

q -BINOMIALS AND NON-CONTINUITY OF THE p -ADIC FOURIER TRANSFORM

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ABSTRACT. Let F be a finite extension of \mathbb{Q}_p . We show that every Schwartz function on F , with values in $\overline{\mathbb{Q}_p}$, is the p -adic uniform limit of a sequence of Schwartz functions, whose Fourier transforms tend uniformly to 0. The proof uses the notion of q -binomial coefficients and some classical identities between these polynomials.

1. INTRODUCTION

Let p be a prime, and let F be a finite extension of \mathbb{Q}_p (the field of p -adic numbers). Let C be an algebraically closed field of characteristic 0. The Fourier transform \mathcal{F} (defined in section 2.1) maps the space \mathcal{S} of locally constant, compactly supported, C -valued functions on F , to itself. When C is the field of complex numbers, this is the basis for harmonic analysis on F , and it is well-known that \mathcal{F} extends to a unitary operator on $L^2(F, dx)$ where dx is the normalized Haar measure.

When C is an algebraic closure of \mathbb{Q}_p , equipped with the p -adic norm $|\cdot|_p$ (normalized in this paper so that $|p|_p = p^{-1}$), no such L^2 -theory is available. One may ask how badly \mathcal{F} behaves when we equip \mathcal{S} with the simplest available norm, the sup norm. Our main result, which turns out to be trickier than anticipated, is the following.

Theorem 1. *Let $\phi \in \mathcal{S}$. Then there are $\phi_n \in \mathcal{S}$ which tend uniformly to ϕ , such that $\mathcal{F}(\phi_n)$ tend uniformly to 0.*

A few remarks are in order.

(i) By the Stone-von Neumann theorem (see the discussion in section 4.1), our theorem is valid for all ϕ if and only if it is valid for one non-zero ϕ .

(ii) The completion of \mathcal{S} in the sup norm is the space $C_0(F)$ of continuous functions on F “vanishing at infinity”. Consider the graph $\Gamma_{\mathcal{F}}$ of \mathcal{F} , as a subspace of $\mathcal{S} \times \mathcal{S} \subset C_0(F) \times C_0(F)$. Equip the space $C_0(F) \times C_0(F)$ with the norm

$$\|(f_1, f_2)\| = \max\{\|f_1\|_{sup}, \|f_2\|_{sup}\}.$$

An equivalent statement is that $\Gamma_{\mathcal{F}}$ is dense in $C_0(F) \times C_0(F)$.

(iii) Let $\Lambda \subset \mathcal{S}$ be the unit ball for the sup norm, and $\widehat{\Lambda}$ its image under the Fourier transform. Both are integral structures for the action of the Heisenberg group $H_\psi(F)$ (see section 2.3). Another formulation of Theorem 1 is that

$$\Lambda + \widehat{\Lambda} = \mathcal{S}.$$

(iv) Let π be a uniformizer of F , let $r \in \mathbb{Z}$, and consider the subspaces $\mathcal{S}_r \subset \mathcal{S}$ and $C_0(F)_r \subset C_0(F)$ of functions which are constant on cosets of the additive subgroup $\pi^r \mathcal{O}_F$. While the Fourier transform is not a bounded linear operator on the whole of \mathcal{S} , it *is* bounded when restricted to \mathcal{S}_r , and maps it to the subspace $\mathcal{S}^r \subset \mathcal{S}$ of functions supported on $\pi^{-r} \mathcal{O}_F$. It therefore extends by continuity to a map of $C_0(F)_r$ into $C_0(F)^r$ (similarly defined). In particular, when $r = 0$, the Fourier transform of $f \in C_0(F)_0 = C_0(F/\mathcal{O}_F)$ is the “Fourier series”

$$\widehat{f}(x) = \sum_{y \in F/\mathcal{O}_F} f(y)\psi(xy)$$

($x \in \mathcal{O}_F$). In this context, Fresnel and de Mathan [2, 3], and, independently, Amice and Escassut [1] showed already in 1973 that the map $f \mapsto \widehat{f}$ is non-injective. In other words, they have constructed an $f \in C_0(F)_0$ and a sequence $\phi_n \in \mathcal{S}_0$ converging uniformly to f , for which $\mathcal{F}(\phi_n)$ converged uniformly to 0. Our theorem is different in the sense that it demands the limit function f to be a Schwartz function ϕ . It is easy to see that in this context, the approximating sequence ϕ_n does *not* belong to any fixed subspace \mathcal{S}_r (or else, by continuity, we would get the contradiction $\mathcal{F}(\phi) = 0$).

(v) There does not seem to be any direct way to adapt the methods of Fresnel and de Mathan (or Amice and Escassut) to prove our theorem. If we could prove a Banach-space version of the Stone-von Neumann theorem, on the topological irreducibility of $C_0(F)$ as a p -adic Banach representation of the Heisenberg group $H(F)$, we would be able to deduce our theorem from the example of Fresnel and de Mathan. This remains, at present, out of reach, so our theorem can only be regarded as an indirect evidence for such a theorem.

(vi) When C is the field of complex numbers, and \mathcal{S} is endowed with the sup norm rather than the L^2 -norm, the Fourier transform is again not continuous. Strangely, the analogous theorem is false then, see section 3, where we also discuss l -adic coefficients for $l \neq p$.

(vii) We do not treat in the present paper a local field F of characteristic p .

The proof proceeds along some unexpected lines. We reduce easily to the case $F = \mathbb{Q}_p$, where, as noted above, it suffices to prove the theorem for $\phi_0 = 1_{\mathbb{Z}_p}$, the characteristic function of \mathbb{Z}_p . Fix an integer $r \geq 0$ and consider the subspace \mathcal{S}_r^r of functions in \mathcal{S} which are supported on $p^{-r}\mathbb{Z}_p$ and are constant modulo $p^r\mathbb{Z}_p$. This is a finite dimensional space, of dimension p^{2r} , and \mathcal{F} preserves it. Let

$$\gamma_r = \inf_{\phi \in \mathcal{S}_r^r} \{\max\{\|\phi - \phi_0\|_{sup}, \|\mathcal{F}(\phi)\|_{sup}\}\}.$$

Our goal is to show that $\gamma_r \rightarrow 0$.

For that purpose we compute the matrix of \mathcal{F} in a suitable basis of \mathcal{S}_r^r , where it comes out to be a multiple of a Vandermonde matrix Z . The key tool in the proof is the use of q -binomial coefficients and the q -binomial theorem, reviewed in section 4.2. These allow us to decompose

$$Z = {}^tUDU$$

as a product of a lower-triangular unipotent matrix tU , a diagonal matrix D , and an upper-triangular unipotent matrix U . While this decomposition must be known, and is closely related to the q -Vandermonde identity, we have not found it in the literature in the shape needed here, so we gave a self-contained account of it. The

matrix U has coefficients in the ring $\mathbb{Z}[\zeta]$ where ζ is a primitive p^{2r} root of unity in C . The decomposition of Z allows us to give a criterion for γ_r to be less than a fixed positive ϵ . If this is the case, we say that ϵ is *attainable* in S_r^r . The criterion is expressed in terms of inequalities for certain quantities $\Omega_{r,n}$ for $0 \leq n \leq p^{2r} - 1$. There are “dual” equivalent inequalities for quantities that we denote $\omega_{r,n}$. Taking both types of inequalities into account, we show that for any given ϵ , our criterion holds if r is large enough, as long as n is outside the “critical interval” $[\frac{1-\delta}{2}(p^{2r} - 1), \frac{1+\delta}{2}(p^{2r} - 1)]$. Here δ is a small pre-fixed positive number. To show that the inequalities hold, for sufficiently large r , even if n belongs to the critical interval, we study certain Laurent polynomials $\Omega_{r,n}(q) \in \mathbb{Z}[q, q^{-1}]$ for which $\Omega_{r,n} = \Omega_{r,n}(\zeta)$. Studying the divisibility of the $\Omega_{r,n}(q)$ by the cyclotomic polynomials $\Phi_{p^j}(q)$ for various values of j , allows us to obtain the required estimates on $\Omega_{r,n}$ for n in the critical interval.

Searching the arXive, we found that q -analogues of classical polynomials and congruences between them have recently attracted the attention of a group of authors in China. This interest arose in connection with combinatorial identities, hypergeometric series, and conjectures of Rodriguez-Villegas. See [5, 6] and the references therein. In fact, our Lemma 12 is taken from [5]. It seems to us that there might be something more fundamental lurking behind this theory, and that its successful application to our problem is not a coincidence.

2. THE FOURIER TRANSFORM

2.1. Definition. Let F be a local field of residue characteristic p . Later on, F itself will be assumed to be of characteristic 0, i.e. a finite extension of \mathbb{Q}_p , but for the meanwhile we allow it to be of characteristic p as well. Let C be an algebraically closed field of characteristic 0 (the *field of coefficients*) and consider the Schwartz space $\mathcal{S} = C_c^\infty(F)$ of locally constant, compactly supported, C -valued functions on F .

Let \mathcal{O}_F be the ring of integers of F and fix, once and for all, an additive character

$$\psi : F \rightarrow C^\times,$$

such that \mathcal{O}_F is equal to its own annihilator under the pairing $\langle x, y \rangle = \psi(xy)$. Let dx be the unique C -valued distribution on the boolean algebra of open and compact sets, which assigns to \mathcal{O}_F the measure 1. The Fourier transform of $\phi \in \mathcal{S}$ is the function $\mathcal{F}(\phi) = \widehat{\phi} \in \mathcal{S}$ given by

$$\widehat{\phi}(x) = \int_F \phi(y)\psi(xy)dy.$$

The Fourier inversion formula says that

$$\mathcal{F}(\mathcal{F}(\phi)) = \dot{\phi}$$

where $\dot{\phi}(x) = \phi(-x)$. Thus \mathcal{F} is “almost an involution”.

2.2. Mesh and support. Let π be a uniformizer of F . To every $\phi \in \mathcal{S}$ we attach two integers. Its *support* $N(\phi)$ is

$$N(\phi) = \min \{n \in \mathbb{Z} \mid \phi(x) = 0 \forall x \notin \pi^{-n}\mathcal{O}_F\},$$

and its *mesh* $M(\phi)$ is

$$M(\phi) = \min \{n \in \mathbb{Z} \mid \phi(x+y) = \phi(x) \forall y \in \pi^n\mathcal{O}_F\}.$$

The Fourier transform exchanges the support and the mesh, i.e.

$$N(\mathcal{F}(\phi)) = M(\phi), \quad M(\mathcal{F}(\phi)) = N(\phi).$$

2.3. The Heisenberg group and the Stone-von Neumann theorem. The *Heisenberg group* $H(F)$ is the group of upper-triangular unipotent matrices in $SL_3(F)$. We write

$$(t; b, b') = \begin{pmatrix} 1 & b' & t \\ & 1 & b \\ & & 1 \end{pmatrix}.$$

Its center, which is also equal to the derived group $H(F)'$, is the subgroup of all elements of the form $(t; 0, 0)$, and is isomorphic to F . We therefore have a short exact sequence of groups

$$1 \rightarrow F \rightarrow H(F) \rightarrow F \times F \rightarrow 1,$$

the projection to $F \times F$ given by $(t; b, b') \mapsto (b, b')$.

Let C^1 be the unit circle $\{\lambda \mid |\lambda| = 1\} \subset C^\times$. The push-out $H_\psi(F)$ of $H(F)$ by the character $\psi : F \rightarrow C^1$ is the group of all triples $[\lambda; b, b']$ with $\lambda \in C^1$ and $b, b' \in F$. The group operation is given by the formula

$$[\lambda; b, b'] [\lambda_1; b_1, b'_1] = [\lambda \lambda_1 \psi(b'b_1); b + b_1, b' + b'_1].$$

The group $H(F)$ acts on \mathcal{S} via

$$((t; b, b')\phi)(x) = \psi(t)\psi(bx)\phi(x + b').$$

This action factors through $H_\psi(F)$, and in the induced action of the latter $\lambda \in C^1$ acts via multiplication by λ .

Let V be a vector space over C and G a topological group. Recall that a representation $\rho : G \rightarrow GL(V)$ is called *smooth* if the stabilizer of every vector is open. The celebrated Stone-von Neumann theorem is the following [4]:

Theorem. *The space \mathcal{S} with the action of $H(F)$ given above is smooth and irreducible. Moreover, up to isomorphism it is the unique smooth irreducible representation of $H(F)$ in which the center acts via the character ψ .*

The relation between the action of $H(F)$ and \mathcal{F} is the following:

$$\mathcal{F} \circ (t; b, b') = (t - bb'; -b', b) \circ \mathcal{F}.$$

3. COMPLEX AND l -ADIC COEFFICIENTS

3.1. Complex coefficients. Let C be the field of complex numbers \mathbb{C} . The Haar distribution dx is, in this case, a Haar measure. The Fourier transform preserves the L^2 metric on \mathcal{S} and extends to a unitary operator on $L^2(F, dx)$. In contrast, if we endow \mathcal{S} with the sup norm $\|\phi\| = \sup_{x \in F} |\phi(x)|$, \mathcal{F} is unbounded, so does not extend by continuity to the Banach space $C_0(F)$, the completion of \mathcal{S} . It is natural to consider its graph

$$\Gamma_{\mathcal{F}} = \{(\phi, \widehat{\phi}) \mid \phi \in \mathcal{S}\} \subset \mathcal{S} \times \mathcal{S} \subset C_0(F) \times C_0(F)$$

and the closure $C_{\mathcal{F}}(F)$ of $\Gamma_{\mathcal{F}}$ in $C_0(F) \times C_0(F)$ under the norm

$$\|(f_1, f_2)\| = \max\{\|f_1\|, \|f_2\|\}.$$

This is a linear subspace of $C_0(F) \times C_0(F)$, and if the projection $pr_1 : C_{\mathcal{F}}(F) \rightarrow C_0(F)$ to the first factor is *injective*, we may extend \mathcal{F} to $\text{Im}(pr_1)$, setting

$$\mathcal{F} = pr_2 \circ pr_1^{-1}.$$

In this way the image of pr_1 becomes a natural subspace of $C_0(F)$ to which we can extend the Fourier transform. All hinges on pr_1 being injective. We are indebted to Izzy Katsenelson for pointing out to us that this is indeed the case.

Proposition 2. *The projection $pr_1 : C_{\mathcal{F}}(F) \rightarrow C_0(F)$ is injective.*

Proof. Recall that if $h \in \mathcal{S}$ and $g \in C_0(F)$ their convolution is defined by

$$h * g(x) = \int_F h(y)g(x - y)dy.$$

If also $g \in \mathcal{S}$ then $h * g \in \mathcal{S}$ and

$$\widehat{h * g} = \widehat{h}\widehat{g}, \quad \widehat{hg} = \widehat{h} * \widehat{g}.$$

Now let $\Delta, \phi \in \mathcal{S}$ and $f \in C_0(F)$. We claim that

$$(i) \|\Delta * f\|_{\infty} \leq \|\Delta\|_{L^1} \|f\|_{\infty} \quad (ii) \|\Delta * \phi\|_{\infty} \leq \|\widehat{\Delta}\|_{L^1} \|\widehat{\phi}\|_{\infty}.$$

Point (i) is clear. For (ii) note that

$$\|\widehat{\Delta} * \widehat{\phi}\|_{\infty} = \|\widehat{\Delta\phi}\|_{\infty} \leq \|\Delta\phi\|_{L^1} \leq \|\Delta\|_{L^1} \|\phi\|_{\infty},$$

substitute $\widehat{\Delta}$ for Δ and $\widehat{\phi}$ for ϕ , and use the Fourier inversion formula.

We can now prove the proposition. Suppose that $(f, 0) \in C_{\mathcal{F}}(F)$ with $f \neq 0$ and pick a sequence $\phi_n \in \mathcal{S}$ such that $\phi_n \rightarrow f$ but $\widehat{\phi}_n \rightarrow 0$ uniformly. Take any $\Delta \in \mathcal{S}$ such that $\Delta * f \neq 0$. Then

$$\Delta * f = \Delta * (f - \phi_n) + \Delta * \phi_n.$$

By (i) $\Delta * (f - \phi_n) \rightarrow 0$ uniformly and by (ii) $\Delta * \phi_n \rightarrow 0$ uniformly. This is a contradiction. \square

3.2. l -adic coefficients. Let $l \neq p$ be a prime and take for C an algebraic closure of \mathbb{Q}_l , endowed with the l -adic metric. Let $\|\phi\|$ be the sup norm on \mathcal{S} and as usual, $C_0(F)$ its (l -adic) Banach completion. In this case the Fourier transform is unitary,

$$\|\widehat{\phi}\| = \|\phi\|$$

for every $\phi \in \mathcal{S}$. Indeed, the ultra-metric inequality and the fact that p is an l -adic unit, imply $\|\widehat{\phi}\| \leq \|\phi\|$, and by Fourier inversion we get an equality. If $\|\phi - 1_{\mathcal{O}_F}\| < 1$ it follows that $\|\widehat{\phi} - 1_{\mathcal{O}_F}\| < 1$, so $\|\widehat{\phi}\| = 1$. The analogue of Theorem 1 is blatantly false. To the contrary, \mathcal{F} extends by continuity to a unitary isomorphism of $C_0(F)$ onto itself.

4. p -ADIC COEFFICIENTS AND THE MAIN THEOREM

4.1. An easy reduction. From now on F will be assumed to be a finite extension of \mathbb{Q}_p , and C an algebraic closure of \mathbb{Q}_p . We normalize the p -adic absolute value on C by $|p| = p^{-1}$ and let again $\|\phi\|$ denote the sup norm of ϕ . The remainder of our work will be devoted to a proof of Theorem 1.

It is easy to check that the space of functions $\phi \in \mathcal{S}$ for which there exists a sequence ϕ_n as in Theorem 1 is closed under translation and also under multiplication by $\psi(bx)$ for any $b \in F$. It follows from the Stone-von Neumann theorem that

this space is either 0 or the whole of \mathcal{S} . It is thus enough to prove the theorem for one specific choice $\phi = \phi_0 \neq 0$, for example $\phi_0 = 1_{\mathcal{O}_F}$.

Next, view F as a vector space of dimension $d = [F : \mathbb{Q}_p]$ over \mathbb{Q}_p and fix a basis $\alpha_1, \dots, \alpha_d$. Then we obtain an isomorphism

$$C_c^\infty(F) \simeq C_c^\infty(\mathbb{Q}_p) \otimes \cdots \otimes C_c^\infty(\mathbb{Q}_p)$$

and it becomes clear that it is enough to prove the theorem for $F = \mathbb{Q}_p$. For example, if $d = 2$, assume that α_1, α_2 is also a basis of \mathcal{O}_F over \mathbb{Z}_p . The characteristic function Φ of \mathcal{O}_F is then

$$\Phi(x_1\alpha_1 + x_2\alpha_2) = \phi(x_1)\phi(x_2)$$

where ϕ is the characteristic function of \mathbb{Z}_p . For the additive character of F we take $\Psi = \psi \circ Tr_{F/\mathbb{Q}_p}$ where ψ is an additive character of \mathbb{Q}_p . This may not make \mathcal{O}_F self-dual, but the validity of the theorem will not be affected by the resulting modification in the Fourier transform. Let β_1, β_2 be the basis dual to α_1, α_2 under the trace pairing. Assume that $\phi_n \rightarrow \phi$ but $\widehat{\phi}_n \rightarrow 0$ and let $\Phi_n(x_1\alpha_1 + x_2\alpha_2) = \phi_n(x_1)\phi_n(x_2)$. Then $\Phi_n \rightarrow \Phi$ and

$$\begin{aligned} \widehat{\Phi}_n(x_1\beta_1 + x_2\beta_2) &= \int \int \phi_n(y_1)\phi_n(y_2)\Psi((x_1\beta_1 + x_2\beta_2)(y_1\alpha_1 + y_2\alpha_2))dy_1dy_2 \\ &= \widehat{\phi}_n(x_1)\widehat{\phi}_n(x_2) \rightarrow 0. \end{aligned}$$

We therefore assume that $F = \mathbb{Q}_p$ and $\phi_0 = 1_{\mathbb{Z}_p}$ is the characteristic function of \mathbb{Z}_p . For every integer $r \geq 0$ let \mathcal{S}_r^r be the vector space of C -valued functions supported on $p^{-r}\mathbb{Z}_p$ and constant modulo $p^r\mathbb{Z}_p$. It is invariant under \mathcal{F} , of dimension p^{2r} , and admits the basis

$$v_i = 1_{\frac{i}{p^r} + p^r\mathbb{Z}_p} \quad (0 \leq i \leq p^{2r} - 1).$$

Clearly

$$\phi_0 = \sum_{j=0}^{p^r-1} v_j p^r.$$

We introduce the positive quantity

$$\gamma_r = \inf_{\phi \in \mathcal{S}_r^r} \left\{ \max\{\|\phi - \phi_0\|, \|\widehat{\phi}\|\} \right\}.$$

The sequence γ_r is ≤ 1 and decreasing, and we say that $\epsilon > 0$ is *attainable* in \mathcal{S}_r^r if $\gamma_r \leq \epsilon$. Theorem 1 is equivalent to the statement

Every $\epsilon > 0$ is attainable in \mathcal{S}_r^r if r is large enough.

Fix ψ and let $\zeta = \psi(1/p^{2r})$, a primitive p^{2r} root of unity. The Fourier transform of v_j is

$$\widehat{v}_j(x) = \int_{p^r\mathbb{Z}_p} \psi(x(\frac{j}{p^r} + y))dy = \frac{1}{p^r} \sum_{i=0}^{p^{2r}-1} \zeta^{ij} v_i(x).$$

It follows that in the basis (v_i) the matrix of \mathcal{F} is $p^{-r}Z$, where Z is the Vandermonde matrix

$$Z = (\zeta^{ij})_{0 \leq i, j \leq p^{2r}-1}.$$

4.2. q -binomial coefficients. We review some well-known identities, working in the ring $\mathbb{Z}[q, q^{-1}]$ of Laurent polynomials. Substituting $q = 1$ yields the classical identities between binomial coefficients, but we shall be concerned later with $q = \zeta$, a primitive p^{2r} root of unity.

For $n \geq 0$ let

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}$$

$$[n]_q! = [1]_q \cdots [n]_q$$

($[0]_q! = 1$) and define the *Pochhammer symbol*

$$(X; q)_n = \prod_{k=0}^{n-1} (1 - Xq^k) \in \mathbb{Z}[q][X]$$

($(X; q)_0 = 1$). Substituting $X = q$ yields

$$(q; q)_n = (1 - q)^n [n]_q! = \prod_{k=1}^n (1 - q^k).$$

The q -binomial coefficients are

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

These are in fact *polynomials* in $\mathbb{Z}[q]$ of degree $k(n-k)$. One way to see it is via the q -Pascal identity:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n+1 \\ k \end{bmatrix}_q.$$

When k is an integer outside the interval $[0, n]$ we put $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$.

Lemma 3. (*q -binomial theorem*) *We have the identity*

$$(Xq; q)_n = \prod_{k=1}^n (1 - Xq^k) = \sum_{k=0}^n \left((-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \right) X^k.$$

In the proof of Theorem 1, the Laurent polynomials

$$(4.1) \quad \Omega_{r,n}(q) = \sum_{k=0}^{\lfloor n/p^r \rfloor} (-1)^{kp^r} q^{\binom{kp^r+1}{2} - nk p^r} \begin{bmatrix} n \\ kp^r \end{bmatrix}_q \in \mathbb{Z}[q, q^{-1}]$$

($0 \leq r, 0 \leq n \leq p^{2r} - 1$), will play an important role.

Lemma 4. *We have the formula*

$$(4.2) \quad \Omega_{r,n}(q) = \frac{1}{p^r} \sum_{\eta^{p^r}=1} \prod_{l=0}^{n-1} (1 - q^{-l} \eta).$$

Proof. We may write

$$\Omega_{r,n}(q) = \frac{1}{p^r} \sum_{\eta^{p^r}=1} \sum_{k=0}^n (-1)^k q^{\binom{k+1}{2} - nk} \begin{bmatrix} n \\ k \end{bmatrix}_q \eta^k.$$

Using the q -binomial theorem, substituting $q^{-n}\eta$ for X , we get the expression

$$\Omega_{r,n}(q) = \frac{1}{p^r} \sum_{\eta^{p^r}=1} \prod_{k=1}^n (1 - q^{k-n}\eta) = \frac{1}{p^r} \sum_{\eta^{p^r}=1} \prod_{l=0}^{n-1} (1 - q^{-l}\eta).$$

□

Observe that it is not clear from the last formula that $\Omega_{r,n}(q)$ has integral coefficients. Nevertheless, it will allow us to prove that $\Omega_{r,n}$ is divisible by certain cyclotomic polynomials over \mathbb{Q} , and by Gauss' lemma this divisibility will hold also over \mathbb{Z} .

4.3. A decomposition of the Vandermonde. Let $d \geq 1$ be an integer and consider the matrix

$$Z = Z_{q,d} = (q^{ij})_{0 \leq i,j \leq d-1}.$$

Let $U = U_{q,d} = (u_{n,k})$ be the upper-triangular unipotent matrix with entries in $\mathbb{Z}[q]$ defined by

$$u_{n,k} = \begin{bmatrix} k \\ n \end{bmatrix}_q$$

for $0 \leq n \leq k \leq d-1$. Similarly let $V = V_{q,d} = (v_{n,k})$ be the lower-triangular unipotent matrix with entries in $\mathbb{Z}[q]$ defined by

$$v_{n,k} = (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

for $0 \leq k \leq n \leq d-1$. Let $D = D_{q,d} = (w_{i,j})$ be the diagonal matrix with

$$w_{n,n} = (-1)^n q^{\binom{n}{2}} (q; q)_n$$

for $0 \leq n \leq d-1$.

Proposition 5. (i) One has $V_{q,d} = {}^t U_{q,d}^{-1}$. Equivalently,

$$\sum_{k=0}^n (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ m \end{bmatrix}_q = \delta_{n,m}.$$

(ii) One has the decomposition

$$Z_{q,d} = {}^t U_{q,d} D_{q,d} U_{q,d}.$$

Corollary 6. Let $L_{q,d} = D_{q,d}^{-1} V_{q,d}$. Then $U_{q,d} = L_{q,d} Z_{q,d}$.

Proof. Define $L_{q,d}$ as in the corollary. Thus, noting that $\binom{n-l}{2} - \binom{n}{2} = \binom{l+1}{2} - nl$,

$$(L_{q,d})_{n,l} = (-1)^l \frac{q^{\binom{l+1}{2} - nl}}{(q; q)_n} \begin{bmatrix} n \\ l \end{bmatrix}_q$$

and

$$\begin{aligned} \sum_{l=0}^{d-1} (L_{q,d})_{n,l} (Z_{q,d})_{l,k} &= \sum_{l=0}^{d-1} (-1)^l \frac{q^{\binom{l+1}{2} - nl}}{(q; q)_n} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{lk} \\ &= \frac{1}{(q; q)_n} \sum_{l=0}^n (-1)^l q^{\binom{l+1}{2}} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{(k-n)l} \end{aligned}$$

$$= \frac{1}{(q; q)_n} \prod_{l=1}^n (1 - q^{k-n+l}) = \frac{(q; q)_k}{(q; q)_n (q; q)_{k-n}} = \begin{bmatrix} k \\ n \end{bmatrix}_q = (U_{q,d})_{n,k}.$$

We have used the q -binomial theorem in the passage from the second to third line. Note also that if $k < n$ the product over $(1 - q^{k-n+l})$ vanishes because of the term with $l = n - k$.

This verifies the corollary, from which we deduce that

$$Z_{q,d} = V_{q,d}^{-1} D_{q,d} U_{q,d}.$$

Since $Z_{q,d}$ is a symmetric matrix, we obtain also $V_{q,d} = {}^t U_{q,d}^{-1}$, because the decomposition into a product of lower-triangular unipotent, diagonal and upper-triangular unipotent must be unique. \square

4.4. A criterion for attainability. Write $d = p^{2r}$. If we identify \mathcal{S}_r^r with the space C^d via the basis $(v_i)_{0 \leq i \leq d-1}$, then the sup norm on \mathcal{S}_r^r corresponds to the sup norm on C^d . The function $\phi_0 = 1_{\mathbb{Z}_p}$ corresponds to the column

$$J = \sum_{j=0}^{p^r-1} v_{jp^r} = {}^t(1, 0, \dots, 0; 1, 0, \dots, 0; \dots; 1, 0, \dots, 0).$$

Thus ϵ is attainable in \mathcal{S}_r^r if and only if there are $x, y \in C^d$ with $\|x\| \leq \epsilon$, $\|y\| \leq \epsilon$ and

$$\frac{1}{p^r} Z_{\zeta,d} x = y + J.$$

Equivalently,

$$U_{\zeta,d} x = p^r L_{\zeta,d} (y + J).$$

Note that $U_{\zeta,d}$ is an isometry of C^d , i.e. $\|U_{\zeta,d} x\| = \|x\|$. Indeed, as all its entries are in \mathcal{O}_C it is contracting, but the same is true of $U_{\zeta,d}^{-1}$. We conclude that ϵ is attainable in \mathcal{S}_r^r if and only if there exists a $y \in C^d$ with $\|y\| \leq \epsilon$ and

$$(4.3) \quad \|L_{\zeta,d} (y + J)\| \leq p^r \epsilon.$$

The advantage of the latter equation over the original one (with the $Z_{\zeta,d}$) is that $L_{\zeta,d}$ is triangular, so the equation renders itself to an inductive argument. The following lemma is our key criterion.

Lemma 7. For $0 \leq n \leq d - 1$ let

$$(4.4) \quad \Omega_{r,n} = \Omega_{r,n}(\zeta) = \sum_{k=0}^{\lfloor n/p^r \rfloor} (-1)^{kp^r} \zeta^{\binom{kp^r+1}{2} - nkp^r} \begin{bmatrix} n \\ kp^r \end{bmatrix}_{\zeta}.$$

Then ϵ is attainable in \mathcal{S}_r^r if and only if

$$(4.5) \quad |\Omega_{r,n}| \leq \max \{1, p^r |(\zeta; \zeta)_n|\} \cdot \epsilon$$

for all $0 \leq n \leq d - 1$.

Proof. We start by showing sufficiency. The n th entry of $L_{\zeta,d} (y + J)$ is

$$L_{\zeta,d} (y + J)_n = \sum_{k=0}^{d-1} (-1)^k \frac{\zeta^{\binom{k+1}{2} - nk}}{(\zeta; \zeta)_n} \begin{bmatrix} n \\ k \end{bmatrix}_{\zeta} (y_k + J_k)$$

$$= \frac{1}{(\zeta; \zeta)_n} \left\{ \sum_{k=0}^n (-1)^k \zeta^{\binom{k+1}{2} - nk} \begin{bmatrix} n \\ k \end{bmatrix}_\zeta y_k + \Omega_{r,n} \right\}.$$

Suppose y_0, \dots, y_{n-1} have been chosen so that $|y_i| \leq \epsilon$ and also $|L_{\zeta,d}(y+J)_i| \leq p^r \epsilon$ for all $0 \leq i \leq n-1$ (if $n=0$ this condition is empty). If $|\Omega_{r,n}| \leq \epsilon$, we can select y_n so that $L_{\zeta,d}(y+J)_n = 0$ and $|y_n| \leq \epsilon$. If $|\Omega_{r,n}| > \epsilon$, choose y_n arbitrarily so that $|y_n| \leq \epsilon$. Then

$$|L_{\zeta,d}(y+J)_n| = |\Omega_{r,n}/(\zeta; \zeta)_n|,$$

so if (4.5) is satisfied, $|L_{\zeta,d}(y+J)_n| \leq p^r \epsilon$ too. Proceeding by induction on n we construct a vector y such that $\|y\| \leq \epsilon$ and (4.3) holds.

Necessity is proved in the same way. Suppose that a y as above has been constructed. Then if $|\Omega_{r,n}| > \epsilon$, we must have $|\Omega_{r,n}| = |(\zeta; \zeta)_n| |L_{\zeta,d}(y+J)_n| \leq p^r |(\zeta; \zeta)_n| \epsilon$. \square

4.5. A dual criterion. For $0 \leq n \leq d-1$ let

$$\omega_{r,n} = \zeta^{\binom{n+1}{2}} \frac{p^r}{(\zeta; \zeta)_n} \Omega_{r,n}.$$

Lemma 8. (i) *We have*

$$(4.6) \quad \omega_{r,n} = \zeta^{\binom{n+1}{2}} \sum_{k=0}^{p^r-1} \begin{bmatrix} kp^r \\ n \end{bmatrix}_\zeta.$$

(ii) *For any r, n the criterion (4.5) is equivalent to*

$$|\omega_{r,n}| \leq \max \{ p^{-r} |(\zeta; \zeta)_n|^{-1}, 1 \} \cdot \epsilon.$$

Remark. Part (ii), the reformulation of the criterion, is of course obvious. What is new is the expression given in part (i). Note that both $\Omega_{r,n}$ and $\omega_{r,n}$ lie in $\mathbb{Z}[\zeta]$. Note that in (4.4) the running index appears in the denominators of the ζ -binomial coefficients, while in (4.6) it appears in the numerators. Once we obtain an estimate on $|(\zeta; \zeta)_n|$, the expression (4.4) will be useful for $n \in [0, \frac{1-\delta}{2}(d-1)]$, while (4.6) will be useful for $n \in [\frac{1+\delta}{2}(d-1), d-1]$. Here $\delta > 0$ is fixed. The smaller we make δ the larger r will have to be taken to guarantee the criterion for attainability in these two regions. Treating n in the ‘‘critical interval’’ $[\frac{1-\delta}{2}(d-1), \frac{1+\delta}{2}(d-1)]$ will take some more effort.

Proof. We only have to prove (i). Using (4.2) we may write

$$(4.7) \quad \Omega_{r,n} = \frac{1}{p^r} \sum_{\eta^{p^r}=1} \prod_{l=0}^{n-1} (1 - \zeta^{-l}\eta).$$

The p^r roots of unity η are just the ζ^{kp^r} for $0 \leq k \leq p^r - 1$. We get

$$\Omega_{r,n} = \frac{1}{p^r} \sum_{k=0}^{p^r-1} \prod_{l=0}^{n-1} (1 - \zeta^{kp^r-l}) = \frac{1}{p^r} \sum_{k=0}^{p^r-1} \frac{(\zeta; \zeta)_{kp^r}}{(\zeta; \zeta)_{kp^r-n}} = \frac{(\zeta; \zeta)_n}{p^r} \sum_{k=0}^{p^r-1} \begin{bmatrix} kp^r \\ n \end{bmatrix}_\zeta.$$

Note that the terms with $kp^r < n$ drop out from the sum.

An alternative proof is to observe that since $1_{\mathbb{Z}_p}$ is invariant under the Fourier transform, $Z_{\zeta,d}J = p^r J$, hence from Corollary 6

$$p^r L_{\zeta,d}J = L_{\zeta,d}Z_{\zeta,d}J = U_{\zeta,d}J.$$

Noting that the n th entry on the left is $\zeta^{-\binom{n+1}{2}}\omega_{r,n}$, we get (4.6). \square

The formulae for $\omega_{r,n}$ can be summarized in the following expression for the generating function:

$$\sum_{n=0}^{d-1} (-1)^n \omega_{r,n} X^n = \sum_{k=0}^{p^r-1} \prod_{l=1}^{kp^r} (1 - \zeta^l X).$$

Simply expand the products using the q -binomial theorem and collect terms.

4.6. An estimate for $|(\zeta; \zeta)_n|$. We turn our attention to

$$(\zeta; \zeta)_n = \prod_{k=1}^n (1 - \zeta^k).$$

Normalize the p -adic valuation $v_p : C^\times \rightarrow \mathbb{Q}$ so that $v_p(p) = 1$.

Lemma 9. *Let $0 \leq n \leq d - 1$. Write*

$$n = \sum_{i=0}^s a_i p^i, \quad 0 \leq a_i \leq p - 1$$

($0 \leq s \leq 2r - 1$) and

$$v_p((\zeta; \zeta)_n) = \frac{\beta_p(n)}{(p-1)p^{2r-1}}.$$

Then

$$\beta_p(n) = n + \frac{p-1}{p} \sum_{i=0}^s i a_i p^i.$$

Proof. Clearly

$$\beta_p(n) = \sum_{k=0}^{\infty} p^k \left(\lfloor \frac{n}{p^k} \rfloor - \lfloor \frac{n}{p^{k+1}} \rfloor \right).$$

By Abel's summation formula

$$\beta_p(n) = n + \frac{p-1}{p} \sum_{k=1}^{\infty} p^k \lfloor \frac{n}{p^k} \rfloor = n + \frac{p-1}{p} \sum_{i=0}^s i a_i p^i.$$

\square

Corollary 10. *We have the estimates*

$$\frac{p-1}{p} n \lfloor \log_p n \rfloor + 1 \leq \beta_p(n) \leq \frac{p-1}{p} n \lfloor \log_p n \rfloor + n.$$

Proof. The corollary follows from the inequalities

$$n \lfloor \log_p n \rfloor - \frac{p}{p-1} (n-1) \leq n \lfloor \log_p n \rfloor - \sum_{k=1}^{\lfloor \log_p n \rfloor} p^k \leq \sum_{k=1}^{\infty} p^k \lfloor \frac{n}{p^k} \rfloor \leq n \lfloor \log_p n \rfloor.$$

\square

4.7. Verification of the criterion (4.5) outside the critical interval.

Lemma 11. *Let $\epsilon > 0$ and $0 < \delta < 1$. Then if r is large enough and $n \notin [\frac{1-\delta}{2}(d-1), \frac{1+\delta}{2}(d-1)]$ the inequality (4.5) holds.*

Proof. Let us treat first the range $n \leq \frac{1-\delta}{2}(p^{2r}-1)$. Since $\log_p |\Omega_{r,n}| \leq 0$, it is enough to show that for r large enough and all n in this range

$$0 \leq r + \log_p |(\zeta; \zeta)_n| + \log_p \epsilon.$$

Now $\log_p |(\zeta; \zeta)_n| = -v_p((\zeta; \zeta)_n)$ so it is enough to show that

$$v_p((\zeta; \zeta)_n) \leq r + \log_p \epsilon.$$

Using the last corollary we have

$$v_p((\zeta; \zeta)_n) = \frac{\beta_p(n)}{(p-1)p^{2r-1}} \leq \frac{n}{p^{2r}} \lfloor \log_p n \rfloor + \frac{n}{(p-1)p^{2r-1}} \leq \frac{1-\delta}{2} \cdot 2r+1 = (1-\delta)r+1.$$

Since $\delta > 0$, for r large enough we are done, no matter how negative $\log_p \epsilon$ is.

For $n \geq \frac{1+\delta}{2}(p^{2r}-1)$ we use the dual criterion in terms of (4.6). Since $\log_p |\omega_{r,n}| \leq 0$, it is enough to show that for r large enough and all n in this range

$$0 \leq -r - \log_p |(\zeta; \zeta)_n| + \log_p \epsilon.$$

Equivalently, we have to show

$$r - \log_p \epsilon \leq v_p((\zeta; \zeta)_n).$$

Now we use the lower bound from Corollary 10:

$$v_p((\zeta; \zeta)_n) = \frac{\beta_p(n)}{(p-1)p^{2r-1}} \geq \frac{n}{p^{2r}} \lfloor \log_p n \rfloor \geq \frac{1+\delta}{2} \cdot (1 - \frac{1}{p^{2r}}) \cdot (2r-1) \geq (1+\delta)r-1.$$

Once again, since $\delta > 0$, for r large enough we are done. \square

4.8. Conclusion of the proof of the main theorem. For n near the center of the interval $[0, d-1]$, the quantity $p^r |(\zeta; \zeta)_n|$, which appears (with its inverse) in both versions of our key criterion, is roughly 1. It is therefore neither very large, nor very small (when $r \rightarrow \infty$) and the integrality of $\Omega_{r,n}$ and $\omega_{r,n}$ alone does not suffice. In order to estimate $\Omega_{r,n}$ in the range $n \in [\frac{1-\delta}{2}(d-1), \frac{1+\delta}{2}(d-1)]$, we study, following Hao-Pan, the divisibility of the polynomials $\Omega_{r,n}(q)$ (4.1) by certain cyclotomic polynomials.

Let $\Phi_m(q)$ be the m th cyclotomic polynomial, the product of $(q - \alpha)$ for all the primitive m th roots of unity α . It is a monic irreducible polynomial in $\mathbb{Z}[q]$. The following is Theorem 1.1 of [5].

Lemma 12. *In $\mathbb{Z}[q, q^{-1}]$ we have*

$$\prod_{j=r}^{2r-1} \Phi_{p^j}(q)^{\lfloor n/p^j \rfloor} | \Omega_{r,n}(q).$$

Proof. We use the expression derived in (4.2). As the Φ_{p^j} are irreducible, monic and distinct, it is enough to check that $\Phi_{p^j}(q)^{\lfloor n/p^j \rfloor} | \Omega_{r,n}(q)$ over \mathbb{Q} . Let ξ be a primitive root of unity of order p^j with $r \leq j \leq 2r-1$. Then any p^r root of 1 η is also a p^j root of 1, hence $\eta = \xi^l$ for a unique $0 \leq l \leq p^j - 1$. It follows that the multiplicity of ξ in $\prod_{l=0}^{n-1} (1 - q^{-l}\eta)$ is $\lfloor n/p^j \rfloor$. Since this is true for every η and every ξ the lemma follows. \square

Substitute ζ for q . If ξ is a p^j root of unity and $j \leq 2r - 1$, $v_p(\zeta - \xi) = 1/(p - 1)p^{2r-1}$, so

$$v_p(\Phi_{p^j}(\zeta)) = \frac{1}{p^{2r-j}}.$$

It follows that

$$(4.8) \quad v_p(\Omega_{r,n}) \geq \sum_{j=r}^{2r-1} \lfloor \frac{n}{p^j} \rfloor \cdot \frac{1}{p^{2r-j}} \geq \sum_{j=r}^{2r-1} \left(\frac{n}{p^j} - 1 \right) \cdot \frac{1}{p^{2r-j}} \geq \frac{rn}{p^{2r}} - \frac{1}{p-1}.$$

We can now conclude the proof of Theorem 1. Fix $\delta > 0$ and let I_r be the interval $[\frac{1-\delta}{2}(d-1), \frac{1+\delta}{2}(d-1)]$. The last formula shows that

$$\max_{n \in I_r} |\Omega_{r,n}| \rightarrow 0$$

as $r \rightarrow \infty$, hence (4.5) holds in I_r if r is large enough. As it was previously shown to hold outside I_r , the proof is concluded. Our estimate (4.8) can be used also to verify the criterion for n in the “right half” $[\frac{1+\delta}{2}(d-1), d-1]$, where previously we have used the “dual criterion” by means of $\omega_{r,n}$, but this is not surprising, since both arguments are based on formula (4.2).

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