KIRILLOV MODELS AND THE BREUIL-SCHNEIDER CONJECTURE FOR $GL_2(F)$

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Abstract. Let F be a local field of characteristic 0. The Breuil-Schneider conjecture for $GL_2(F)$ predicts which locally algebraic representations of this group admit an integral structure. We extend the methods of [K-dS12], which treated smooth representations only, to prove the conjecture for some locally algebraic representations as well.

1. Introduction

1.1. **Background.** Let F be a local field of characteristic 0 and residue characteristic p, π a fixed uniformizer of F, and q the cardinality of its residue field $\mathcal{O}_F/\pi\mathcal{O}_F$. Let E be an algebraic closure of F.

Let $\mathbf{G}$ be a reductive group over F and $G = \mathbf{G}(F)$. A *locally algebraic* representation (ρ, V_ρ) of G over E is a representation of the type

$$\rho = \tau \otimes \sigma \tag{1.1}$$

where (τ, V_τ) is (the E-points of) a finite dimensional rational representation of $\mathbf{G}$, and (σ, V_σ) is a smooth representation of G over E. An *integral structure* V_ρ^0 in V_ρ is an $\mathcal{O}_E[G]$-submodule which spans V_ρ over E, but does not contain any E-line.

If τ and σ are irreducible then ρ is irreducible as well ([P01], Theorem 1). In such a case, a non-zero $\mathcal{O}_E[G]$-submodule V_ρ^0 of V_ρ is an integral structure if and only if it is properly contained in V_ρ. Indeed, the union of all E-lines in V_ρ^0, as well as the subspace of V_ρ spanned by V_ρ^0 over E, are both $E[G]$-submodules of V_ρ. If $0 \subset V_\rho^0 \subset V_\rho$ (both inclusions being proper), the irreducibility of ρ implies that the first is 0, and the second is V_ρ.

Two integral structures in V_ρ are *commensurable* if each of them is contained in a scalar multiple of the other. In general, V_ρ need not contain an integral structure. When such an integral structure exists, it need not be unique, even up to commensurability. However, if ρ is irreducible, and an integral structure does exist, there is a unique commensurability class of *minimal integral structures*, namely the class of any cyclic $\mathcal{O}_E[G]$-module. Thus, when ρ is irreducible, to test whether integral structures exist at all, it is enough to check that for some $0 \neq v \in V_\rho$, $\mathcal{O}_E[G]v$ is not the whole of V_ρ.

The existence (and classification) of integral structures in irreducible locally algebraic representations is a natural and important question for the *p-adic local Langlands programme* (see [Br10]). When $\mathbf{G} = GL_n$, a precise conjecture for the conditions on τ and σ under which an integral structure should exist in ρ was proposed by Breuil and Schneider in [Br-Sch07], and became known as *the Breuil-Schneider conjecture.* The *necessity* of these conditions was proved there in some special cases, and by Hu [Hu09] in general. The *sufficiency* tends to be, in the words of Vigneras [V], either "obvious" or "very hard", even for GL_2.

Quite generally, if $\mathbf{G}$ is an arbitrary reductive group, the simpler σ is algebraicly, the harder the question becomes. An obvious necessary condition is for the central character of ρ to be unitary[1]. Assume therefore that this is the case. If σ is supercuspidal (its matrix coefficients are compactly supported modulo the center), the existence of an integral structure is obvious. Using global methods and the trace formula, existence of an integral structure can also be proved when σ, realized over $\mathbb{C}$ by means of some field embedding $E \hookrightarrow \mathbb{C}$, is essentially discrete series (its matrix coefficients are square integrable modulo the center)[2] [So13]. In these cases, no further restrictions are imposed on τ. At the other extreme stand principal series representations, where one should impose severe restrictions on τ, and the problem becomes very hard.

We warn the reader that for arithmetic applications, the minimal integral structures in an irreducible V_ρ are often insufficient. In particular, they may be non-admissible, in the sense that their reduction modulo the maximal ideal of $\mathcal{O}_E$ is a non-admissible smooth representation over $\bar{\mathbb{F}}_q$. In such a case, even if minimal integral structures are known to exist, the existence of larger admissible integral structures is a mystery, which is resolved only in special cases, again by global methods. See [Br04].

1.2. **The main result.** We now specialize to $\mathbf{G} = GL_2$. In this case the full Breuil-Schneider conjecture is known when $F = \mathbb{Q}_p$, but only by indirect methods involving (ϕ, Γ)-modules and Galois representations. It comes as a by-product of the proof of the p-adic local Langlands correspondence ($pLLC$). This large-scale project [B-B-C] depends so far crucially on the assumption $F = \mathbb{Q}_p$. It is therefore desirable to have a *direct local proof* of the Breuil-Schneider conjecture, which does not depend on $pLLC$, and which holds for arbitrary F. As mentioned above, if σ is either supercuspidal or special, there are no restrictions on τ and integral structures are known to exist. We therefore assume that $\sigma = Ind(\chi_1, \chi_2)$ is an irreducible principal series representation.

In this work we prove the Breuil-Schneider conjecture for $GL_2(F)$ in the following cases: (1) The characters χ_1 and χ_2 are unramified, $\tau = \det(.)^m \otimes Sym^n$, and the weight is low: $n < q$ (2) The χ_i are tamely ramified, and $\tau = \det(.)^m$. The second case has been done in [K-dS12] already, but the proof presented here is somewhat cleaner.

To formulate our theorem, let χ_i be smooth characters of $F^\times$ with values in $E^\times$, and ω the unramified character[3] for which $\omega(\pi) = q^{-1}$. Let B be the Borel subgroup of upper triangular matrices in G, and consider the principal series representation

$$(V_\sigma, \sigma) = Ind_B^G(\chi_1, \chi_2). \tag{1.2}$$

[1] A character $\chi : F^\times \to E^\times$ is *unitary* if its values lie in $\mathcal{O}_E^\times$.

[2] The notion of "essentially discrete series" should be invariant under $Aut(\mathbb{C})$, hence independent of the embedding of E in $\mathbb{C}$. This is known for GL_n by the work of Bernstein-Zelevinski, and for the classical groups by Tadic.

[3] This character is usually denoted $|.|$ over $\mathbb{C}$. We will have to consider $|\omega(\pi)|$, the absolute value of q^{-1} as an element of E, and we found the notation $||\pi||$ too confusing.

This is the space of functions $f : G \to E$ for which (i)

$$f\left(\begin{pmatrix} t_1 & s \\ 0 & t_2 \end{pmatrix} g\right) = \chi_1(t_1)\chi_2(t_2)f(g) \tag{1.3}$$

and (ii) there exists an open subgroup $H \subset G$, depending on f, such that $f(gh) = f(g)$ for all $h \in H$. The group G acts by right translation:

$$\sigma(g)f(g') = f(g'g). \tag{1.4}$$

The central character of σ is $\chi_1\chi_2$, and $Ind_B^G(\chi_1, \chi_2) \simeq Ind_B^G(\omega\chi_2, \omega^{-1}\chi_1)$, unless this representation is reducible. In fact, σ is reducible precisely when $\chi_1/\omega\chi_2 = \omega^{\pm 1}$. In this "special" case σ is indecomposable of length 2, and its irreducible constituents are a one-dimensional character and a twist of the Steinberg representation by a character. Since the Breuil-Schneider conjecture for a twist of Steinberg, and any τ, is known (for $GL_2(F)$, see [T93] or [V08]), we exclude this case from now on, and assume that σ is *irreducible*.

Next, fix integers m and $n \geq 0$, and consider the rational representation

$$(V_\tau, \tau) = \det(.)^m \otimes Sym^n, \tag{1.5}$$

where Sym^n denotes the nth symmetric power of the standard representation of $\mathbf{G}$. Put

$$\begin{aligned} \lambda &= \chi_1(\pi),\ \mu = \omega\chi_2(\pi), \\ \tilde{\lambda} &= \lambda\pi^m,\ \tilde{\mu} = \mu\pi^m. \end{aligned} \tag{1.6}$$

The Breuil-Schneider conjecture for $\rho = \tau \otimes \sigma$ predicts that ρ has an integral structure if and only if the following two conditions are satisfied:

$$\text{(i) } |\tilde{\lambda}\tilde{\mu}q\pi^n| = 1 \quad \text{(ii) } |\tilde{\lambda}| \leq |q^{-1}\pi^{-n}|,\ |\tilde{\mu}| \leq |q^{-1}\pi^{-n}|. \tag{1.7}$$

Condition (i) means that the central character of ρ is unitary. Given (i), (ii) is equivalent to $1 \leq |\tilde{\lambda}| \leq |q^{-1}\pi^{-n}|$ or to the symmetric condition for $\tilde{\mu}$. It is known (and easy to prove) that these two conditions are necessary.

Theorem 1.1. *Assume that (i) and (ii) are satisfied. Assume, in addition, that* either *(1) χ_1 and χ_2 are unramified and $n < q$,* or *(2) that χ_1 and χ_2 are tamely ramified and $n = 0$. Then ρ has an integral structure.*

Although our method is new, and gives some new insight into the minimal integral structure (see Theorem 1.2 below), the two cases have been known before: case (1) by Breuil [Br03] (for $\mathbb{Q}_p$) and de Ieso [dI12] (for general F), and case (2) by Vigneras [V08]. It is interesting to note that the restriction $n < q$ in case (1) and the restriction on tame ramification in case (2) are also needed in the above mentioned works. In fact, Breuil, de Ieso and Vigneras all use, in one way or another, the method of *compact induction*, replacing the representation ρ by a local system on the tree of G. Our approach takes place in a certain *dual* space of functions on F. Any attempt to translate it to the set-up of the tree involves the p-adic Fourier transform, which is unbounded, and makes it impossible to trace back the arguments. The way in which the weight and ramification restrictions are brought to bear on the problem are also not similar, yet the very same restrictions turn out to be necessary for the proofs to work.

1.3. **An outline of the proof.** As in [K-dS12], our approach is based on a study of the Kirillov model of ρ. For the sake of exposition we now exclude the case $\chi_1 = \omega\chi_2$, which requires special attention. Assuming $\chi_1 \neq \omega\chi_2$, the *Kirillov model* of ρ is then the following space of functions on $F - \{0\}$:

$$\mathcal{K} = C_c^\infty(F, \tau)\chi_1 + C_c^\infty(F, \tau)\omega\chi_2. \tag{1.8}$$

Here $C_c^\infty(F, \tau)$ is the space of V_τ-valued locally constant functions of compact support on F. The model $\mathcal{K}$ is obtained by tensoring τ with the classical Kirillov model of the smooth representation σ (see [Bu98]). It contains $\mathcal{K}_0 = C_c^\infty(F^\times, \tau)$, the subspace of functions vanishing near 0, and $\mathcal{K}/\mathcal{K}_0$ consists of two copies of V_τ. When $\tau = 1$, this is just the *Jacquet module* of $\mathcal{K}$. The characters χ_1 and $\omega\chi_2$ are the *exponents* of the Jacquet module, the two characters by which the torus of diagonal matrices acts on it.

We record the action of an element

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in B \tag{1.9}$$

on $\phi \in \mathcal{K}$. Fix an additive character $\psi : F \to E^\times$ under which $\mathcal{O}_F$ is its own annihilator. Then

$$\rho(g)\phi(x) = \tau(g)\left(\psi(bx)\phi(ax)\right). \tag{1.10}$$

The action of G in the model $\mathcal{K}$ depends on the choice of ψ, but only up to isomorphism.

At this point, we must introduce more notation and recall some easy facts. Let 1_S be the characteristic function of $S \subset F$, and $\phi_l = 1_{\pi^l U_F}$ $(l \in \mathbb{Z})$. If $b \in F$, write $\psi_b(x) = \psi(bx)$. The function $\psi_b(\pi^{-l}x)\phi_l(x)$ depends only on β, the image of b in $W = F/\mathcal{O}_F$, so from now on we denote it by $\psi_\beta(\pi^{-l}x)\phi_l(x)$. Any locally constant function on the annulus $\pi^l U_F$ can be expanded as a finite linear combination of these functions. Moreover, Fourier analysis on the disk $\pi^l\mathcal{O}_F$ implies that

$$\sum_{\beta \in W} C_l(\beta)\psi_\beta(\pi^{-l}x)\phi_l(x) = 0 \tag{1.11}$$

if and only if $C_l(\beta)$ depends only on $\pi\beta$, i.e.

$$C_l(\beta) = C_l(\beta') \text{ if } \beta - \beta' \in W_1 = \pi^{-1}\mathcal{O}_F/\mathcal{O}_F. \tag{1.12}$$

The same applies of course to V_τ-valued functions, except that now the coefficients $C_l(\beta) \in V_\tau$.

An arbitrary function $\phi \in \mathcal{K}$ may be expanded annulus-by-annulus as

$$\phi = \sum_{l=l_0}^{\infty} \sum_{\beta \in W} C_l(\beta)\psi_\beta(\pi^{-l}x)\phi_l(x), \tag{1.13}$$

where $C_l(\beta) \in V_\tau$, and for every l only finitely many $C_l(\beta) \neq 0$. The only restriction on ϕ is imposed by the asymptotics as $x \to 0$. In particular, finite linear combinations as above represent the elements of $\mathcal{K}_0$. One should think of the β as frequencies, and of the $C_l(\beta)$ as the amplitudes attached to these frequencies on the annulus $\pi^l U_F$. These amplitutes are not uniquely defined since we may add to $C_l(\beta)$ a perturbation $\tilde{C}_l(\beta)$ without affecting $\phi|\pi^l U_F$, provided $\tilde{C}_l(\beta) = \tilde{C}_l(\beta')$ whenever $\beta - \beta' \in W_1$. But as explained above, this is the only ambiguity.

Theorem 1.1 follows from the following more precise result, which makes the integral structure on V_ρ “visible”.

Theorem 1.2. *Let the assumptions be as in Theorem 1.1. Let V_ρ^0 be the $\mathcal{O}_E[G]$-submodule of $V_\rho = \mathcal{K}$ spanned by a non-zero vector. Then there exist $\mathcal{O}_E$-lattices $M_0(\beta) \subset V_\tau$ such that if $\phi \in V_\rho^0$ vanishes outside $\mathcal{O}_F$, it has an expansion as above with $C_0(\beta) \in M_0(\beta)$ for every β.*

Note that we do not claim that the values of $\phi \in V_\rho^0$ are bounded on U_F, nor at any other point. The amplitudes can be bounded only separately, and only on the first annulus where ϕ does not vanish. Since the $C_0(\beta)$ are not uniquely defined, one still needs a simple argument to show that this is good enough.

Proposition 1.3. *Theorem 1.2 implies Theorem 1.1.*

Proof. We shall show that $V_\rho^0 \neq V_\rho$, so in view of the irreducibility of ρ, V_ρ^0 will be an integral structure. Consider the function $\phi = C\phi_0$ where $C \in V_\tau$ *lies outside* $M = \sum_{\beta \in W_1} M_0(\beta)$. Suppose, by way of contradiction, that $\phi \in V_\rho^0$. Then ϕ is also given by an expansion as in Theorem 1.2. For $x \in U_F$ we must have then

$$C = \sum_{\beta \in W} C_0(\beta)\psi_\beta(x). \tag{1.14}$$

This forces, as we have seen, the equality $C_0(0) - C = C_0(\beta)$ for $\beta \in W_1 - \{0\}$. But this contradicts the choice of C. ∎

We now make some comments on the proof of Theorem 1.2. The first step is standard. Using the decomposition $G = BK$, $K = GL_2(\mathcal{O}_F)$, we show that V_ρ^0 is commensurable with a certain $\mathcal{O}_E[B]$-module of finite type Λ which also spans V_ρ over E. We may therefore prove the assertion of the theorem for Λ instead of V_ρ^0. Our Λ will be spanned over $\mathcal{O}_E$ by an explicit infinite set $\mathcal{E}$ of nice functions.

Pick a $\phi \in \Lambda$, express it as a linear combination of the functions in $\mathcal{E}$, and expand it annulus-by-annulus as above. The coefficients $C_l(\beta)$ then satisfy *recursive relations*, in which the coefficients used to express ϕ as a linear combination of $\mathcal{E}$ figure out.

Suppose that ϕ vanishes off $\mathcal{O}_F$. It may still be the case that $C_l(\beta) \neq 0$ for some β and $l < 0$. However, cancellation must take place, and as we have seen, $C_l(\beta)$ depends then, for $l < 0$, on $\pi\beta$ only. We proceed by increasing induction on l and show that $C_l(\beta)$ must belong, for $l \leq 0$, to a certain $\mathcal{O}_E$-lattice $M_l(\beta) \subset V_\tau$, depending on l and β, but not on ϕ. When $l = 0$ we reach the desired conclusion.

Two phenomena assist us in establishing these bounds on the coefficients. The first, which has already been utilized in our previous work [K-dS12], is that in the recursive relations for $C_l(\beta)$ we encounter terms such as

$$\sum_{\pi\alpha = \beta} C_{l-1}(\alpha). \tag{1.15}$$

As long as $l \leq 0$, the q summands are all equal, so their sum is equal to $qC_{l-1}(\alpha_\beta)$, where α_β is any one of the α's. The factor q is small, and helps to control $C_l(\beta)$.

The second phenomenon is new, and more subtle. The information that $C_l(\beta)$ depends only on $\pi\beta$, puts a further restriction on $C_l(\beta)$, beyond lying in $M_l(\beta)$, which is vital for the deduction that the $C_{l+1}(\gamma)$ lie in $M_{l+1}(\gamma)$. For example, assume that $m = 0$ and $n = 1$, so τ is the standard representation of G on E^2, and let e_1 and e_2 be the standard basis. In this example, up to scaling,

$$M_l(\beta) = Span_{\mathcal{O}_E}\left\{\pi^{-l}e_1, e_2 - \pi^{-l}\beta e_1\right\} \tag{1.16}$$

(note that this is indeed well defined, i.e. depends only on $\beta \bmod \mathcal{O}_F$). It is easily checked that if $C_l(\beta) \in M_l(\beta)$ for all β, and *in addition,* $C_l(\beta)$ depends only on $\pi\beta$, then in fact

$$C_l(\beta) \in Span_{\mathcal{O}_E} \left\{ \pi^{-l}e_1, \pi(e_2 - \pi^{-l}\beta e_1) \right\}. \tag{1.17}$$

This minor improvement on $C_l(\beta) \in M_l$ is crucial for our method to work. Roughly speaking, the first phenomenon described above takes care of the factor q^{-1} in condition (1.7)(ii), while the second one takes care of the π^{-n}.

The inductive procedure requires also the relation $M_l(\beta) \subset M_{l+1}(\pi\beta)$. It is here that we need the condition $n < q$. We may modify the definition of $M_l(\beta)$ to guarantee this relation without any restriction on n, but we then lose the subtle phenomenon to which we alluded in the previous paragraph. At present, we are unable to hold the rope at both ends simultaneously.

When χ_1 and χ_2 are unramified this is the end of the story. When χ_1 and χ_2 are ramified, two types of complications occur. First, we must give up the algebraic part τ (except for the benign twist by the determinant). Second, in the recursive relations used to define $C_l(\beta)$, Gauss sums intervene. These Gauss sums have denominators which are still under control if the characters are only tamely ramified, but if the χ_i are wildly ramified, our method breaks down. It is interesting to note that the well-known estimates on Gauss sums intervene also in Vigneras' proof of the tamely-ramified smooth case of the conjecture.

In the remaining cases, not covered by (1) or (2), it is possible that Theorem 1.2 fails, yet Theorem 1.1 continues to hold, for a different reason. It will be interesting to check numerically whether one should expect Theorem 1.2 in general. Even for $F = \mathbb{Q}_p$, where, as mentioned above, the full conjecture is known, it is unclear to us whether Theorem 1.2 holds beyond cases (1) and (2).

2. Preliminary results

2.1. **Fourier analysis on $\mathcal{O}_F$.** The discrete group $W = F/\mathcal{O}_F$ is the topological dual of $\mathcal{O}_F$ via the pairing

$$(\beta, x) \mapsto \psi_\beta(x) = \psi(\beta x). \tag{2.1}$$

Every locally constant E-valued function on $\mathcal{O}_F$ has a unique finite Fourier expansion

$$\phi = \sum_{\beta \in W} c(\beta)\psi_\beta(x). \tag{2.2}$$

The proof of the following easy lemma is left to the reader.

Lemma 2.1. *(i) $\phi|U_F = 0$ if and only if $c(\beta)$ depends only on $\pi\beta$.*
(ii) $\phi|\pi\mathcal{O}_F = 0$ if and only if $\sum_{\pi\beta=\gamma} c(\beta) = 0$ for every $\gamma \in W$.

The lemma is immediately translated to a similar one in the disk $\pi^l\mathcal{O}_F$ using the functions $\psi_\beta(\pi^{-l}x)$ as a basis for the expansion.

2.2. **Lattices in V_τ.** If $\beta \in W$ and $l \in \mathbb{Z}$ let

$$D_l(\beta) = \left\{ u \in F |\, |u - \pi^{-l}\beta| \le |\pi^{-l}| \right\}. \tag{2.3}$$

This disk indeed depends only on $\beta \bmod \mathcal{O}_F$. Note that

$$D_{l+1}(\gamma) = \coprod_{\pi\beta=\gamma} D_l(\beta). \tag{2.4}$$

Let $\tau = \det(.)^m \otimes Sym^n$. Identify V_τ with $E[u]^{\leq n}$, the space of polynomials of degree at most n, with the action

$$\tau\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) u^i = (ad-bc)^m (a+cu)^{n-i}(b+du)^i. \tag{2.5}$$

Let

$$N_l(\beta) = \left\{ P \in V_\tau \,|\, |P(u)| \leq |\pi|^{-nl} \ \forall u \in D_l(\beta) \right\}. \tag{2.6}$$

These are lattices in V_τ.

Lemma 2.2. *(i) For any $\gamma \in W$*

$$\bigcap_{\pi\beta=\gamma} N_l(\beta) = \pi^n N_{l+1}(\gamma). \tag{2.7}$$

(ii) Assume that $n < q$. Then

$$N_l(\beta) = Span_{\mathcal{O}_E} \left\{ (\pi^{-l})^{n-i}(u-\pi^{-l}\beta)^i \ \ (0 \leq i \leq n) \right\}. \tag{2.8}$$

(iii) Assume that $n < q$. Then

$$N_l(\beta) \subset N_{l+1}(\pi\beta). \tag{2.9}$$

Proof. (i) If $P \in N_l(\beta)$ then it is bounded by $|\pi|^{-nl}$ on $D_l(\beta)$. But the q disks $D_l(\beta)$, for the β satisfying $\pi\beta = \gamma$, cover $D_{l+1}(\gamma)$. The result follows.

(ii) Clearly $P \in N_l(\beta)$ if and only if $\pi^{nl}P(\pi^{-l}u + \pi^{-l}\beta) \in N_0(0)$. It is therefore enough to prove that $|P(u)| \leq 1$ for all $u \in \mathcal{O}_F$ if and only if $P \in \mathcal{O}_E[u]^{\leq n}$. This is well-known, but note that it fails if $n \geq q$ (consider $\pi^{-1}(u^q - u)$).

(iii) This is an immediate consequence of (ii). ∎

2.3. **Passing from $\mathcal{O}_E[B]$-modules to $\mathcal{O}_E[G]$-modules.** Consider the representation V_ρ, where $\rho = \tau \otimes \sigma$, $\tau = \det(.)^m \otimes Sym^n$, and $\sigma = Ind_B^G(\chi_1, \chi_2)$ are as in the introduction.

Proposition 2.3. *Let $v_1, \ldots, v_r \in V_\sigma$ be such that the module $\Lambda_\sigma = \sum_{j=1}^r \mathcal{O}_E[B]v_j$ spans V_σ over E. Let*

$$\Lambda = \sum_{i=0}^n \sum_{j=1}^r \mathcal{O}_E[B]\left(u^i \otimes v_j\right) \subset V_\rho. \tag{2.10}$$

Then Λ is commensurable with every cyclic $\mathcal{O}_E[G]$-submodule of V_ρ.

Proof. Let $K = GL_2(\mathcal{O}_F)$ and recall that $G = BK$. If $N \leq K$ is a subgroup of finite index fixing all the v_j, then N preserves the finitely generated $\mathcal{O}_E$-submodule

$$\sum_{i,j} \mathcal{O}_E(u^i \otimes v_j), \tag{2.11}$$

because $\tau(K)$ preserves $\mathcal{O}_E[u]^{\leq n}$. It follows that $\sum_{i,j} \mathcal{O}_E[K](u^i \otimes v_j)$ is finitely generated over $\mathcal{O}_E$. Since Λ spans V_ρ over E, there is a constant $c \in E$ such that

$$\sum_{i,j} \mathcal{O}_E[K](u^i \otimes v_j) \subset c\Lambda. \tag{2.12}$$

But then

$$\begin{aligned} \sum_{i,j} \mathcal{O}_E[G](u^i \otimes v_j) &= \mathcal{O}_E[B] \sum_{i,j} \mathcal{O}_E[K](u^i \otimes v_j) \\ &\subset \mathcal{O}_E[B](c\Lambda) = c\Lambda. \end{aligned} \tag{2.13}$$

On the other hand, $\Lambda \subset \sum_{i,j} \mathcal{O}_E[G](u^i \otimes v_j)$. The two inclusions prove the proposition, since the sum of a finite number of cyclic modules, all being commensurable, is again commensurable with any cyclic module. ∎

Corollary 2.4. *To prove Theorem 1.2 we may replace V_ρ^0 by Λ.*

2.4. **The Kirillov model and a choice of Λ.** Assume from now on that $\chi_1 \neq \omega\chi_2$. The exceptional case $\chi_1 = \omega\chi_2$ requires special attention and will be dealt with in the end. Let $\mathcal{K}$ be the model of V_ρ described in the introduction. For $\{v_j\}$ we choose the two functions

$$v_1 = F_0'(x) = 1_{\mathcal{O}_F}\chi_1, \;\; v_2 = F_0'' = 1_{\mathcal{O}_F}\omega\chi_2. \tag{2.14}$$

Let $F_k'(x) = F_0'(\pi^{-k}x)$ and similarly $F_k''(x) = F_0''(\pi^{-k}x)$. Since

$$\sigma\left(\begin{pmatrix} \pi^{-k} & -\pi^{-k}\beta \\ & 1 \end{pmatrix}\right) F_0'(x) = \psi_\beta(-\pi^{-k}x)F_k'(x) \tag{2.15}$$

and similarly for $F_0''(x)$, we see that $\Lambda_\sigma = \mathcal{O}_E[B]F_0' + \mathcal{O}_E[B]F_0''$ spans V_σ over E.

Lemma 2.5. *Let $\Lambda = \sum_{i=0}^n \sum_{j=1}^2 \mathcal{O}_E[B]\left(u^i \otimes v_j\right)$, where $v_1 = F_0'$ and $v_2 = F_0''$. Then every element of Λ can be written as a finite sum*

$$\phi = \sum_{k=k_0}^{\infty} \sum_{\beta \in W} c_k'(\beta)\psi_\beta(-\pi^{-k}x)F_k'(x) + c_k''(\beta)\psi_\beta(-\pi^{-k}x)F_k''(x), \tag{2.16}$$

where $c_k'(\beta), c_k''(\beta) \in \pi^{-km}N_k(\beta)$.

Proof. Since the central character of ρ is unitary (condition (1.7)(i)), it is enough to span Λ by matrices in the mirabolic subgroup

$$\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right\} \leq B. \tag{2.17}$$

Furthermore, as $B \cap K$ stabilizes $\sum_{i=0}^n \sum_{j=1}^2 \mathcal{O}_E\left(u^i \otimes v_j\right)$, we see that

$$\begin{aligned} \Lambda &= \sum_{k\in\mathbb{Z}} \sum_{\beta\in W} \mathcal{O}_E\rho\left(\begin{pmatrix} \pi^{-k} & -\pi^{-k}\beta \\ & 1 \end{pmatrix}\right)\left(u^i \otimes v_j\right) \\ &= \sum_{k\in\mathbb{Z}} \sum_{\beta\in W} \pi^{-km}(\pi^{-k})^{n-i}(u-\pi^{-k}\beta)^i \otimes \psi_\beta(-\pi^{-k}x)\left(\mathcal{O}_E F_k'(x) + \mathcal{O}_E F_k''(x)\right). \end{aligned}$$

The coefficients $(\pi^{-k})^{n-i}(u-\pi^{-k}\beta)^i \in N_k(\beta)$, see Lemma 2.2(ii). ∎

3. The unramified case

3.1. **The recursion relations.** Assume now that χ_1 and χ_2 are unramified. Then

$$F_k'(x) = \sum_{l=k}^{\infty} \lambda^{l-k}\phi_l, \;\; F_k''(x) = \sum_{l=k}^{\infty} \mu^{l-k}\phi_l. \tag{3.1}$$

Pick a $\phi \in \Lambda$. Substituting (3.1) in the expression (2.16), and rearranging the sum "by annuli" we get

$$\phi = \sum_{l=k_0}^{\infty} \sum_{\beta\in W} C_l(\beta)\psi_\beta(-\pi^{-l}x)\phi_l(x), \tag{3.2}$$

where

$$
\begin{aligned}
C_l(\beta) &= C'_l(\beta) + C''_l(\beta), \\
C'_l(\beta) &= \sum_{k=k_0}^{l} \lambda^{l-k} \sum_{\pi^{l-k}\alpha=\beta} c'_k(\alpha), \\
C''_l(\beta) &= \sum_{k=k_0}^{l} \mu^{l-k} \sum_{\pi^{l-k}\alpha=\beta} c''_k(\alpha).
\end{aligned}
\tag{3.3}
$$

We deduce that

$$
\begin{aligned}
C'_{k_0}(\beta) &= c'_{k_0}(\beta) \\
C'_l(\beta) &= \lambda \sum_{\pi\alpha=\beta} C'_{l-1}(\alpha) + c'_l(\beta),
\end{aligned}
\tag{3.4}
$$

and similarly for $C''_l(\beta)$, with μ instead of λ. We now derive from these relations a recursion relation for the $C_l(\beta)$, going two generations backwards.

Lemma 3.1. *Let $c_l = c'_l + c''_l$. Then $C_{k_0}(\beta) = c_{k_0}(\beta)$ and*

$$
\begin{aligned}
C_{l+1}(\gamma) &= (\lambda+\mu) \sum_{\pi\beta=\gamma} C_l(\beta) - \mu\lambda \sum_{\pi\beta=\gamma} \sum_{\pi\alpha=\beta} C_{l-1}(\alpha) \\
&\quad - \sum_{\pi\beta=\gamma} (\lambda c''_l(\beta) + \mu c'_l(\beta)) + c_{l+1}(\gamma).
\end{aligned}
\tag{3.5}
$$

Proof. We add the relations that we have obtained for $C'_l(\beta)$ and $C''_l(\beta)$ and rearrange them. We do the same at level $l+1$. Letting α, β and γ range over W as usual, we get

$$
\begin{aligned}
C_l(\beta) &= \lambda \sum_{\pi\alpha=\beta} C_{l-1}(\alpha) + (\mu-\lambda) \sum_{\pi\alpha=\beta} C''_{l-1}(\alpha) + c_l(\beta), \\
C_{l+1}(\gamma) &= \lambda \sum_{\pi\beta=\gamma} C_l(\beta) + (\mu-\lambda) \sum_{\pi\beta=\gamma} C''_l(\beta) + c_{l+1}(\gamma).
\end{aligned}
\tag{3.6}
$$

To deal with the middle term in the *second* equation we use the recursive relation for $C''_l(\beta)$ and then eliminate $(\mu-\lambda)\sum_{\pi\alpha=\beta} C''_{l-1}(\alpha)$ using the *first* equation:

$$
\begin{aligned}
(\mu-\lambda) \sum_{\pi\beta=\gamma} C''_l(\beta) &= (\mu-\lambda) \sum_{\pi\beta=\gamma} \left(\mu \sum_{\pi\alpha=\beta} C''_{l-1}(\alpha) + c''_l(\beta) \right) \\
&= \mu \sum_{\pi\beta=\gamma} \left(C_l(\beta) - \lambda \sum_{\pi\alpha=\beta} C_{l-1}(\alpha) - c_l(\beta) \right) \\
&\quad + (\mu-\lambda) \sum_{\pi\beta=\gamma} c''_l(\beta) \\
&= \mu \sum_{\pi\beta=\gamma} C_l(\beta) - \mu\lambda \sum_{\pi\beta=\gamma} \sum_{\pi\alpha=\beta} C_{l-1}(\alpha) \\
&\quad - \sum_{\pi\beta=\gamma} (\lambda c''_l(\beta) + \mu c'_l(\beta)).
\end{aligned}
\tag{3.7}
$$

The lemma follows from this. ∎

3.2. **Conclusion of the proof.** Let ρ satisfy the conditions of Thoerem 1.2, i.e. the estimates (1.7)(i) and (ii) on λ and μ, and $n < q$. Pick a $\phi \in \Lambda$ as before, and expand it as in (3.2). Assume that it vanishes outside of $\mathcal{O}_F$. Let

$$M_l(\beta) = q^{-1}\pi^{-n-lm}N_l(\beta). \tag{3.8}$$

Lemma 3.2. *For every $k_0 \leq l \leq 0$ and every $\beta \in W$, $C_l(\beta) \in M_l(\beta)$.*

Proof. We apply Lemma 2.2 and Lemma 2.5, and prove the desired bound on $C_l(\beta)$ by increasing induction on l.

When $l = k_0$, $C_{k_0}(\beta) = c_{k_0}(\beta) \in \pi^{-k_0 m}N_{k_0}(\beta) \subset M_{k_0}(\beta)$. Suppose that the lemma has been established up to index l, and $l+1 \leq 0$. Then $C_l(\beta)$ (resp. $C_{l-1}(\alpha)$) depends only on $\pi\beta$ (resp. $\pi\alpha$), since ϕ vanishes on $F - \mathcal{O}_F$. We invoke the recursion relation (3.5) for $C_{l+1}(\gamma)$. The term

$$\sum_{\pi\beta=\gamma}(\lambda c''_l(\beta) + \mu c'_l(\beta)) \in M_{l+1}(\gamma) \tag{3.9}$$

since $c'_l(\beta), c''_l(\beta) \in \pi^{-lm}N_l(\beta)$, $|\mu|, |\lambda| \leq |q^{-1}\pi^{-n-m}|$, and because of the relation $N_l(\beta) \subset N_{l+1}(\gamma)$, that holds whenever $\pi\beta = \gamma$. That

$$c_{l+1}(\gamma) \in M_{l+1}(\gamma) \tag{3.10}$$

is clear. The term

$$(\lambda + \mu)\sum_{\pi\beta=\gamma} C_l(\beta) \in M_{l+1}(\gamma) \tag{3.11}$$

because the q summands $C_l(\beta)$ are *equal*, hence belong to

$$\bigcap_{\pi\beta=\gamma} M_l(\beta) = q^{-1}\pi^{-n-lm}\bigcap_{\pi\beta=\gamma} N_l(\beta) = q^{-1}\pi^{-lm}N_{l+1}(\gamma). \tag{3.12}$$

Thus $\sum_{\pi\beta=\gamma} C_l(\beta) \in \pi^{-lm}N_{l+1}(\gamma)$, while $|\lambda + \mu| \leq |q^{-1}\pi^{-n-m}|$. Finally,

$$\mu\lambda \sum_{\pi\beta=\gamma}\sum_{\pi\alpha=\beta} C_{l-1}(\alpha) \in M_{l+1}(\gamma) \tag{3.13}$$

for similar reasons: For a given β, the q summands $C_{l-1}(\alpha)$ are equal, so belong to

$$\bigcap_{\pi\alpha=\beta} M_{l-1}(\alpha) = q^{-1}\pi^{-n-(l-1)m}\bigcap_{\pi\alpha=\beta} N_{l-1}(\alpha) = q^{-1}\pi^{-(l-1)m}N_l(\beta). \tag{3.14}$$

This implies that their sum, $\sum_{\pi\alpha=\beta} C_{l-1}(\alpha) \in \pi^{-(l-1)m}N_l(\beta) \subset \pi^{-(l-1)m}N_{l+1}(\gamma)$. But $|\mu\lambda| = |q^{-1}\pi^{-n-2m}|$, so for every β,

$$\mu\lambda \sum_{\pi\alpha=\beta} C_{l-1}(\alpha) \in q^{-1}\pi^{-n-(l+1)m}N_{l+1}(\gamma) = M_{l+1}(\gamma). \tag{3.15}$$

Since each of the four terms in (3.6) has been shown to lie in $M_{l+1}(\gamma)$, the proof of the induction step is complete. ∎

When $l = 0$, $C_0(\beta) \in M_0(\beta)$, and this proves Theorem 1.2.

4. The tamely ramified case

For the sake of completeness we treat also case (2) of the theorem, which is covered by [K-dS12]. The proof is the same, except that we have cleaned up the computations.

4.1. **The recursion relations.** Assume from now on that at least one of the characters χ_1 and χ_2 is ramified, but $\tau = \det(.)^m$, i.e. $n = 0$. Since a twist of ρ by a character of finite order does not affect the validity of Theorem 1.2, we may assume that χ_2 is unramified. We let ε be the restriction of χ_1 to U_F, and extend it to a character of $F^\times$ so that $\varepsilon(\pi) = 1$. We denote by $\nu \geq 1$ the conductor of ε. Letting $\lambda = \chi_1(\pi)$ and $\mu = \chi_2(\pi)$ as before, we have

$$\chi_1(u\pi^k) = \varepsilon(u)\lambda^k,\ \chi_2(u\pi^k) = \mu^k \tag{4.1}$$

if $u \in U_F$.

Recall that

$$F'_k = \varepsilon \sum_{l=k}^{\infty} \lambda^{l-k}\phi_l,\ \ F''_k = \sum_{l=k}^{\infty} \mu^{l-k}\phi_l. \tag{4.2}$$

The module Λ consists this time of functions of the form

$$\begin{aligned} \phi(x) &= \sum_{k=k_0}^{\infty} \sum_{\beta\in W} c'_k(\beta)\psi_\beta(-\pi^{-k}x)F'_k(x) + c''_k(\beta)\psi_\beta(-\pi^{-k}x)F''_k(x) \\ &= \sum_{l=k_0}^{\infty} \sum_{\beta\in W} C_l(\beta)\psi_\beta(-\pi^{-l}x)\phi_l(x), \end{aligned} \tag{4.3}$$

with $c'_k(\beta), c''_k(\beta) \in \pi^{-mk}\mathcal{O}_E$, and some $C_l(\beta)$ which we are now going to compute. Let, as before

$$\begin{aligned} C'_l(\beta) &= \sum_{k=k_0}^{l} \lambda^{l-k} \sum_{\pi^{l-k}\alpha=\beta} c'_k(\alpha) \\ C''_l(\beta) &= \sum_{k=k_0}^{l} \mu^{l-k} \sum_{\pi^{l-k}\alpha=\beta} c''_k(\alpha). \end{aligned} \tag{4.4}$$

These coefficients satisfy the *recursion relations*

$$\begin{aligned} C'_{k_0}(\beta) &= c'_{k_0}(\beta) \\ C'_l(\beta) &= \lambda \sum_{\pi\alpha=\beta} C'_{l-1}(\alpha) + c'_l(\beta), \end{aligned} \tag{4.5}$$

and similarly for $C''_l(\beta)$, with μ instead of λ. In terms of the $C'_l(\beta)$ and the $C''_l(\beta)$ we have

(4.6)

$$\phi(x) = \varepsilon(x) \sum_{l=k_0}^{\infty} \sum_{\beta\in W} C'_l(\beta)\psi_\beta(-\pi^{-l}x)\phi_l(x) + \sum_{l=k_0}^{\infty} \sum_{\beta\in W} C''_l(\beta)\psi_\beta(-\pi^{-l}x)\phi_l(x).$$

Invoking the Fourier expansion of $\varepsilon(x)\phi_l(x)$ (see [K-dS12], Corollary 2.2) we finally get the formula

$$C_l(\beta) = \frac{\tau(\varepsilon^{-1})}{q^\nu} \sum_{u\in U_F/U_F^\nu} \varepsilon^{-1}(u)C'_l(\beta - \pi^{-\nu}u) + C''_l(\beta). \tag{4.7}$$

Here U_F^ν denotes the group of units which are congruent to 1 modulo π^ν, and $\tau(\varepsilon^{-1})$ is the Gauss sum

$$\tau(\varepsilon^{-1}) = \sum_{u \in U_F/U_F^\nu} \psi(\pi^{-\nu}u)\varepsilon(u). \tag{4.8}$$

We recall the well-known identity

$$\tau(\varepsilon)\tau(\varepsilon^{-1}) = \varepsilon(-1)q^\nu. \tag{4.9}$$

4.2. **Operators on functions on W.** As in [K-dS12], Section 3.4, we introduce some operators on the space $\mathcal{C}$ of E-valued functions on W with finite support. If $f \in \mathcal{C}$ we define

- The *suspension* of f

$$Sf(\beta) = \sum_{\pi\alpha=\beta} f(\alpha). \tag{4.10}$$

- The *convolution* of f with a character ξ of U_F, of conductor $\nu \geq 1$

$$E_\xi f(\beta) = \frac{\tau(\xi^{-1})}{q^\nu} \sum_{u \in U_F/U_F^\nu} \xi^{-1}(u) f(\beta - \pi^{-\nu}u). \tag{4.11}$$

- The operator Π

$$\Pi f(\beta) = f(\pi\beta). \tag{4.12}$$

We decompose $\mathcal{C}$ as a direct sum $\mathcal{C} = \mathcal{C}_0 \bigoplus \mathcal{C}_1$, where

$$\begin{aligned} \mathcal{C}_0 &= \left\{ f \,|\, \forall\beta,\ \sum_{\pi t=0} f(\beta+t) = 0 \right\} \\ \mathcal{C}_1 &= \{ f \,|\, f(\beta) \text{ depends only on } \pi\beta \}. \end{aligned} \tag{4.13}$$

Lemma 4.1. *(i) The projection onto $\mathcal{C}_1$ is*

$$P_1 = \frac{1}{q}\Pi S. \tag{4.14}$$

(ii) Let ξ be any non-trivial character. Then the projection onto $\mathcal{C}_0$ is

$$P_0 = E_\xi E_{\xi^{-1}} = E_{\xi^{-1}} E_\xi. \tag{4.15}$$

(iii) If ξ is non-trivial then $SE_\xi = 0$ and $E_\xi E_{\xi^{-1}} E_\xi = E_\xi$.

Proof. All the statements are elementary, and best understood if we associate to f its Fourier transform

$$\hat{f}(x) = \sum_{\beta \in W} f(\beta)\psi_\beta(x) \tag{4.16}$$

($x \in \mathcal{O}_F$) and apply Lemma 2.1. See [K-dS12], Section 3.4. ∎

For $f, g_1, \ldots, g_r \in \mathcal{C}$ we write $f = O(g_1, \ldots, g_r)$ to mean that in the sup norm $||f|| \leq \max ||g_i||$.

4.3. **Conclusion of the proof in the tamely ramified case.** We assume from now on that $\nu = 1$, i.e. ε is tamely ramified. The Breuil-Schneider estimates on λ and μ are

$$\begin{aligned} |\pi^{-m}| &\leq |\lambda|, |\mu| \leq |q^{-1}\pi^{-m}| \\ |\lambda\mu| &= |q^{-1}\pi^{-2m}|. \end{aligned}$$

Fix a $\phi \in \Lambda$ as in (4.3), so that

$$c'_k,\ c''_k = O(\pi^{-mk}), \tag{4.17}$$

and assume that it vanishes off $\mathcal{O}_F$. We shall prove by increasing induction on l that for $l \leq 0$

$$C'_l,\ C''_l = O(q^{-1}\pi^{-ml}). \tag{4.18}$$

When we reach $l = 0$ this will imply Theorem 1.2, even uniformly in β, thanks to the fact that the algebraic part of ρ is essentially trivial.

Using the notation of the last sub-section, we can write the recursion relations (4.5) as

$$\begin{aligned} C'_{k_0} &= c'_{k_0},\ C''_{k_0} = c''_{k_0} \\ C'_l &= \lambda S C'_{l-1} + c'_l \\ C''_l &= \mu S C''_{l-1} + c''_l. \end{aligned} \tag{4.19}$$

Besides $C_l(\beta)$ we introduce $\tilde{C}_l(\beta)$ so that the following formulae hold

$$\begin{aligned} C_l &= E_\varepsilon C'_l + C''_l \\ \tilde{C}_l &= E_{\varepsilon^{-1}} C''_l + C'_l. \end{aligned} \tag{4.20}$$

Here the first formula is just (4.7). The second shows that the amplitudes $\tilde{C}_l(\beta)$ are analogously associated with the function $\tilde{\phi}(x) = \varepsilon^{-1}(x)\phi(x)$.

Next, we observe that since $SE_\varepsilon = SE_{\varepsilon^{-1}} = 0$, we can rewrite the recursion relations as

$$\begin{aligned} C'_l &= \lambda S \tilde{C}_{l-1} + c'_l \\ C''_l &= \mu S C_{l-1} + c''_l. \end{aligned} \tag{4.21}$$

For $l \leq 0$ the functions C_{l-1} and $\tilde{C}_{l-1}$ belong to the subspace that we have called $\mathcal{C}_1$, because ϕ and $\tilde{\phi}$ vanish on $\pi^{l-1}U_F$. This implies the following result.

Lemma 4.2. *For $l \leq 0$,*

$$\begin{aligned} C'_l &= O(\lambda q \tilde{C}_{l-1}, c'_l) \\ C''_l &= O(\mu q C_{l-1}, c''_l). \end{aligned} \tag{4.22}$$

We can now proceed with the induction. When $l = k_0$ (4.17) clearly implies (4.18). Assume that $l \leq 0$ and that (4.18) has been established up to index $l-1$. As $C'_{l-2} = O(q^{-1}\pi^{-m(l-2)})$, and as $C''_{l-2} = O(q^{-1}\pi^{-m(l-2)}) = O(q^{-2}\tau(\varepsilon^{-1})\pi^{-m(l-2)})$, we obtain from (4.20) and the fact that $\nu = 1$ the estimate

$$C_{l-2} = O(q^{-2}\tau(\varepsilon^{-1})\pi^{-m(l-2)}). \tag{4.23}$$

By the lemma, this gives

$$C''_{l-1} = O(\mu q^{-1}\tau(\varepsilon^{-1})\pi^{-m(l-2)}, c''_{l-1}) = O(\mu q^{-1}\tau(\varepsilon^{-1})\pi^{-m(l-2)}) \tag{4.24}$$

(the last equality coming from $|\mu q^{-1}\tau(\varepsilon^{-1})| \geq |\pi^{-m}|$). A second application of (4.20), the identity (4.9), and the induction hypothesis for C'_{l-1} (recall $|\mu| \geq |\pi^{-m}|$) yield

$$\tilde{C}_{l-1} = O(\mu q^{-1}\pi^{-m(l-2)}). \tag{4.25}$$

A second application of the lemma finally gives

$$\begin{aligned} C'_l &= O(\lambda\mu\pi^{-m(l-2)}, c''_l) \\ &= O(q^{-1}\pi^{-ml}, c''_l) = O(q^{-1}\pi^{-ml}). \end{aligned} \tag{4.26}$$

Symmetrically, we get the same estimate on C''_l. This completes the proof of (4.18) at level l, and with it, the proof of Theorem 1.2.

5. The case $\chi_1 = \omega\chi_2$

We finally deal with the one excluded case, when $\chi_1 = \omega\chi_2$. After a twist by a character of finite order we may assume that χ_1 is unramified. In this case $\lambda = \mu$ and the Kirillov model is the space

$$\mathcal{K} = C_c^\infty(F,\tau)\chi_1 + C_c^\infty(F,\tau)v\chi_1, \tag{5.1}$$

where $v : F^\times \to \mathbb{Z} \subset E$ is the normalized valuation. The action of B is still given by (1.10). Once more, $\mathcal{K}$ contains $\mathcal{K}_0 = C_c^\infty(F^\times,\tau)$ as a subspace. When $\tau = 1$, the quotient $\mathcal{K}/\mathcal{K}_0$ is the Jacquet module. The torus acts on it non-semisimply, by

$$\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mapsto \chi_1(t_1t_2)\begin{pmatrix} 1 & v(t_1/t_2) \\ & 1 \end{pmatrix}. \tag{5.2}$$

Following the notation of Section 3, we let

$$F'_0 = \chi_1 1_{\mathcal{O}_F}, \quad F''_0 = -v\chi_1 1_{\mathcal{O}_F} \tag{5.3}$$

and

$$F'_k = \sum_{l=k}^{\infty} \lambda^{l-k}\phi_l, \quad F''_k = \sum_{l=k}^{\infty}(k-l)\lambda^{l-k}\phi_l. \tag{5.4}$$

The module Λ consists of all the functions ϕ as in (2.16), and any such ϕ can be expanded "by annuli" as in (3.2). The coefficients of the expansion are given by (3.3), except that the last equation now takes the shape

$$C''_l(\beta) = \sum_{k=k_0}^{l}(k-l)\lambda^{l-k}\sum_{\pi^{l-k}\alpha=\beta} c''_k(\alpha). \tag{5.5}$$

The recursion relation for $C'_l(\beta)$ is given by (3.4) but $C''_l(\beta)$ needs a modification.

Lemma 5.1. *We have $C''_{k_0}(\beta) = 0$, $C''_{k_0+1}(\beta) = -\lambda\sum_{\pi\alpha=\beta} c''_{k_0}(\alpha)$, and for $l > k_0$*

$$C''_{l+1}(\gamma) = 2\lambda\sum_{\pi\beta=\gamma} C''_l(\beta) - \lambda^2\sum_{\pi\beta=\gamma}\sum_{\pi\alpha=\beta} C''_{l-1}(\alpha) - \lambda\sum_{\pi\beta=\gamma} c''_l(\beta). \tag{5.6}$$

Proof. A straightforward exercise. ∎

Lemma 5.2. *The following recursion relation holds:*

$$C_{l+1}(\gamma) = 2\lambda\sum_{\pi\beta=\gamma} C_l(\beta) - \lambda^2\sum_{\pi\beta=\gamma}\sum_{\pi\alpha=\beta} C_{l-1}(\alpha) - \lambda\sum_{\pi\beta=\gamma}(c''_l(\beta) + c'_l(\beta)) + c'_{l+1}(\gamma). \tag{5.7}$$

Proof. We write

$$\begin{aligned}
C'_{l+1}(\gamma) &= \lambda \sum_{\pi\beta=\gamma} C'_l(\beta) + c'_{l+1}(\gamma) \\
&= 2\lambda \sum_{\pi\beta=\gamma} C'_l(\beta) - \lambda \sum_{\pi\beta=\gamma} \left(\lambda \sum_{\pi\alpha=\beta} C'_{l-1}(\alpha) + c'_l(\beta) \right) + c'_{l+1}(\gamma) \\
&= 2\lambda \sum_{\pi\beta=\gamma} C'_l(\beta) - \lambda^2 \sum_{\pi\beta=\gamma} \sum_{\pi\alpha=\beta} C'_{l-1}(\alpha) - \lambda \sum_{\pi\beta=\gamma} c'_l(\beta) + c'_{l+1}(\gamma) \qquad (5.8)
\end{aligned}$$

and we add the result to the recursive relation for $C''_{l+1}(\gamma)$. ∎

Note the similarity with Lemma 3.1. The rest of the proof of Theorem 1.2 is now identical to that given in the case $\lambda \neq \mu$ in Section 3.2.

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