag{3.78}$$

Using the isomorphism $Lie(\mathcal{A})(\bar{\Sigma})^{(p^2)} \simeq Lie(\mathcal{A})(\bar{\Sigma})^{p^2}$ to which we alluded before, we get the commutative diagram

$$\begin{array}{ccc} Lie(\mathcal{A})(\bar{\Sigma})^{p^2} & \stackrel{h_{\bar{\Sigma}}}{\to} & Lie(\mathcal{A})(\bar{\Sigma}) \\ \downarrow a(p^2) & & \downarrow a(1) \\ \delta_{\mathcal{K}}^{-1} \otimes Lie(\mathbb{G}_m)(\bar{\Sigma}) & = & \delta_{\mathcal{K}}^{-1} \otimes Lie(\mathbb{G}_m)(\bar{\Sigma}) \end{array}, \tag{3.79}$$

from which we deduce that $h_{\bar{\Sigma}} = a(p^2 - 1)$. Note that we can interprete $a(1)$ as an arrow as above rather than as a section of $\mathcal{L} = Lie(\mathcal{A})(\bar{\Sigma})^\vee$ because of the identity $Hom(M, N) = Hom(N^\vee, M^\vee)$. ∎

3.6.2. *The compactification $\overline{Ig}$ of $\overline{Ig}_\mu$.* Quite generally, let $L \to X$ be a line bundle associated with an invertible sheaf $\mathcal{L}$ on a scheme X. Write L^n for the line bundle $L^{\otimes n}$ over X. Let $s : X \to L^n$ be a section. Consider the fiber product

$$Y = L \times_{L^n} X \tag{3.80}$$

where the two maps to L^n are $\lambda \mapsto \lambda^n$ and s. Let $p : Y \stackrel{pr_2}{\to} X$ be the projection which can also factor as $Y \stackrel{pr_1}{\to} L \to L^n \to X$ (since $X \stackrel{s}{\to} L^n \to X$ is the identity). Consider

$$p^*L = L \times_X (L \times_{L^n} X). \tag{3.81}$$

This line bundle on Y has a tautological section $t : Y \to p^*L$,

$$t : y = (\lambda, x) \mapsto (\lambda, y) = (\lambda, (\lambda, x)) \tag{3.82}$$

Here $s(x) = \lambda^n$ and

$$t^n(y) = (\lambda^n, y) = (s(x), y) = p^*s(y) \tag{3.83}$$

so t is an nth root of p^*s. Moreover, Y has the *universal property* with respect to extracting nth roots from s: If $p_1 : Y_1 \to X$, and $t_1 \in \Gamma(Y_1, p_1^*L)$ is such that $t_1^n = p_1^*s$, then there exists a unique morphism $h : Y_1 \to Y$ covering the two maps to X such that $t_1 = h^*t$.

The map $L \to L^n$ is *finite flat of degree* n and if n is invertible on the base, finite étale away from the zero section. Indeed, locally on X it is the map $\mathbb{A}^1 \times X \to \mathbb{A}^1 \times X$ which is just raising to nth power in the first coordinate. By base-change, it follows that the same is true for the map $p : Y \to X$: this map is finite flat of degree n and étale away from the vanishing locus of the section s (assuming n is invertible). We remark that if L is the trivial line bundle, we recover usual Kummer theory.

Applying this in our example with $n = p^2 - 1$ we define the *complete Igusa surface of level* p, $\overline{Ig} = \overline{Ig}(p)$ as

$$\overline{Ig} = \mathcal{L} \times_{\mathcal{L}^{p^2-1}} \bar{S} \tag{3.84}$$

where the map $\overline{S} \to \mathcal{L}^{p^2-1}$ is $h_{\bar{\Sigma}}$. From the universal property and part (ii) of Proposition 3.22 we get a map of $\bar{S}$-schemes

$$\overline{Ig}_\mu \to \overline{Ig}. \tag{3.85}$$

This map is an isomorphism over $\bar{S}_\mu$ because both schemes are étale torsors for $\Delta(p) = (\mathcal{O}_\mathcal{K}/p\mathcal{O}_\mathcal{K})^\times$ and the map respects the action of this group. We summarize the discussion in the following theorem (for the last point, consult [Mu2], Proposition 2, p.198).

Theorem 3.23. *The morphism $\tau : \overline{Ig} \to \bar{S}$ satisfies the following properties:*

(i) It is finite flat of degree $p^2 - 1$, étale over $\bar{S}_\mu$, totally ramified over S_{ss}.

(ii) Δ acts on $\overline{Ig}$ as a group of deck transformations and the quotient is $\bar{S}$.

(iii) Let $s_0 \in S_{gss}(\bar{\mathbb{F}}_p)$. Then there exist local parameters u, v at s_0 such that $\widehat{\mathcal{O}}_{S,s_0} = \bar{\mathbb{F}}_p[[u,v]]$, $S_{gss} \subset S$ is formally defined by $u = 0$, and if $\tilde{s}_0 \in Ig$ maps to s_0 under τ, then $\widehat{\mathcal{O}}_{Ig,\tilde{s}_0} = \bar{\mathbb{F}}_p[[w,v]]$ where $w^{p^2-1} = u$. In particular, Ig is regular in codimension 1.

(iv) Let $s_0 \in S_{ssp}(\bar{\mathbb{F}}_p)$. Then there exist local parameters u, v at s_0 such that $\widehat{\mathcal{O}}_{S,s_0} = \bar{\mathbb{F}}_p[[u,v]]$, $S_{gss} \subset S$ is formally defined by $u^{p+1} + v^{p+1} = 0$, and if $\tilde{s}_0 \in Ig$ maps to s_0 under τ, then

$$\widehat{\mathcal{O}}_{Ig,\tilde{s}_0} = \bar{\mathbb{F}}_p[[w,u,v]]/(w^{p^2-1} - u^{p+1} - v^{p+1}) \tag{3.86}$$

In particular, $\tilde{s}_0$ is a normal singularity of Ig.

3.6.3. *Irreducibility of Ig.* So far we have avoided the delicate question of whether $\overline{Ig}$ is "relatively irreducible", i.e. whether $\tau^{-1}(T)$ is irreducible if $T \subset \bar{S}$ is an irreducible (equivalently, connected) component. Using an idea of Katz, and following the approach taken by Ribet in [Ri], the irreducibility of $\tau^{-1}(T)$ could be proven for any level p^n if we could prove the following:

- Let $q = p^2$. For any r sufficiently large and for any $\gamma \in (\mathcal{O}_\mathcal{K}/p^n\mathcal{O}_\mathcal{K})^\times$ there exists a μ-ordinary abelian variety with PEL structure $\underline{A} \in S_\mu(\mathbb{F}_{q^r})$ such that the image of $Gal(\bar{\mathbb{F}}_q/\mathbb{F}_{q^r})$ in

$$Aut\left(Isom_{\bar{\mathbb{F}}_q}(\delta_\mathcal{K}^{-1} \otimes \mu_{p^n}, A[p^n]^\mu)\right) = (\mathcal{O}_\mathcal{K}/p^n\mathcal{O}_\mathcal{K})^\times \tag{3.87}$$

  contains γ.

Instead, we shall give a different argument valid for the case $n = 1$.

Proposition 3.24. *The morphism $\tau : \overline{Ig} \to \bar{S}$ induces a bijection on irreducible components.*

Proof. Since $\overline{Ig}$ is a normal surface, connected components and irreducible components are the same. Let T be a connected component of $\bar{S}$ and $T_{ss} = T \cap S_{ss}$. Let $\tau^{-1}(T) = \coprod Y_i$ be the decomposition into connected components. As τ is finite and flat, each $\tau(Y_i) = T$. Since τ is totally ramified over T_{ss}, there is only one Y_i. ∎

4. Modular forms modulo p

4.1. Modular forms $\bmod p$ as functions on Ig.

4.1.1. *Representing modular forms by functions on Ig.* The Galois group $\Delta(p) = (\mathcal{O}_\mathcal{K}/p\mathcal{O}_\mathcal{K})^\times$ acts on the coordinate ring $H^0(Ig_\mu, \mathcal{O})$ and we let $H^0(Ig_\mu, \mathcal{O})^{(k)}$ be the subspace where it acts via the character $\bar{\Sigma}^k$. Then

$$H^0(Ig_\mu, \mathcal{O}) = \bigoplus_{k=0}^{p^2-2} H^0(Ig_\mu, \mathcal{O})^{(k)} \tag{4.1}$$

and each $H^0(Ig_\mu, \mathcal{O})^{(k)}$ is free of rank 1 over $H^0(S_\mu, \mathcal{O}) = H^0(Ig_\mu, \mathcal{O})^{(0)}$.

For any $0 \le k$ the map $f \mapsto f/a(k)$ is an embedding

$$M_k(N, R_0/pR_0) \hookrightarrow H^0(Ig_\mu, \mathcal{O})^{(k)}. \tag{4.2}$$

Lemma 4.1. *Fix* $0 \le k < p^2 - 1$. *Then we have a* surjective *homomorphism*

$$\bigoplus_{n \ge 0} M_{k+n(p^2-1)}(N, R_0/pR_0) \twoheadrightarrow H^0(Ig_\mu, \mathcal{O})^{(k)}. \tag{4.3}$$

Proof. Take $f \in H^0(Ig_\mu, \mathcal{O})^{(k)}$, so that $f \cdot a(k) \in H^0(Ig_\mu, \mathcal{L}^k)^{(0)}$, hence descends to $g \in H^0(S_\mu, \mathcal{L}^k)$. This g may have poles along S_{ss}, but some $h_{\bar{\Sigma}}^n g$ will extend holomorphically to S, hence represents a modular form of weight $k + n(p^2 - 1)$, which will map to f because $a(k + n(p^2 - 1)) = h_{\bar{\Sigma}}^n a(k)$. ∎

Proposition 4.2. *The resulting ring homomorphism*

$$r : \bigoplus_{k \ge 0} M_k(N, R_0/pR_0) \twoheadrightarrow H^0(Ig_\mu, \mathcal{O}) \tag{4.4}$$

obtained by dividing a modular form of weight k by $a(k)$ is surjective, respects the $\mathbb{Z}/(p^2-1)\mathbb{Z}$-grading on both sides, and its kernel is the ideal generated by $(h_{\bar{\Sigma}} - 1)$.

Proof. We only have to prove that anything in $\ker(r)$ is a multiple of $h_{\bar{\Sigma}} - 1$, the rest being clear. Since r respects the grading, we may assume that for some $k \ge 0$ we have $f_j \in M_{k+j(p^2-1)}(S, R_0/pR_0)$ and

$$\sum_{j=0}^{m} a(k)^{-1} h_{\bar{\Sigma}}^{-j} f_j = 0. \tag{4.5}$$

But then $f_m = -h_{\bar{\Sigma}}^m \left(\sum_{j=0}^{m-1} h_{\bar{\Sigma}}^{-j} f_j \right)$, so $\sum_{j=0}^m f_j = \sum_{j=0}^{m-1} (1 - h_{\bar{\Sigma}}^{m-j}) f_j$ belongs to $(1 - h_{\bar{\Sigma}})$. ∎

As a result we get that

$$Ig_\mu^* = Spec \left(\bigoplus_{k \ge 0} M_k(N, R_0/pR_0)/(h_{\bar{\Sigma}} - 1) \right) \tag{4.6}$$

and

$$S_\mu^* = Spec \left(\bigoplus_{k \ge 0} M_{k(p^2-1)}(N, R_0/pR_0)/(h_{\bar{\Sigma}} - 1) \right). \tag{4.7}$$

4.1.2. *Fourier-Jacobi expansions modulo p.* The arithmetic Fourier-Jacobi expansion (2.48) depended on a choice of a nowhere vanishing section s of $\mathcal{L}$ along the boundary $C = \bar{S} - S$ of $\bar{S}$. As the boundary $\tilde{C} = \overline{Ig}_\mu - Ig_\mu$ is (non-canonically) identified with $\Delta(p) \times C$, we may "compute" the Fourier-Jacobi expansion on the Igusa surface rather than on S. But on the Igusa surface, $a(1)$ is a canonical choice for such an s. We may therefore associate a *canonical* Fourier-Jacobi expansion

$$FJ(f) = \sum_{m=0}^{\infty} c_m(f) \in \prod_{m=0}^{\infty} H^0(\tilde{C}, \mathcal{N}^m) \tag{4.8}$$

along the boundary of Ig, for every

$$f \in M_*(N, R) = \bigoplus_{k=0}^{\infty} M_k(N, R) \tag{4.9}$$

(R an R_0/pR_0-algebra). The following proposition becomes almost a *tautology.*

Proposition 4.3. *The Fourier-Jacobi expansion $FJ(h_{\bar{\Sigma}})$ of the Hasse invariant is 1. Moreover, for f_1 and f_2 in the graded ring $M_*(N,R)$, $r(f_1)=r(f_2)$ if and only if $FJ(f_1)=FJ(f_2)$.*

Proof. The first statement is tautologically true. For the second, note that for $f\in M_k(N,R)$, $FJ(f)$ is the (expansion of the) image of $f/a(k)$ in $H^0(\tilde{C},\mathcal{O}_{\widehat{Ig}})$ where $\widehat{Ig}$ is the formal completion of Ig along $\tilde{C}$, while $r(f)$ is the image of $f/a(k)$ in $H^0(\overline{Ig}_\mu,\mathcal{O})$. The proposition follows from the fact that by Proposition 3.24 the irreducible components of $\overline{Ig}_\mu$ are in bijection with the connected components of $\bar{S}$, so every irreducible component of $\overline{Ig}_\mu$ contains at least one cuspidal component ("q-expansion principle"). A function on $\overline{Ig}_\mu$ that vanishes in the formal neighborhood of any cuspidal component must therefore vanish on any irreducible component, so is identically 0. ∎

4.1.3. *The filtration of a modular form modulo p.* Let $f\in M_k(N,R)$, where R is an R_0/pR_0-algebra as before. Define the *filtration* $\omega(f)$ to be the minimal j such that $r(f)=r(f')$ (equivalently $FJ(f)=FJ(f')$) for some $f'\in M_j(N,R)$. The following proposition follows immediately from previous results.

Proposition 4.4. *Let $f\in M_k(N,R)$. Then $\omega(f)\le k$ and*

$$\omega(f)\equiv k \bmod (p^2-1). \tag{4.10}$$

Let $\omega(f)=k-(p^2-1)n$. Then n is the order of vanishing of f *along S_{ss}. Equivalently, $k-\omega(f)$ is the order of vanishing of the pull-back of f to Ig along Ig_{ss}. In addition, $\omega(f^m)=m\omega(f)$.*

4.2. **The theta operator.**

4.2.1. *Definition of $\Theta(f)$.* From now on we work over $\bar{\mathbb{F}}_p$. We first recall some notation. Let S be the (open) Picard surface over $\bar{\mathbb{F}}_p$ and $Ig=Ig(p)$ the Igusa surface of level p over S (completed along the supersingular locus as explained above). We denote by $Z=S_{ss}=S-S_\mu$ the supersingular locus of S, by $\tilde{Z}=Ig_{ss}=Ig-Ig_\mu$ its pre-image under the covering map $\tau: Ig\to S$, by $Z'=S_{gss}=S_{ss}-S_{ssp}$ the smooth part of Z, and by $\tilde{Z}'=Ig_{gss}=Ig_{ss}-Ig_{ssp}$ the pre-image of Z' under τ. When we need to compactify these schemes at the cusps, we let $\bar{S}$ and $\overline{Ig}$ stand for the smooth compactifications. By the Koecher principle, the space of $\bmod\, p$ modular forms of weight $k\ge 0$ and level N can be regarded as sections of $\mathcal{L}^k$ over either S or $\bar{S}$:

$$M_k(N,\bar{\mathbb{F}}_p)=H^0(S,\mathcal{L}^k)=H^0(\bar{S},\mathcal{L}^k). \tag{4.11}$$

We recall (Theorem 3.23) that $\tau:\overline{Ig}\to\bar{S}$ is finite Galois of degree p^2-1, that it is étale outside Z, and fully ramified over Z. The Galois group of the covering may be canonically identified with

$$\Delta=(\mathcal{O}_{\mathcal{K}}/p\mathcal{O}_{\mathcal{K}})^\times. \tag{4.12}$$

If M is an $\bar{\mathbb{F}}_p[\Delta]$ module, and $\chi:\Delta\to\bar{\mathbb{F}}_p^\times$ a character, we let M^χ be the submodule on which Δ acts via χ. We continue to denote by $\mathcal{A}$ the universal abelian scheme over Ig (or its extension to a semi-abelian scheme over $\overline{Ig}$) and by $\mathcal{P}$ and $\mathcal{L}$ the Σ- and $\bar{\Sigma}$-parts of $\omega_{\mathcal{A}/Ig}$. These vector bundles are just the base-change by τ^* of their counterparts over $\bar{S}$. We let $a(1)$ be the canonical section of $\mathcal{L}$ over $\overline{Ig}$, trivializing

$\mathcal{L}$ over $\overline{Ig}_\mu = \overline{Ig} - \tilde{Z}$, and vanishing to first order along $\tilde{Z}$. The Galois group Δ acts on $a(1)$ via $\bar{\Sigma}^{-1}$.

Let $f \in H^0(S, \mathcal{L}^k)$. Let $g = r(f) = \tau^* f / a(1)^k \in H^0(Ig_\mu, \mathcal{O})$. This function has a pole of order k along $\tilde{Z}$, and the Galois group acts on it via $\bar{\Sigma}^k$. Let $dg \in H^0(Ig_\mu, \Omega^1_{Ig}) = H^0(Ig_\mu, \tau^* \Omega^1_S)$. The Kodaira-Spencer isomorphism $KS(\Sigma)$ is an isomorphism

$$KS(\Sigma) : \mathcal{P} \otimes \mathcal{L} \simeq \Omega^1_S. \tag{4.13}$$

Write $KS(\Sigma)^{-1}$ for its inverse: $\Omega^1_S \simeq \mathcal{P} \otimes \mathcal{L}$. Let

$$\kappa = (V \otimes 1) \circ KS(\Sigma)^{-1} : \Omega^1_S \to \mathcal{L}^{(p)} \otimes \mathcal{L} \simeq \mathcal{L}^{p+1} \tag{4.14}$$

be the map obtained from $KS(\Sigma)^{-1}$ when we apply V to $\mathcal{P}$. Over $S - S_{ssp}$ this is the same as dividing out by the line sub-bundle $\mathcal{P}_0 = \ker V_{\mathcal{P}}$, since

$$\mathcal{P}/\mathcal{P}_0 = \mathcal{P}_\mu \simeq V(\mathcal{P}) = \mathcal{L}^{(p)}. \tag{4.15}$$

We denote by κ also the map induced on the base-change of these vector bundles by τ^* to $Ig - Ig_{ssp}$, and consider $\kappa(dg)$. As Δ still acts on $\kappa(dg)$ via $\bar{\Sigma}^k$, its action on $a(1)^k \kappa(dg)$ is trivial, so it descends to S_μ. We define

$$\Theta(f) = a(1)^k \kappa(dg) \in H^0(S_\mu, \mathcal{L}^{k+p+1}). \tag{4.16}$$

A priori, this extends only to a meromorphic modular form of weight $k + p + 1$, as it may have poles along Z.

4.2.2. *The main theorem.* For the formulation of the next theorem we need to define what we mean by the *standard cuspidal component* of $\bar{S}$ or $\overline{Ig}$. Recall that according to [Bel] and [La1] the cuspidal scheme $C = \bar{S} - S$ classifies $\mathcal{O}_{\mathcal{K}}$-semi-abelian varieties with level N structure. The *standard component* of C is the component (over $\mathbb{C}$) which classifies extensions of the elliptic curve $\mathbb{C}/\mathcal{O}_{\mathcal{K}}$ by the $\mathcal{O}_{\mathcal{K}}$-torus $\mathcal{O}_{\mathcal{K}} \otimes \mathbb{C}^\times$ (thus sits over a cusp of type $(\mathcal{O}_{\mathcal{K}}, \mathcal{O}_{\mathcal{K}})$ in $S^*_{\mathbb{C}}$), together with a level-N structure (α, β, γ) (see [Bel], I.4.2 and Section 1.6.2), where

$$\alpha : \mathcal{O}_{\mathcal{K}}/N\mathcal{O}_{\mathcal{K}} = \mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}/N\mathbb{Z} \to \mathcal{O}_{\mathcal{K}} \otimes \mathbb{C}^\times \tag{4.17}$$

is given by $1 \otimes (a \mapsto \exp(2\pi i a/N))$ and

$$\beta : \mathcal{O}_{\mathcal{K}}/N\mathcal{O}_{\mathcal{K}} = N^{-1}\mathcal{O}_{\mathcal{K}}/\mathcal{O}_{\mathcal{K}} \to \mathbb{C}/\mathcal{O}_{\mathcal{K}} \tag{4.18}$$

is the canonical embedding. (The splitting γ varies along the component.) The standard component of C over R_N is the one which becomes this component after base change to $\mathbb{C}$ (our convention is that all number fields are contained in $\mathbb{C}$). Let P be a prime of R_N above p at which we reduce the Picard surface. The standard component of C over R_N/PR_N is the reduction modulo P of the standard component of C over R_N. Finally, $\overline{Ig}$ maps to $\bar{S}$ (over R_N/PR_N) and the cuspidal components mapping to a given component E of C are classified by the embedding of $\delta_{\mathcal{K}}^{-1}\mathcal{O}_{\mathcal{K}} \otimes \mu_p$ in the toric part of $\mathcal{A}$. Since the toric part of the universal semi-abelian variety over the standard component is $\mathcal{O}_{\mathcal{K}} \otimes \mathbb{G}_m$, we may define the standard cuspidal component of $\overline{Ig}$ to be the component where the map

$$\varepsilon : \delta_{\mathcal{K}}^{-1}\mathcal{O}_{\mathcal{K}} \otimes \mu_p \to \mathcal{O}_{\mathcal{K}} \otimes \mathbb{G}_m \tag{4.19}$$

is the natural embedding. Here we use the fact that

$$\delta_{\mathcal{K}}^{-1}\mathcal{O}_{\mathcal{K}} \otimes \mu_p = \mathcal{O}_{\mathcal{K}} \otimes \mu_p \tag{4.20}$$

since $\delta_{\mathcal{K}}$ is invertible in $\mathcal{O}_{\mathcal{K}}/p\mathcal{O}_{\mathcal{K}}$. Let $\tilde{E} \subset \tilde{C} = \overline{Ig} - Ig$ be this standard component.

Theorem 4.5. *(i) The operator Θ maps $H^0(S,\mathcal{L}^k)$ to $H^0(S,\mathcal{L}^{k+p+1})$.*

(ii) The effect of Θ on Fourier-Jacobi expansions is a "Tate twist". More precisely, let

$$FJ(f)=\sum_{m=0}^{\infty}c_m(f) \tag{4.21}$$

be the Fourier-Jacobi expanison of f along $\tilde{E}$ (thus $c_m(f)\in H^0(\tilde{E},\mathcal{N}^m)$). Then

$$FJ(\Theta(f))=M^{-1}\sum_{m=0}^{\infty}mc_m(f). \tag{4.22}$$

Here M (equal to N or $2^{-1}N$) is the width of the cusp.

(iii) If $f\in H^0(S,\mathcal{L}^k)$ and $g\in H^0(S,\mathcal{L}^l)$ then

$$\Theta(fg)=f\Theta(g)+\Theta(f)g. \tag{4.23}$$

(iv) $\Theta(h_{\bar{\Sigma}}f)=h_{\bar{\Sigma}}\Theta(f)$ (equivalently, $\Theta(h_{\bar{\Sigma}})=0$).

Corollary 4.6. *The operator Θ extends to a derivation of the graded ring of modular forms, and for any f, $\Theta(f)$ is a cusp form.*

Parts (iii) and (iv) of the theorem are clear from the construction. The proof of (i), that $\Theta(f)$ is in fact *holomorphic* along S_{ss}, will be given in the next section. We shall now study its effect on Fourier-Jacobi expansions, i.e. part (ii). That a factor like M^{-1} is necessary in (ii) becomes evident if we consider what happens to FJ expansions under level change. If N is replaced by $N'=NQ$ then the conormal bundle becomes the Q-th power of the conormal bundle of level N', i.e. $\mathcal{N}=\mathcal{N}'^{Q}$ (see Section 1.4.3). It follows that what was the m-th FJ coefficient at level N becomes the Qm-th coefficient at level N'. The operator Θ commutes with level-change, but the factor M^{-1}, which changes to $(QM)^{-1}$, takes care of this.

4.2.3. *The effect of Θ on FJ expansions.* Let E be the standard cuspidal component of $\bar{S}$ (over the ring R_N). As the reader probably noticed, we have trivialized the line bundle $\mathcal{L}$ along E on two different occasions in two seemingly different ways, that we now have to compare. On the one hand, after reducing modulo P and pulling $\mathcal{L}$ back to the Igusa surface, we got a *canonical* nowhere vanishing section $a(1)$ trivializing $\mathcal{L}$ over $\overline{Ig}_\mu$, and in particular along any of the p^2-1 cuspidal components lying over E in $\overline{Ig}_\mu$. On the other hand, extending scalars from R_N to $\mathbb{C}$, shifting to the analytic category, restricting to the connected component $\bar{X}_\Gamma$ on which E lies, and then pulling back to the unit ball $\mathfrak{X}$, we have trivialized $\mathcal{L}|_E$ by means of the section $2\pi i d\zeta_3$, which we showed to be $\mathcal{K}_N$-rational.

Lemma 4.7. *The sections $a(1)$ and $2\pi i d\zeta_3$ "coincide" in the sense that they come from the same section in $H^0(E,\mathcal{L})$.*

Proof. Let A be the universal semi-abelian variety over E. Its toric part is $\mathcal{O}_\mathcal{K}\otimes\mathbb{G}_m$, hence, taking $\bar{\Sigma}$-component of the cotangent space at the origin

$$\mathcal{L}|_{\tilde{E}}=\omega_{A/\tilde{E}}(\bar{\Sigma})=(\delta_\mathcal{K}^{-1}\mathcal{O}_\mathcal{K}\otimes\omega_{\mathbb{G}_m})(\bar{\Sigma}) \tag{4.24}$$

admits the canonical section $e_{\bar{\Sigma}}\cdot(1\otimes dT/T)$. Tracing back the definitions and using (1.69), this section becomes, under the base change $R_N\hookrightarrow\mathbb{C}$, just $2\pi i d\zeta_3$. On the other hand, when we reduce it modulo P and use the Igusa level structure ε at the standard cusp, it pulls back to the section "with the same name" $e_{\bar{\Sigma}}\cdot(1\otimes dT/T)$, because along $\tilde{E}$ (4.19) induces the identity on cotangent spaces. The lemma follows from the fact that, by definition, $\varepsilon^* a(1) = e_{\bar{\Sigma}} \cdot (1 \otimes dT/T)$ too. ∎

Lemma 4.8. *The sections $a(1)^{p+1}$ and $2\pi i d\zeta_2 \otimes 2\pi i d\zeta_3$ "coincide" in the sense that they come from the same section in $H^0(E, \mathcal{P}_\mu \otimes \mathcal{L})$.*

Proof. Let σ_2 (resp. σ_3) be the $\mathcal{K}_N$-rational section of $\mathcal{P}_\mu$ (resp. $\mathcal{L}$) along E, which over $\mathbb{C}$ becomes the section $2\pi i d\zeta_2$ (resp. $2\pi i d\zeta_3$). We have just seen that modulo P, when we identify $\tilde{E}$ with E (via the covering map $\tau : \overline{Ig} \to \bar{S}$), σ_3 reduces to $a(1)$. To conclude, we must show that the map

$$V : \mathcal{P}/\mathcal{P}_0 = \mathcal{P}_\mu \simeq \mathcal{L}^{(p)} \tag{4.25}$$

carries σ_2 to $\sigma_3^{(p)}$. This will map, under $\mathcal{L}^{(p)} \simeq \mathcal{L}^p$, to $a(1)^p$. Along E the line bundles $\mathcal{P}_\mu$ and $\mathcal{L}$ are just the Σ- and $\bar{\Sigma}$-parts of the cotangent space at the origin of the torus $\mathcal{O}_\mathcal{K} \otimes \mathbb{G}_m$, and σ_2 and σ_3 are the sections

$$\sigma_2 = e_\Sigma \cdot (1 \otimes dT/T), \ \sigma_3 = e_{\bar{\Sigma}} \cdot (1 \otimes dT/T). \tag{4.26}$$

Since in characteristic p, $V = Ver^* : \omega_{\mathbb{G}_m} \to \omega_{\mathbb{G}_m}^{(p)}$ maps dT/T to $(dT/T)^{(p)}$, for the $\mathcal{O}_\mathcal{K}$-torus, $V(\sigma_2) = \sigma_3^{(p)}$, and we are done. ∎

To prove part (ii) we argue as follows. Let $g = f/a(1)^k$ be the function on $\overline{Ig}_\mu$ obtained by trivializing the line bundle $\mathcal{L}$. We have to study the FJ expansion along $\tilde{E}$ of $\kappa(dg)/a(1)^{p+1}$, where κ is the map defined in (4.14). For that purpose we may restrict to a formal neighborhood of $\tilde{E}$. This formal neighborhood is isomorphic, under the covering map $\tau : \overline{Ig}_\mu \to \bar{S}_\mu$, to the formal neighborhood $\widehat{S}$ of E in S. We may therefore regard dg as an element of $\Omega^1_{\widehat{S}}$. Now

$$\kappa : \Omega^1_{\widehat{S}} \to \mathcal{P}_\mu \otimes \mathcal{L} \tag{4.27}$$

is a homomorphism of $\mathcal{O}_{\widehat{S}}$-modules defined over R_N so, having restricted to $\widehat{S}$, we may study the effect of κ on FJ expansions by embedding $\widehat{S}_\mathbb{C}$ in a tubular neighborhood $\bar{S}(\varepsilon)$ of E and using complex analytic Fourier-Jacobi expansions. We are thus reduced to a complex-analytic computation, near the standard cusp at infinity.

Let

$$g(z, u) = \sum_{m=0}^{\infty} \theta_m(u) q^m \tag{4.28}$$

where $q = e^{2\pi i z/M}$ and θ_m is a theta function, so that $\theta_m(u)q^m$ is a section of $\mathcal{N}^m$ along E (now over $\mathbb{C}$). Then

$$dg = 2\pi i M^{-1} \sum_{m=0}^{\infty} m\theta_m(u) q^m dz + \sum_{m=0}^{\infty} \theta'_m(u) q^m du. \tag{4.29}$$

According to Corollary 2.17, $\kappa(du) = 0$, and $\kappa(dz) = 2\pi i d\zeta_2 \otimes d\zeta_3$. It follows that

$$\kappa(dg) = M^{-1} \sum_{m=0}^{\infty} m\theta_m(u) q^m \cdot 2\pi i d\zeta_2 \otimes 2\pi i d\zeta_3. \tag{4.30}$$

Recalling that in characteristic p, $2\pi i d\zeta_2 \otimes 2\pi i d\zeta_3$ reduced to $a(1)^{p+1}$, the proof of part (ii) of the theorem is now complete. For the convenience of the reader we summarize the transitions between complex and p-adic maps in the following diagram:

$$
\begin{array}{lllll}
/\bar{\mathbb{F}}_p & \Omega^1_{\bar{S}_\mu/\bar{\mathbb{F}}_p} & \overset{KS(\Sigma)^{-1}}{\longrightarrow} & \mathcal{P}\otimes\mathcal{L} & & \\
 & \cap & & \downarrow \bmod \mathcal{P}_0 & & \\
/\bar{\mathbb{F}}_p & \Omega^1_{\widehat{S}/\bar{\mathbb{F}}_p} & \overset{\kappa}{\to} & \mathcal{P}_\mu\otimes\mathcal{L} & \overset{V\otimes 1}{\simeq} & \mathcal{L}^{p+1} \\
 & \uparrow \bmod P & & \uparrow & & \\
/R_N & \Omega^1_{\widehat{S}/R_N} & \overset{\kappa}{\to} & \mathcal{P}_\mu\otimes\mathcal{L} & & \\
 & \downarrow & & \downarrow & & \\
/\mathbb{C} & \Omega^1_{\widehat{S}/\mathbb{C}} & \overset{\kappa}{\to} & \mathcal{P}_\mu\otimes\mathcal{L} & & \\
 & \cup & & \uparrow \bmod \mathcal{P}_0 & & \\
/\mathbb{C} & \Omega^1_{\bar{S}(\varepsilon)/\mathbb{C}} & \overset{KS(\Sigma)^{-1}_{an}}{\longrightarrow} & \mathcal{P}\otimes\mathcal{L} & &
\end{array}
\quad . \tag{4.31}
$$

We next turn to part (i).

4.3. **A study of the theta operator along the supersingular locus.**

4.3.1. *De Rham cohomology in characteristic p.* We continue to consider the Picard surface S over $\bar{\mathbb{F}}_p$. Let $U = Spec(R) \hookrightarrow S$ be a closed point s_0 ($R = \bar{\mathbb{F}}_p = \mathcal{O}_{S,s_0}/\mathfrak{m}_{S,s_0}$), a nilpotent thickening of a closed point, or an affine open subset of S. We consider the restriction of the universal abelian scheme to R and denote it by A/R. Let $A^{(p)} = R \otimes_{\phi,R} A$ be its base change with respect to the map $\phi(x) = x^p$. Let

$$D = H^1_{dR}(A/R), \tag{4.32}$$

a locally free R-module of rank 6. The de Rham cohomology of $A^{(p)}$ is

$$D^{(p)} = R \otimes_{\phi,R} D. \tag{4.33}$$

The R-linear Frobenius and Verschiebung morphisms $Frob : A \to A^{(p)}$, $Ver : A^{(p)} \to A$ induce (by pull-back) linear maps

$$F : D^{(p)} \to D,\ V : D \to D^{(p)}. \tag{4.34}$$

Both F and V are everywhere of rank 3, which implies that their kernel and image are locally free direct summands. Moreover, $\operatorname{Im} F = \ker V$ and $\operatorname{Im} V = \ker F = \omega_{A^{(p)}/R}$. The maps F and V preserve the types $\Sigma, \bar{\Sigma}$, but note that $D^{(p)}(\Sigma) = D(\bar{\Sigma})^{(p)}$ etc.

The principal polarization on A induces one on $A^{(p)}$, and these polarizations induce symplectic forms

$$\langle , \rangle : D \times D \to R,\ \langle , \rangle^{(p)} : D^{(p)} \times D^{(p)} \to R \tag{4.35}$$

where the second form is just the base-change of the first. For $x \in D^{(p)}, y \in D$ we have

$$\langle Fx, y\rangle = \langle x, Vy\rangle^{(p)}. \tag{4.36}$$

In addition, for $a \in \mathcal{O}_{\mathcal{K}}$

$$\langle \iota(a)x, y\rangle = \langle x, \iota(\bar{a})y\rangle. \tag{4.37}$$

As $VF = FV = 0$, the first relation implies that $\operatorname{Im} F$ and $\operatorname{Im} V$ are isotropic subspaces. So is $\omega_{A/R}$.

The Gauss-Manin connection is an integrable connection

$$\nabla : D \to \Omega^1_R \otimes D. \tag{4.38}$$

It is a priori defined (e.g. in [Ka-O]) when R is smooth over $\bar{\mathbb{F}}_p$, but we can define it by base change also when R is a nilpotent thickening of a point of S (see [Kob], where R is a local Artinian ring). Note however that if $R = \mathcal{O}_{S,s_0}$ and $R_m = \mathcal{O}_{S,s_0}/\mathfrak{m}^{m+1}_{S,s_0}$ ($m \geq 0$), and if we extend scalars from R to R_m we get by base change the connection

$$\nabla : D_{R_m} \to (\Omega^1_R \otimes_R R_m) \otimes_{R_m} D_{R_m} \to \Omega^1_{R_m} \otimes_{R_m} D_{R_m} \tag{4.39}$$

but $\Omega^1_R \otimes_R R_m \to \Omega^1_{R_m}$ is *not* an isomorphism. In fact, $\Omega^1_{R_m}$ is not R_m-free. As a result the Kodaira-Spencer map

$$KS(\Sigma) : \mathcal{P}_{R_m} \otimes_{R_m} \mathcal{L}_{R_m} \to \Omega^1_{R_m} \tag{4.40}$$

will not be an isomorphism over R_m, as it is over R. It is nevertheless true that $KS(\Sigma)$ induces an isomorphism

$$KS(\Sigma) : \mathcal{P}_{R_{m-1}} \otimes_{R_{m-1}} \mathcal{L}_{R_{m-1}} \simeq \Omega^1_{R_m} \otimes_{R_m} R_{m-1}. \tag{4.41}$$

This follows from the fact that $\Omega^1_{R_m} \otimes_{R_m} R_{m-1} = \Omega^1_R \otimes_R R_{m-1}$ (if u, v are local parameters at s_0 then $\{u^i v^j du,\, u^i v^j dv\}_{i+j \leq m-2}$ are linearly independent over $\bar{\mathbb{F}}_p$ in $\Omega^1_{R_m}$, the only dependencies between these differentials occuring for $i + j = m - 1$).

We shall need to deal only with the first infinitesimal neighborhood of a point, $R = \mathcal{O}_{S,s_0}/\mathfrak{m}^2_{S,s_0}$. In this case, D has a basis of horizontal sections. Indeed, $R = \bar{\mathbb{F}}_p[u,v]/(u^2, uv, v^2)$ where u and v are local parameters at s_0, and

$$\Omega^1_R = (Rdu + Rdv)/\langle udu,\, vdv,\, udv + vdu\rangle$$

(p is odd). If $x \in D$ and

$$\nabla x = du \otimes x_1 + dv \otimes x_2 \tag{4.42}$$

then $\tilde{x} = x - ux_1 - vx_2$ satisfies

$$\nabla \tilde{x} = -u\nabla x_1 - v\nabla x_2. \tag{4.43}$$

But if $\nabla x_1 = du \otimes x_{11} + dv \otimes x_{12}$ and $\nabla x_2 = du \otimes x_{21} + dv \otimes x_{22}$ then

$$0 = \nabla^2 x = du \wedge dv \otimes (x_{21} - x_{12}) \tag{4.44}$$

hence $x_{21} - x_{12} \in \mathfrak{m}_{S,s_0} D$. It follows that

$$\begin{aligned} \nabla \tilde{x} &= -udv \otimes x_{12} - vdu \otimes x_{21} \\ &= du \otimes v(x_{12} - x_{21}) = 0. \end{aligned} \tag{4.45}$$

This means that $\tilde{x}$ is a horizontal section having the same specialization as x in the special fiber, so the horizontal sections span D over R by Nakayama's lemma.

Let $e_1, e_2, f_3, f_1, f_2, e_3$ be any six horizontal sections over R, specializing to a basis of $H^1_{dR}(A_{s_0}/\bar{\mathbb{F}}_p)$. Let D_0 be their $\bar{\mathbb{F}}_p$-span. As we have just seen,

$$R \otimes_{\bar{\mathbb{F}}_p} D_0 = D \tag{4.46}$$

and $\nabla = d \otimes 1$. Since $R^{d=0} = \bar{\mathbb{F}}_p$, it follows that $D_0 = D^\nabla$, i.e. there are no more horizontal sections besides D_0. Thus every $x \in H^1_{dR}(A_{s_0}/\bar{\mathbb{F}}_p)$ has a *unique* extension to a horizontal section $x \in H^1_{dR}(A/R)$.

There is a similar connection on $D^{(p)}$. The isogenies $Frob$ and Ver, like any isogeny, take horizontal sections with respect to the Gauss-Manin connection to horizontal sections, e.g. if $x \in D$ and $\nabla x = 0$ then $Vx \in D^{(p)}$ satisfies $\nabla(Vx) = 0$.

The pairing $\langle , \rangle$ is horizontal for ∇, i.e.

$$d\langle x, y\rangle = \langle \nabla x, y\rangle + \langle x, \nabla y\rangle . \tag{4.47}$$

Remark 4.1. *In the theory of Dieudonné modules one works over a perfect base. It is then customary to identify D with $D^{(p)}$ via $x \leftrightarrow 1 \otimes x$. This identification is only σ-linear where $\sigma = \phi$, now viewed as an* automorphism *of R. The operator F becomes σ-linear, V becomes σ^{-1}-linear and (4.36) reads $\langle Fx, y\rangle = \langle x, Vy\rangle^{\sigma}$. With this convention F and V switch types, rather than preserve them.*

4.3.2. *The Dieudonné module at a* gss *point.* Assume from now on that $s_0 \in Z' = S_{gss}$ is a closed point of the general supersingular locus. We write D_0 for $H^1_{dR}(A_{s_0}/\bar{\mathbb{F}}_p)$.

Lemma 4.9. *There exists a basis $e_1, e_2, f_3, f_1, f_2, e_3$ of D_0 with the following properties. Denote by $e_1^{(p)} = 1 \otimes e_1 \in D_0^{(p)}$ etc.*

(i) $\mathcal{O}_{\mathcal{K}}$ acts on the e_i via Σ and on the f_i via $\bar{\Sigma}$ (hence it acts on the $e_i^{(p)}$ via $\bar{\Sigma}$ and on the $f_i^{(p)}$ via Σ).

(ii) The symplectic pairing satisfies

$$\langle e_i, f_j\rangle = -\langle f_j, e_i\rangle = \delta_{ij}, \ \ \langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0. \tag{4.48}$$

(iii) The vectors e_1, e_2, f_3 form a basis for the cotangent space $\omega_{A_0/\bar{\mathbb{F}}_p}$. Hence e_1 and e_2 span $\mathcal{P}$ and f_3 spans $\mathcal{L}$.

(iv) $\ker(V)$ *is spanned by e_1, f_2, e_3. Hence $\mathcal{P}_0 = \mathcal{P} \cap \ker(V)$ is spanned by e_1.*

(v) $Ve_2 = f_3^{(p)}$, $Vf_3 = e_1^{(p)}$, $Vf_1 = e_2^{(p)}$.

(vi) $Ff_1^{(p)} = -e_3$, $Ff_2^{(p)} = -e_1$, $Fe_3^{(p)} = -f_2$.

Proof. Up to a slight change of notation, this is the unitary Dieudonné module which Bültel and Wedhorn call a "braid of length 3" and denote by $\bar{B}(3)$, cf [Bu-We] (3.2). The classification in loc. cit. Proposition 3.6 shows that the Dieudonné module of a μ-ordinary abelian variety is isomorphic to $\bar{B}(2) \oplus \bar{S}$, that of a gss abelian variety is isomorphic to $\bar{B}(3)$ and in the superspecial case we get $\bar{B}(1) \oplus \bar{S}^2$. ∎

4.3.3. *Infinitesimal deformations.* Let $\mathcal{O}_{S,s_0}$ be the local ring of S at s_0, $\mathfrak{m}$ its maximal ideal, and $R = \mathcal{O}_{S,s_0}/\mathfrak{m}^2$. This R is a truncated polynomial ring in two variables, isomorphic to $\bar{\mathbb{F}}_p[u, v]/(u^2, uv, v^2)$.

As remarked above, the de Rham cohomology $D = H^1_{dR}(A/R)$ has a basis of horizontal sections,

$$D = R \otimes_{\bar{\mathbb{F}}_p} D^{\nabla}, \ \ \nabla = d \otimes 1 \tag{4.49}$$

and since $D_0 = \bar{\mathbb{F}}_p \otimes_R D$, we may identify D^{∇} canonically with D_0.

Grothendieck tells us that A/R is completely determined by A_0 and by the Hodge filtration $\omega_{A/R} \subset D = R \otimes_{\bar{\mathbb{F}}_p} D_0$. Since A *is* the universal infinitesimal deformation of A_0, we may choose the coordinates u and v so that

$$\mathcal{P} = Span_R\{e_1 + ue_3, e_2 + ve_3\}. \tag{4.50}$$

The fact that $\omega_{A/R}$ is isotropic implies then that

$$\mathcal{L} = Span_R\{f_3 - uf_1 - vf_2\}. \tag{4.51}$$

Consider the abelian scheme $A^{(p)}$. It is *not* the universal deformation of $A_0^{(p)}$ over R. In fact, the map $\phi : R \to R$ factors as

$$R \xrightarrow{\pi} \bar{\mathbb{F}}_p \xrightarrow{\phi} \bar{\mathbb{F}}_p \xrightarrow{i} R, \tag{4.52}$$

and therefore $A^{(p)}$, *unlike* A, is constant: $A^{(p)} = Spec(R) \times_{Spec\bar{\mathbb{F}}_p} A_0^{(p)}$. As with D, $D^{(p)} = R \otimes_{\bar{\mathbb{F}}_p} D_0^{(p)}$, $\nabla = d \otimes 1$, but this time the basis of horizontal sections can be obtained also from the trivalization of $A^{(p)}$, and $\omega_{A^{(p)}/R} = Span_R\{e_1^{(p)}, e_2^{(p)}, f_3^{(p)}\}$.

Since V and F preserve horizontality, e_1, f_2, e_3 span $\ker(V)$ over R in D, and the relations in *(v)* and *(vi)* of Lemma 4.9 continue to hold. Indeed, the matrix of V in the basis at s_0 prescribed by that lemma, continues to represent V over $Spec(R)$ by "horizontal continuation". The matrix of F is then derived from the relation (4.36).

The Hodge filtration nevertheless varies, so we conclude that

$$\mathcal{P}_0 = \mathcal{P} \cap \ker(V) = Span_R\{e_1 + ue_3\}. \tag{4.53}$$

The condition $V(\mathcal{L}) = \mathcal{P}_0^{(p)}$, which is the "equation" of the closed subscheme $Z' \cap Spec(R)$ (see Proposition 3.9) means

$$V(f_3 - uf_1 - vf_2) = e_1^{(p)} - ue_2^{(p)} \in R \cdot e_1^{(p)} \tag{4.54}$$

and this holds if and only if $u = 0$. We have proved the following lemma.

Lemma 4.10. *Let $s_0 \in S_{gss}$ as above. Then the closed subscheme $S_{gss} \cap Spec(R)$ is given by the equation $u = 0$.*

4.3.4. *The Kodaira-Spencer isomorphism along the general supersingular locus.* We keep the assumptions of the previous subsections, and compute what the Gauss-Manin connection does to $\mathcal{P}_0$. A typical element of $\mathcal{P}_0$ is $g(e_1 + ue_3)$ for some $g \in R$. Then

$$\nabla(g(e_1 + ue_3)) = dg \otimes (e_1 + ue_3) + gdu \otimes e_3. \tag{4.55}$$

Note that when we divide by $\omega_{A/R}$ and project $H^1_{dR}(A/R)$ to $H^1(A, \mathcal{O})$, $e_1 + ue_3$ dies, and the image $\overline{e_3}$ of e_3 becomes a basis for the line bundle that we called $\mathcal{L}^\vee(\rho) = H^1(A, \mathcal{O})(\Sigma)$. Recall the definition of κ given in (4.14), but note that this definition only makes sense over $Spec(\mathcal{O}_{S,s_0})$ or its completion, where $KS(\Sigma)$ is an isomorphism, and can be inverted.

Proposition 4.11. *Let $s_0 \in Z' = S_{gss}$. Choose local parameters u and v at s_0 so that in $\mathcal{O}_{S,s_0}$ the local equation of Z' becomes $u = 0$. Then at s_0, $\kappa(du)$ has a zero along Z'.*

Proof. Let $i : Z' \hookrightarrow S$ be the locally closed embedding. We must show that in a suitable Zariski neighborhood of s_0, where $u = 0$ is the local equation of Z', $i^*\kappa(du) = 0$. It is enough to show that the image of $\kappa(du)$ in the *fiber* at every point s of Z' near s_0, vanishes. All points being alike, it is enough to do it at s_0. In other words, we denote by κ_0 the map

$$\kappa_0 : \Omega^1_{S,s_0} \to \mathcal{P}_\mu \otimes \mathcal{L}|_{s_0} \simeq \mathcal{L}^{p+1}|_{s_0}. \tag{4.56}$$

and show that $\kappa_0(du) = 0$. We may now work over $Spec(R)$, where $R = \mathcal{O}_{S,s_0}/\mathfrak{m}^2$. It is enough to show that in the diagram

$$\begin{array}{ccc} \mathcal{P}_R \otimes \mathcal{L}_R & \overset{KS(\Sigma)}{\rightarrow} & \Omega^1_R \\ \downarrow & & \downarrow \\ \mathcal{P}_{s_0} \otimes \mathcal{L}_{s_0} & \simeq & \Omega^1_{S,s_0} \end{array} \tag{4.57}$$

$KS(\Sigma)$ maps the line sub-bundle $\mathcal{P}_{0,R} \otimes \mathcal{L}_R$ onto Rdu. Once we have passed to the infinitesimal neighborhood $Spec(R)$ we can replace the local parameters u, v by any two formal parameters for which $u = 0$ defines $Z' \cap Spec(R)$. We may therefore assume, in view of Lemma 4.10, that u and v have been chosen as in section 4.3.3. But then equation (4.55) shows that the restriction of $KS(\Sigma)$ to Z', i.e. the homomorphism $i^*KS(\Sigma)$, maps $i^*\mathcal{P}_0$ onto $i^*R \cdot du \otimes \overline{e_3}$. This concludes the proof. ∎

4.3.5. *A computation of poles along the supersingular locus.* We are now ready to prove the following.

Proposition 4.12. *Let $k \geq 0$, and let $f \in H^0(S, \mathcal{L}^k)$ be a modular form of weight k in characteristic p. Then $\Theta(f) \in H^0(S, \mathcal{L}^{k+p+1})$.*

Proof. A priori, the definition that we have given for $\Theta(f)$ produces a meromorphic section of $\mathcal{L}^{k+p+1}$ which is holomorphic on the μ-ordinary part S_μ but may have a pole along $Z = S_{ss}$. Since S is a non-singular surface, it is enough to show that $\Theta(f)$ does not have a pole along $Z' = S_{gss}$, the non-singular part of the divisor Z. Consider the degree $p^2 - 1$ covering $\tau : Ig \to S$, which is finite, étale over S_μ and totally ramified along Z. Let $s_0 \in Z'$ and let $\tilde{s}_0 \in Ig$ be the closed point above it. Let u, v be formal parameters at s_0 for which Z' is given by $u = 0$, as in Theorem 3.23. As explained there we may choose formal parameters w, v at $\tilde{s}_0$ where $w^{p^2-1} = u$ (and v is the same function v pulled back from S to Ig). It follows that in Ω^1_{Ig} we have

$$du = -w^{p^2-2}dw. \tag{4.58}$$

We now follow the steps of our construction. Dividing f by $a(1)^k$ we get a function $g = f/a(1)^k$ on Ig with a pole of order k along $\tilde{Z}$, the supersingular divisor on Ig, whose local equation is $w = 0$. In $\widehat{\mathcal{O}}_{Ig,\tilde{s}_0}$ we may write

$$g = \sum_{l=-k}^{\infty} g_l(v)w^l. \tag{4.59}$$

Then

$$\begin{aligned} dg &= \sum_{l=-k}^{\infty} l g_l(v) w^{l-1} dw + \sum_{l=-k}^{\infty} w^l g_l'(v) dv \\ &= -\sum_{l=-k}^{\infty} l g_l w^{l-(p^2-1)} du + \sum_{l=-k}^{\infty} w^l g_l'(v) dv. \end{aligned} \tag{4.60}$$

Applying the map κ (extended $\mathcal{O}_{Ig}$-linearly from S to Ig), and noting that $\kappa(du)$ has a zero along Z', hence a zero of order $p^2 - 1$ along $\tilde{Z}'$, we conclude that $\kappa(dg)$ has a pole of order k (at most) along $\tilde{Z}'$. Finally $\Theta(f) = a(1)^k \cdot \kappa(dg)$ becomes holomorphic along $\tilde{Z}'$, and also descends to S. It is therefore a holomorphic section of $\mathcal{P}_\mu \otimes \mathcal{L}^{k+1} \simeq \mathcal{L}^{k+p+1}$. ∎

It is amusing to compare the reasons for the increase by $p+1$ in the weight in $\Theta(f)$ for modular curves and for Picard modular suraces. In the case of modular curves the Kodaira-Spencer isomorphism is responsible for a shift by 2 in the weight, but the section acquires simple poles at the supersingular points. One has to multiply it by the Hasse invariant, which has weight $p-1$, to make the section holomorphic, hence a total increase by $p+1 = 2 + (p-1)$ in the weight. In our case, the Kodaira-Spencer isomorphism is responsible for a shift by $p+1$ (the p coming from $\mathcal{P}_\mu \simeq \mathcal{L}^p$), but the section turns out to be holomorphic along the supersingular locus. See Section 4.5.

4.4. **Relation to the filtration and theta cycles.** In part (ii) of the main theorem we have described the way Θ acts on Fourier-Jacobi expansions at the standard cusp. A similar formula inevitably exists at any other cusp. We may deduce from it that modular forms in the image of Θ have vanishing FJ coefficients in degrees divisible by p. Moreover, for such a form $f \in \mathrm{Im}(\Theta)$, $\Theta^{p-1}(f)$ and f have the same FJ expansions, and hence the same filtration. Note also that if $r(f_1) = r(f_2)$ then $r(\Theta(f_1)) = r(\Theta(f_2))$. We may therefore define unambiguously

$$\Theta(r(f)) = r(\Theta(f)). \tag{4.61}$$

As we clearly have

$$\omega(\Theta(f)) = \omega(f) + p + 1 - a(p^2 - 1) \tag{4.62}$$

for some $a \geq 0$ we deduce the following result.

Proposition 4.13. *Let $f \in M_k(N, \bar{\mathbb{F}}_p)$ be a modular form modulo p, and assume that $r(f) \in \mathrm{Im}(\Theta)$. Then*

$$r(f) = r(\Theta^{p-1}(f)). \tag{4.63}$$

There exists a unique index $0 \leq i \leq p-2$ such that

$$\omega(\Theta^{i+1}(f)) = \omega(\Theta^i(f)) + p + 1 - (p^2 - 1). \tag{4.64}$$

For any other i in this range

$$\omega(\Theta^{i+1}(f)) = \omega(\Theta^i(f)) + p + 1. \tag{4.65}$$

This is reminiscent of the "theta cycles" for classical (i.e. elliptic) modular forms modulo p, see [Se], [Ka2] and [Joc]. Recall that if f is a $\mathrm{mod}\, p$ modular form of weight k on $\Gamma_0(N)$ with q-expansion $\sum a_n q^n$ ($a_n \in \bar{\mathbb{F}}_p$), then $\Theta(f)$ is a $\mathrm{mod}\, p$ modular form of weight $k+p+1$ with q expansion $\sum n a_n q^n$ (Katz denotes $\Theta(f)$ by $A\theta(f)$). One has $\omega(\Theta(f)) < \omega(f) + p + 1$ *if and only if* $\omega(f) \equiv 0 \bmod p$. In such a case we say that the filtration "drops" and we have

$$\omega(\Theta(f)) = \omega(f) + p + 1 - a(p-1) \tag{4.66}$$

for some $a > 0$. As a corollary, $\omega(f)$ can never equal $1 \bmod p$ for an $f \in \mathrm{Im}(\Theta)$. Assume now that $f \in \mathrm{Im}(\Theta)$ is a "low point" in its "theta cycle", namely, $\omega(f)$ is minimal among all $\omega(\Theta^i(f))$. Then $\omega(\Theta^{i+1}(f)) < \omega(\Theta^i(f)) + p + 1$ for *one* or *two* values of $i \in [0, p-2]$, which are completely determined by $\omega(f) \bmod p$ [Joc].

This is not true anymore for Picard modular forms. Not only the drop in the theta cycle is unique, but the question of when exactly it occurs is mysterious and deserves further study. We make the following elementary observation showing that whether a drop in the filtration occurs in passing from f to $\Theta(f)$ can *not* be determined by $\omega(f)$ modulo p alone. Let f and k be as in the Proposition.

1. If $k \leq p^2 - 1$ then $\omega(f) = k$.
2. If $k < p+1$ then $\omega(\Theta^i(f)) = k + i(p+1)$ for $0 \leq i \leq p-2$, so the drop occurs at the last step of the theta cycle, i.e. at weight $k + (p-2)(p+1)$, which is congruent to $k-2$ modulo p.
3. If $k < p+1$ but $f \notin \mathrm{Im}(\Theta)$ then starting with $\Theta(f)$ instead of f, one sees that the drop in the theta cycle of $\Theta(f)$ occurs either in passing from $\Theta^{p-2}(f)$ to $\Theta^{p-1}(f)$, or in passing from $\Theta^{p-1}(f)$ to $\Theta^p(f)$.

4.5. **Compatability between theta operators for elliptic and Picard modular forms.**

4.5.1. *The theta operator for elliptic modular forms.* The theta operator for elliptic modular forms modulo p was introduced by Serre and Swinnerton-Dyer in terms of q-expansions, *cf.* [Se], but its geometric construction was given by Katz in [Ka1] and [Ka2], where it is denoted $A\theta$. Katz' construction relied on a canonical splitting of the Hodge filtration over the ordinary locus, but it coincides with a slightly modified construction, similar to the one we have been using over the Picard modular surface. This construction was suggested in [Gr], Proposition 5.8, see also [An-Go] in the Hilbert modular case.

Let X be the modular curve $X(N)$ over $\bar{\mathbb{F}}_p$ ($N \geq 3$, $p \nmid N$) and I_{ord} the Igusa curve of level p lying over $X_{ord} = X - X_{ss}$, the ordinary part of X. Let $\bar{X}$ and $\bar{I}_{ord}$ be the curves obtained by adjoing the cusps to X and I_{ord} respectively. Let $\mathcal{L} = \omega_{E/X}$ be the cotangent bundle of the universal elliptic curve, extended over the cusps as usual. Classical modular forms of weight k and level N are sections of $\mathcal{L}^k$ over $\bar{X}$. Let $a(1)$ be the tautological nowehere vanishing section of $\mathcal{L}$ over $\bar{I}_{ord}$. Given a modular form f of weight k, we consider $r(f) = \tau^* f / a(1)^k$ where $\tau : \bar{I}_{ord} \to \bar{X}$ is the covering map, and apply the inverse of the Kodaira-Spencer isomorphism $KS : \mathcal{L}^2 \to \Omega^1_{\bar{I}_{ord}}$ to get a section $\kappa(dr(f))$ of $\mathcal{L}^2$ over $\bar{I}_{ord}$. When multiplied by $a(1)^k$ it descends to $\bar{X}_{ord}$, and when this is multiplied further by $h = a(1)^{p-1}$, the Hasse invariant for elliptic modular forms, it extends holomorphically over X_{ss} to an element

$$\theta(f) = a(1)^{k+p-1} \kappa(dr(f)) \in \Gamma(\bar{X}, \mathcal{L}^{k+p+1}). \tag{4.67}$$

4.5.2. *An embedding of a modular curve in $\bar{S}$.* To illustrate our idea, and to simplify the computations, we assume that $N = 1$ and $d_{\mathcal{K}} \equiv 1 \bmod 4$, so that $D = D_{\mathcal{K}} = d_{\mathcal{K}}$. We shall treat only one special embedding of the modular curve $\bar{X} = X_0(D)$ into $\bar{S}$ (there are many more).

Embed $SL_2(\mathbb{R})$ in G'_∞ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}. \tag{4.68}$$

This embedding induces an embedding of symmetric spaces $\mathfrak{H} \hookrightarrow \mathfrak{X}$, $z \mapsto {}^t(z, 0)$. One can easily compute that the intersection of Γ, the stabilizer of the lattice L_0 in G'_∞, with $SL_2(\mathbb{R})$, is the subgroup of $SL_2(\mathbb{Z})$ given by

$$\Gamma^0(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : D | b \right\}. \tag{4.69}$$

Let $E_0 = \mathbb{C}/\mathcal{O}_{\mathcal{K}}$, endowed with the canonical principal polarization and CM type Σ. For $z \in \mathfrak{H}$ let $\Lambda_z = \mathbb{Z} + \mathbb{Z}z$ and $E_z = \mathbb{C}/\Lambda_z$. Let M_z be the cyclic subgroup of order D of E_z generated by $D^{-1}z \bmod \Lambda_z$. Using the model (1.27) of the abelian variety A_z associated to the point ${}^t(z,0) \in \mathfrak{X}$, we compute that

$$A_z \simeq E_0 \times (\mathcal{O}_{\mathcal{K}} \otimes E_z/\delta_{\mathcal{K}} \otimes M_z) \tag{4.70}$$

with the obvious $\mathcal{O}_{\mathcal{K}}$-structure. The principal polarization on A_z provided by the complex uniformization is the product of the canonical polarization of E_0 and the principal polarization of $\mathcal{O}_{\mathcal{K}} \otimes E_z/\delta_{\mathcal{K}} \otimes M_z$ obtained by descending the polarization

$$\lambda_{can} : \mathcal{O}_{\mathcal{K}} \otimes E_z \to \delta_{\mathcal{K}}^{-1} \otimes E_z = (\mathcal{O}_{\mathcal{K}} \otimes E_z)^t \tag{4.71}$$

of degree D^2, modulo the maximal isotropic subgroup $\delta_{\mathcal{K}} \otimes M_z$ of $\ker(\lambda_{can})$.

It is now clear that over any R_0-algebra R we have the same moduli theoretic construction, sending a pair (E, M) where M is a cyclic subgroup of degree D to $A(E, M)$, with $\mathcal{O}_{\mathcal{K}}$ structure and polarization given by the same formulae. This gives a modular embedding $X \to S$ which is generically injective. To make this precise at the level of schemes (rather than stacks) one would have to add a level N structure and replace the base ring R_0 by R_N.

4.5.3. *Comparison of the two theta operators.* From now on we work over $\bar{\mathbb{F}}_p$. The modular interpretation of the embedding $j : \bar{X} \to \bar{S}$ allows us to complete it to a diagram

$$\begin{array}{ccc} \bar{I}_{ord} & \xrightarrow{j} & \overline{Ig}_{\mu} \\ \tau \downarrow & & \downarrow \tau \\ \bar{X}_{ord} & \xrightarrow{j} & \bar{S}_{\mu} \end{array} . \tag{4.72}$$

Lemma 4.14. *The pull-back $j^*\omega_{\mathcal{A}/S}$ decomposes as a product $\omega_{E_0} \times (\mathcal{O}_{\mathcal{K}} \otimes \omega_{E/X})$. Under this isomorphism*

$$\begin{aligned} j^*\mathcal{L} &= (\mathcal{O}_{\mathcal{K}} \otimes \omega_{E/X})(\bar{\Sigma}) \\ j^*\mathcal{P}_0 &= \omega_{E_0} \\ j^*\mathcal{P}_{\mu} &\simeq (\mathcal{O}_{\mathcal{K}} \otimes \omega_{E/X})(\Sigma). \end{aligned} \tag{4.73}$$

The line bundle $j^*\mathcal{P}_0$ is constant, and $\mathcal{P}_{\mu}$, originally a quotient bundle of $\mathcal{P}$, becomes a direct summand when restricted to $\bar{X}$.

Proof. This is straightforward from the construction of j, and the fact that E_0 is supersingular, while E is ordinary over $\bar{X}_{ord}$. Note that $\mathcal{O}_{\mathcal{K}} \otimes E/\delta_{\mathcal{K}} \otimes M$ and $\mathcal{O}_{\mathcal{K}} \otimes E$ have the same cotangent space. ∎

Proposition 4.15. *Identify $j^*\mathcal{L}$ with $\omega_{E/X}$ ($\mathcal{O}_{\mathcal{K}}$ acting via $\bar{\Sigma}$). Then for $f \in \Gamma(\bar{S}, \mathcal{L}^k) = M_k(N, \bar{\mathbb{F}}_p)$*

$$\theta(j^*(f)) = j^*(\Theta(f)). \tag{4.74}$$

Proof. We abbreviate I_{ord} by I and Ig_{μ} by Ig. The pull-back via j of the tautological section $a(1)$ of $\mathcal{L}$ over Ig is the tautological section $a(1)$ of $j^*\mathcal{L} = \omega_{E/X}$. We therefore have

$$j^*(dr(f)) = dr(j^*(f)) \tag{4.75}$$

($r(f) = \tau^* f/a(1)^k$ is the function on Ig denoted earlier also by g). It remains to check the commutativity of the following diagram

$$\begin{array}{ccccc} \Omega^1_{Ig} & \stackrel{KS(\Sigma)^{-1}}{\to} & \mathcal{P}\otimes\mathcal{L} & \stackrel{V\otimes 1}{\to} & \mathcal{L}^{p+1} \\ \downarrow j_0^* & & & & \downarrow j^* \\ \Omega^1_I & \stackrel{KS^{-1}}{\to} & j^*\mathcal{L}^2 & \stackrel{\times h}{\to} & j^*\mathcal{L}^{p+1} \end{array}. \tag{4.76}$$

Here j_0^* is the map $j^*\Omega^1_{Ig} \to \Omega^1_I$ on differentials whose kernel is the conormal bundle of I in Ig. For that we have to compare the Kodaira-Spencer maps on S and on X. As we have seen in the lemma, $\mathcal{P}/\mathcal{P}_0 = \mathcal{P}_\mu$ pulls back under j to $\mathcal{L}(\rho)$ (the line bundle $\mathcal{L}$ with the $\mathcal{O}_\mathcal{K}$ action conjugated). But, $KS(\Sigma)(\mathcal{P}_0 \otimes \mathcal{L})$ maps under j^* to the conormal bundle, so we obtain a commutative diagram

$$\begin{array}{ccc} \Omega^1_{Ig} & \stackrel{KS(\Sigma)}{\leftarrow} & \mathcal{P}\otimes\mathcal{L} \\ \downarrow j_0^* & & \downarrow \mathrm{mod}\,\mathcal{P}_0 \\ \Omega^1_I & \stackrel{KS}{\leftarrow} & j^*\mathcal{L}(\rho)\otimes j^*\mathcal{L} \end{array}. \tag{4.77}$$

The commutativity of the diagram

$$\begin{array}{ccc} \mathcal{P}_\mu & \stackrel{V}{\to} & \mathcal{L}^{(p)} \\ \downarrow & & \downarrow \\ j^*\mathcal{L}(\rho) & \stackrel{\times h}{\to} & j^*\mathcal{L}^{(p)} \end{array} \tag{4.78}$$

follows from the definition of the Hasse invariant h on X. Identifying $\mathcal{L}^{(p)}$ with $\mathcal{L}^p$ as usual and tensoring the last diagram with $\mathcal{L}$ provides the last piece of the puzzle. ∎

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