BOUNDED COHOMOLOGY OF THE p-ADIC UPPER HALF PLANE

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1. Introduction

Let K be a p-adic field, and $\mathbb{C}_p$ the completion of its algebraic closure. Let $\mathfrak{X}$ be the p-adic "upper half plane" over K introduced by Drinfel'd in [Dr]. The rigid analytic de-Rham cohomology of $\mathfrak{X}$ is the space of rigid analytic $\mathbb{C}_p$-valued 1-forms on $\mathfrak{X}$, modulo exact ones,

$$H^1_{dR}(\mathfrak{X}) = \Omega/d\mathcal{O}. \tag{1.1}$$

In contrast to the complex upper half plane, this cohomology does not vanish. In fact, it is a classical result that it is isomorphic to the space of *harmonic* ($\mathbb{C}_p$-valued) 1-cochains on the Bruhat-Tits tree $\mathcal{T}$ of the group $G = SL_2(K)$. (See [D-T] for background.) Recal that $\mathcal{T}$ is a $q+1$ regular tree (where q is the cardinality of the residue field of K), and a 1-cochain c on $\mathcal{T}$ is harmonic if (i) $c(\bar{e}) = -c(e)$ if $\bar{e}$ is the edge e with reversed orientation, and (ii) the sum of $c(e)$ for all the edges flowing into any given vertex, is zero. We denote the space of harmonic 1-cochains by C^1_{har}. The isomorphism

$$H^1_{dR}(\mathfrak{X}) \simeq C^1_{har} \tag{1.2}$$

is given by the residue homomorphism. If $\omega \in \Omega$, we let

$$c_\omega(e) = res_e(\omega) \tag{1.3}$$

be the residue of ω along the oriented edge e (see [G-vdP] p.94). Then c_ω is a harmonic cochain, it vanishes if and only if ω is exact, and every harmonic cochain is of this type. Both Ω and C^1_{har} are inverse limits of Banach spaces (in the case of C^1_{har} these are finite dimensional Banach spaces), and carry a natural p-adic Fréchet topology in which they are reflexive (see [Sch2]). The group G acts on the two spaces. The residue homomorphism respects both the topology and the G-action.

If M is a finite dimensional representation of G over K, we may tensor the above spaces with M to get the notions of M-valued rigid analytic functions or forms, cohomology with coefficients in M, and M-valued harmonic cochains. The isomorphism

$$H^1_{dR}(\mathfrak{X}; M) \simeq C^1_{har}(M) \tag{1.4}$$

(where the notation is self-explanatory) is obtained from the case of trivial coefficients by tensoring with M. One should only be aware that the G-action now involves both the geometric action on $\mathfrak{X}$, and the action on the coefficients.

An *integral structure* $\mathbf{M}$ in M is the assignment, for every simplex σ of $\mathcal{T}$, of an $\mathcal{O}_K$-lattice $\mathbf{M}(\sigma)$ in M, such that $\mathbf{M}(\gamma\sigma) = \gamma\mathbf{M}(\sigma)$ for every $\gamma \in G$, and $\mathbf{M}(v) \subset \mathbf{M}(\varepsilon)$ if the vertex v belongs to the edge ε. Integral structures exist, and any two are commensurable. An M-valued harmonic cochain c is *bounded* if there exists an integral structure $\mathbf{M}$ such that $c(e) \in \mathbf{M}(e)$ for every e. The space of bounded M-valued harmonic cochains is preserved by the action of G. We denote it by $C^1_{har}(M)^{bnd}$. It is not a-priori clear that bounded harmonic cochains exist, namely that the two conditions of harmonicity and boundedness can coexist. The simplest way to produce plenty of bounded harmonic cochains is to consider a discrete cocompact subgroup Γ in G, and note that $H^0(\Gamma, C^1_{har}(M)) \subset C^1_{har}(M)^{bnd}$. The spaces $H^0(\Gamma, C^1_{har}(M))$ have been studied in [dS1], among other places.

The cotangent space to $\mathfrak{X}$ at z_0 has a canonical integral structure $\mathbf{\Omega}|_{z_0}$ which is the $\mathcal{O}_{\mathbb{C}_p}$-span of $dz/(z_0-\zeta)$ for $\zeta \in K$ (this integral structure does not depend on the choice of the coordinate z and is G-equivariant). Let $r : \mathfrak{X} \to |\mathcal{T}|$ be the reduction map from $\mathfrak{X}$ to the real realization of the tree (see [G-vdP] or [dS1]). An M-valued 1-form $\omega \in \Omega(M)$ is called *bounded* if there exists an integral structure $\mathbf{M}$ such that

$$\omega|_z \in \mathbf{\Omega}|_z \otimes \mathbf{M}(\sigma) \tag{1.5}$$

whenever $r(z) \in |\sigma|$. The space of bounded M-valued forms is denoted by $\Omega(M)^{bnd}$. It is G-stable. Once again, the simplest way to produce such forms is to produce Γ-invariant ones for some Γ.

We shall describe below a filtration, due to Schneider and Stuhler, of the module $\Omega(M)$, by certain coherent submodules $F^i\Omega(M)$ of decreasing ranks, which are direct summands (i.e. give rise to subvectorbundles), and which are G-stable. The last step in the filtration, $F^n\Omega(M)$, still maps surjectively onto $H^1_{dR}(\mathfrak{X}; M)$, so every cohomology class is represented by an M-valued form from that last piece.

We shall check that bounded M-valued forms get mapped, by the residue homomorphism, to bounded M-valued harmonic cochains.

Theorem 1.1. *The residue homomorphism induces an isomorphism*

$$F^n\Omega(M) \cap \Omega(M)^{bnd} \simeq C^1_{har}(M)^{bnd}. \tag{1.6}$$

In other words, every bounded M-valued harmonic cochain is of the form c_ω for an $\omega \in F^n\Omega(M) \cap \Omega(M)^{bnd}$, and if a bounded ω which lies in the last step of the filtration is exact, then it is zero.

The theorem is not new. For a survey see [D-T], Theorem 2.3.2, Corollary 2.3.4 and the references therein. However, our approach to Morita duality, to the notion of bounded differential forms, and to the injectivity statement in the theorem, is different than what may be found in the literature. Except for a certain technical verification of convergence at the very end, our exposition is self contained.

The surjectivity statement in the theorem asserts that the map from forms to cohomology splits G-equivariantly over the bounded cohomology. The existence of such a splitting has been known for some time and follows from the theorem of Amice-Velu-Vishik. Roughly speaking (we shall make everything precise below), to a bounded M-valued harmonic cochain c one attaches a (scalar valued) distribution λ_c on $\mathbb{P}^1_K$ with "growth conditions" of order n. This distribution extends in a natural way to a tempered distribution $\tilde{\lambda}_c$, i.e. to a linear functional on the space of the locally analytic functions on $\mathbb{P}^1_K$ (this is where one needs the estimates of Amice-Velu-Vishik). Teitelbaum's Poisson integral

$$f(z) = \int \frac{d\tilde{\lambda}_c(\zeta)}{z-\zeta} \tag{1.7}$$

makes sense since the integrand, $1/(z-\zeta)$, is locally analytic in $\zeta \in \mathbb{P}^1_K$. As a function of $z \in \mathfrak{X}$, $f(z)$ is rigid analytic. The M-valued form

$$\omega = f(z)(u - zv)^n dz \tag{1.8}$$

(where we have realized M as the space of homogeneous polynomials of degree n in the two variables u and v) lies in the last step of the filtration, is bounded, and c_ω is our original c.

It is tempting to think about generalizations of the questions discussed in this paper to higher dimensions. Let us only mention that the notions of boundedness, either for differential forms, or for cohomology with coefficients in a rational representation M, can be defined in precisely the same way, and the higher dimensional residue homomorphism from k-forms to harmonic k-cochains (see [dS2]) carries bounded closed M-valued k-forms to bounded M-valued harmonic k-cochains. The paper [Sch1] discusses an analogue of the filtration $F^\cdot$ studied here, but an analogue of the theorem mentioned above is not known to hold. Such an analogue would probably be non-void for forms of top degree only. For lower degree forms, and non-trivial coefficients, the notion of boundedness has to be modified.

For trivial coefficients, we expect that the bounded k-forms are nothing else but the Iovita-Spiess forms (see [I-S] and [A-dS]). This will be the case if we can show that a bounded exact form must vanish, and it would be enough to check this last assertion for forms of top degree. For that see [EGK].

1.1. **Notation.** Let K be a p-adic field (a finite extension of $\mathbb{Q}_p$), V a two dimensional vector space over K, and $G = SL(V)$. Fix a basis, and identify V with column vectors, and G with $SL_2(K)$ acting from the left. Let u, v be the dual basis, so that $V^\vee = Ku + Kv$. For $n \geq 0$, let $M = M_n = Symm^n(V^\vee)$. Then M is the unique irreducible rational representation of G (viewed as an algebraic group over K) of dimension $n+1$. In coordinates,

$$M = K[u, v]_{\text{hom . deg .}=n} \tag{1.9}$$

and the action of $\gamma \in G$ on $P \in M$ is

$$\gamma P = P \circ \gamma^{-1} = P(au + bv, cu + dv) \tag{1.10}$$

where

$$\gamma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \gamma = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{1.11}$$

Let $\mathbb{C}_p$ be the completion of a fixed algebraic closure of K. The *p-adic upper half plane* is the rigid analytic space $\mathfrak{X} \subset \mathbb{P}(V^\vee)$ which is the complement of the K-rational points:

$$\mathfrak{X} = \mathbb{P}(V^\vee) \backslash \mathbb{P}(V^\vee_K). \tag{1.12}$$

We do not distinguish between $\mathfrak{X}$ (as a ringed space in a certain category) and the set of its $\mathbb{C}_p$-points. The latter is identified with the complement of K in $\mathbb{C}_p$ by sending the line through $tu + sv$ to its *coordinate* $z = -st^{-1}$. With this convention the action of γ on z is the usual action

$$\gamma(z) = \frac{dz - b}{-cz + a}. \tag{1.13}$$

We denote by $\mathcal{A}$ the *set* of K-rational lines in V with its p-adic topology. Thus $\mathcal{A} = \mathbb{P}(V_K)$. If $a \in \mathcal{A}$ the "hyperplane" H_a is the point $b \in \mathbb{P}(V^\vee_K)$ given by $\langle a, b \rangle = 0$, and the complement of $\mathfrak{X}$ is the union of the H_a for $a \in \mathcal{A}$. In coordinates, if $a = (x:y)^t$ (column vector up to homothety) then H_a is the line through $-yu+xv$, so its coordinate is

$$z(H_a) = xy^{-1} \tag{1.14}$$

and we see that the standard (left) G action on a corresponds to the action on z, namely $z(H_{\gamma a}) = \gamma(z(H_a))$. This allows us to identify $\mathcal{A}$ and $\mathbb{P}(V^\vee)\backslash\mathfrak{X}$ as G-sets. Of course, we could do it without introducing coordinates by observing that $\gamma(H_a) = H_{\gamma a}$. This discussion may seem superfluous, but we shall soon need to extend locally analytic functions on $\mathcal{A}$ to a rigid analytic neighborhood of $\mathcal{A}$ in $\mathfrak{X}$, so it is necessary to view $\mathcal{A}$ and $\mathfrak{X}$ in the same ambient space, despite the fact that a-priori they live in dual projective spaces. The correct generalization of this in higher dimensions requires some care.

We let $\mathcal{T}$ denote the Bruhat-Tits tree of G. Then $\mathcal{A}$ is identified canonically with the set of the *ends* of $\mathcal{T}$, and for an oriented edge e we let $\mathcal{A}_e$ denote the subset of ends in the direction of e. Each $\mathcal{A}_e$ is a disk in $\mathbb{P}(V_K)$ and $\mathcal{A}$ is the disjoint union of $\mathcal{A}_e$ and $\mathcal{A}_{\bar{e}}$. Let

$$\mathcal{O} = \underline{\mathcal{O}}(\mathfrak{X}),\ \Omega = \underline{\Omega}(\mathfrak{X}) \tag{1.15}$$

denote the $\mathbb{C}_p$-algebra of rigid analytic functions on $\mathfrak{X}$, and the $\mathcal{O}$-module of rigid analytic differential forms. The underlined symbols represent the corresponding sheaves in the rigid analytic topology. We let $\mathcal{O}(M)$ and $\Omega(M)$ stand for the same spaces tensored (over K) with M. These are the M-valued functions or forms. An element $\gamma \in G$ acts on $f \in \mathcal{O}(M)$ via the rule

$$(\gamma f)(z) = (1 \otimes \gamma)(f(\gamma^{-1}z)). \tag{1.16}$$

It acts on $\omega \in \Omega(M)$ via

$$(\gamma\omega)|_z = (1 \otimes \gamma)(\gamma^{-1})^*(\omega|_{\gamma^{-1}z}). \tag{1.17}$$

2. Differential forms and cohomology with coefficients in M

2.1. **Filtration on $\mathcal{O}(M)$.** We follow [Sch-St], pp.95-97. Let

$$\mathcal{O}(M) = F^0 \supset F^1 \supset \cdots \supset F^n \supset \{0\} = F^{n+1} \tag{2.1}$$

where

$$F^k = Span_{\mathcal{O}} \left\{ (u-zv)^k u^{n-k-l} v^l;\ 0 \le l \le n-k \right\}. \tag{2.2}$$

Then:

(1) The filtration is G-stable. In fact, if γ^{-1} is as above, $f = (u-zv)^k u^{n-k-l} v^l$

$$(\gamma f)(z) = \gamma(f(\gamma^{-1}z)) = (cz+d)^{-k}(u-zv)^k(au+bv)^{n-k-l}(cu+dv)^l \in F^k. \tag{2.3}$$

(2) $F^k = \mathcal{O}\cdot(u-zv)^k v^{n-k} \oplus F^{k+1}$ (as $\mathcal{O}$-modules, but not as G-modules).

(3) $\mathcal{O}(M)$ is free over $\mathcal{O}$ with basis $(u-zv)^k v^{n-k}$ $(0 \le k \le n)$.

(4) For any $m \in \mathbb{Z}$ let $\mathcal{O}(m)$ be the ring $\mathcal{O}$ with the *twisted* G-action

$$\gamma \cdot f = (cz+d)^m \left(f \circ \gamma^{-1}\right), \tag{2.4}$$

where (c,d) is the bottom row of γ^{-1}. Then $f \mapsto f(z)(u-zv)^k v^{n-k} \bmod F^{k+1}$ is an isomorphism

$$\Theta_k : \mathcal{O}(n-2k) \simeq F^k/F^{k+1}. \tag{2.5}$$

(5) Similarly $F^{\cdot}\Omega = \Omega \otimes_{\mathcal{O}} F^{\cdot}$ is a G-stable filtration on $\Omega(M)$ and

$$f \mapsto f(z)(u - zv)^k v^{n-k} dz \mod \Omega \otimes F^{k+1} \tag{2.6}$$

is an isomorphism

$$\Theta'_k : \mathcal{O}(n - 2k - 2) \simeq \Omega \otimes F^k / \Omega \otimes F^{k+1}. \tag{2.7}$$

(6) For $1 \leq k \leq n$, the G-homomorphism $d : \mathcal{O}(M) \to \Omega(M)$ maps F^k to $\Omega \otimes F^{k-1}$ (*Griffiths transversality*) and induces a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(n-2k) & \stackrel{(-k)\cdot}{\rightarrow} & \mathcal{O}(n-2k) \\ \downarrow \Theta_k & & \downarrow \Theta'_{k-1} \\ F^k/F^{k+1} & \stackrel{d}{\simeq} & \Omega \otimes F^{k-1}/\Omega \otimes F^k \end{array} . \tag{2.8}$$

(7) There is a decomposition as a direct sum of abelian groups with G-action

$$\Omega(M) = d(F^1) \oplus (\Omega \otimes F^n). \tag{2.9}$$

Indeed, (6) shows that
$\Omega(M) = d(F^1) + (\Omega \otimes F^1) = d(F^1) + (\Omega \otimes F^2) = \cdots = d(F^1) + (\Omega \otimes F^n)$.

For every $1 \leq k \leq n$, if $f \in F^k - F^{k+1}$, then $df \in \Omega \otimes F^{k-1} - \Omega \otimes F^k$. Since any non-zero $f \in F^1$ falls in some $F^k - F^{k+1}$, $df \notin \Omega \otimes F^k$, hence clearly $df \notin \Omega \otimes F^n$ and the sum is direct.

Note that $d(F^1)$ is *not* an $\mathcal{O}$-submodule of $\Omega(M)$, so this decomposition is not a splitting of the last step in the filtration of $\Omega(M)$. For example, if $n = 1$ the decomposition is

$$\{\alpha u\, dz\} \oplus \{\beta v\, dz\} = \{-fvdz + f'(u - zv)dz\} \oplus \{g(u - vz)dz\} \tag{2.10}$$

where $(f, g) = (-\beta - z\alpha, 2\alpha + \beta' + z\alpha')$ and $(\alpha, \beta) = (g + f', -f - zf' - zg)$.

(8) Let $n \geq 0$. There is a commutative diagram of additive (but not $\mathcal{O}$-linear) G-homomorphisms

$$\begin{array}{ccc} \mathcal{O}(n) & \stackrel{\frac{1}{n!}\left(\frac{d}{dz}\right)^{n+1}}{\rightarrow} & \mathcal{O}(-n-2) \\ \downarrow \Theta_0 & & \downarrow \Theta'_n \\ \mathcal{O}(M)/F^1 & \stackrel{pr_2 \circ d}{\rightarrow} & \Omega \otimes F^n \end{array} , \tag{2.11}$$

where pr_2 is the projection on the second factor in the decomposition (7).

Indeed, for $1 \leq k \leq n$

$$d \circ \Theta_k(f) = \left\{ f'(z)(u - zv)^k v^{n-k} - kf(z)(u - zv)^{k-1} v^{n-k+1} \right\} dz \mod dF^{k+1} \tag{2.12}$$

implies

$$f'(z)(u - zv)^k v^{n-k} dz \equiv kf(z)(u - zv)^{k-1} v^{n-k+1} dz \mod dF^1 \tag{2.13}$$

hence, iterating,

$$f(z)v^n dz \equiv \frac{1}{n!} f^{(n)}(z)(u - zv)^n dz \mod dF^1. \tag{2.14}$$

From this we get

$$d(g(z)v^n) \equiv \frac{1}{n!} g^{(n+1)}(z)(u - zv)^n dz \mod dF^1, \tag{2.15}$$

or

$$pr_2 \circ d \circ \Theta_0(g) = \Theta'_n\left(\frac{1}{n!} g^{(n+1)}\right). \tag{2.16}$$

Note that the fact that the top arrow is a G-homomorphism is not easy to establish via a direct computation, since differentiation does not commute with the action of G. It is rather a consequence of the fact that the other three arrows commute with G. The surjectivity of the horizontal arrows has the following consequence.

Corollary 2.1. *For $n \geq 0$, let $P_n(n)$ be the space of polynomials of degree at most n, with G-action induced from (4). Then there is an exact sequence of G-representations*

$$0 \to \mathcal{O}(n)/P_n(n) \stackrel{\frac{1}{n!}\left(\frac{d}{dz}\right)^{n+1}}{\longrightarrow} \mathcal{O}(-n-2) \to H^1_{dR}(\mathfrak{X}; M_n) \to 0. \tag{2.17}$$

Proof. De-Rham cohomology with values in M is given by

$$\begin{aligned} \Omega(M)/d\mathcal{O}(M) &= \Omega \otimes F^n / \left(d\mathcal{O}(M) \cap \Omega \otimes F^n\right) \\ &= \Omega \otimes F^n / pr_2 \circ d\left(\mathcal{O}(M)/F^1\right) \end{aligned} \tag{2.18}$$

so we conclude by the commutative diagram of step (8), and by the fact that the kernal of $n+1$-fold differentiation is P_n. ∎

2.2. **Morita duality (trivial coefficients). Locally analytic and meromorphic functions.** Let $\mathcal{C}^{an}$ be the space of locally-analytic, $\mathbb{C}_p$-valued functions on $\mathcal{A}$. Since $\mathcal{A}$ is compact, each $\varphi \in \mathcal{C}^{an}$ admits a decomposition of $\mathcal{A}$ into a finite union of closed disks, such that φ is rigid analytic on each of them. Let $\mathcal{C} = \mathcal{C}^{mer}$ be the space of locally meromorphic $\mathbb{C}_p$-valued functions on $\mathcal{A}$, and $\mathcal{R}$ the subspace of $\mathbb{C}_p$-valued rational functions with poles in $\mathcal{A}$. Note that $\mathcal{R}$ is a subring, but not a field. Of course, $\mathcal{R} \cap \mathcal{C}^{an} = \mathbb{C}_p$, the constants, and $\mathcal{R} + \mathcal{C}^{an} = \mathcal{C}$ (the theorem on principal parts).

Topologies. The space $\mathcal{O}$ (hence $\Omega = \mathcal{O}dz$) is a locally convex topological vector space (for *nonarchimedean functional analysis* consult [Sch2]). The topology is given by the family of norms $|.|_{\mathfrak{X}_n}$ where $\mathfrak{X}_n$ is an increasing sequence of affinoids exhausting $\mathfrak{X}$. In other words,

$$\Omega = \lim_{\leftarrow} \Omega(\mathfrak{X}_n). \tag{2.19}$$

The space $\mathcal{C}^{an}$ on the other hand is topologized as an inductive limit (union)

$$\mathcal{C}^{an} = \lim_{\rightarrow} \prod_{U \in \mathcal{U}} \mathcal{O}(U), \tag{2.20}$$

where the limit is over all the finite coverings $\mathcal{U}$ of $\mathcal{A}$ by disjoint unions of closed disks, and for a closed disk U, $\mathcal{O}(U)$ is given the usual Banach (sup norm) topology.

Pairing. We give $\mathcal{C}/\mathcal{R}$ the natural topology arising from the identification

$$\mathcal{C}/\mathcal{R} = \mathcal{C}^{an}/\mathbb{C}_p. \tag{2.21}$$

If $\mathcal{T}'$ is a finite subtree of $\mathcal{T}$ we denote by $Ends(\mathcal{T}')$ the collection of oriented edges (u, v) such that $u \in \mathcal{T}'$ but $v \notin \mathcal{T}'$. If $\varphi \in \mathcal{C}$ and $\omega \in \Omega$ we define a pairing

$$\langle \varphi, \omega \rangle = \lim_{\mathcal{T}'} \sum_{e \in Ends(\mathcal{T}')} res_e(\varphi\omega). \tag{2.22}$$

The limit means that we take $\mathcal{T}'$ large enough. The sum then makes sense and is independent of $\mathcal{T}'$.

Theorem 2.2. *(Morita duality) The above pairing induces a perfect pairing of topological vector spaces*

$$\mathcal{C}/\mathcal{R} \times \Omega \to \mathbb{C}_p. \tag{2.23}$$

In other words, this pairing identifies each of the two spaces (algebraically) as the space of all continuous linear functionals on the other.

In fact, more is true. Since $\mathcal{A}$ is a compact set, $\mathcal{C}^{an}$, hence $\mathcal{C}/\mathcal{R}$, is a topological vector space *of compact type* (an inductive limit of Banach spaces under inclusion maps which are compact). It is complete and reflexive, and its strong dual

$$(\mathcal{C}^{an})'_b = \lim_{\leftarrow} \left(\prod_{U \in \mathcal{U}} \mathcal{O}(U) \right)' \tag{2.24}$$

is a Fréchet space. The above pairing identifies Ω with $(\mathcal{C}/\mathcal{R})'_b$ and $\mathcal{C}/\mathcal{R}$ with Ω'_b.

Theorem 2.3. *The annihilator of the exact forms $d\mathcal{O}$ under the above pairing is the closed space $\mathcal{C}^\infty/\mathbb{C}_p$ of locally contstant functions on $\mathcal{A}$ modulo constants, so one obtains a duality*

$$\mathcal{C}^\infty/\mathbb{C}_p \times H^1_{dR}(\mathfrak{X}) \to \mathbb{C}_p. \tag{2.25}$$

In fact, if φ is locally constant we have

$$\begin{aligned} \langle \varphi, \omega \rangle &= \lim_{\mathcal{T}'} \sum_{e \in \mathrm{Ends}(\mathcal{T}')} \varphi|_{\mathcal{A}_e} res_e(\omega) \\ &= \int_{\mathcal{A}} \varphi d\mu_\omega. \end{aligned} \tag{2.26}$$

Here μ_ω is the distribution defined by

$$\mu_\omega(\mathcal{A}_e) = c_\omega(e) = res_e(\omega). \tag{2.27}$$

2.3. **Morita duality with values in M, and the filtrations.** Now let $M = M_n$ as before. We introduce the filtration $\Phi^k = \Phi^k\mathcal{C}$ on $\mathcal{C}(M) = \mathcal{C} \otimes_K M$, as follows:

$$\mathcal{C}(M) = \Phi^0 \supset \Phi^1 \supset \cdots \supset \Phi^n \supset \{0\} = \Phi^{n+1} \tag{2.28}$$

where

$$\Phi^k = Span_{\mathcal{C}} \left\{ (u - \zeta v)^k u^{n-k-l} v^l;\ 0 \le l \le n-k \right\}. \tag{2.29}$$

Then:

(1) The filtration is G-stable. In fact, if

$$\gamma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.30}$$

and $\varphi = (u - \zeta v)^k u^{n-k-l} v^l$ then

$$(\gamma\varphi)(\zeta) = \gamma(\varphi(\gamma^{-1}\zeta)) = (c\zeta + d)^{-k} (u - \zeta v)^k (au + bv)^{n-k-l} (cu + dv)^l \in \Phi^k. \tag{2.31}$$

Notice that we could not make this definition if we let $\mathcal{C}$ stand for $\mathcal{C}^{an}$ because, contrary to the situation with $\mathcal{O}(\mathfrak{X})$, $c\zeta + d$ may vanish on $\mathcal{A}$.

(2) $\Phi^k = \mathcal{C}\cdot(u - \zeta v)^k v^{n-k} \oplus \Phi^{k+1}$ (as $\mathcal{C}$-modules, but not as G-modules).

(3) $\mathcal{C}(M)$ is free over $\mathcal{C}$ with basis $(u - \zeta v)^k v^{n-k}$ $(0 \le k \le n)$.

(4) For any $m \in \mathbb{Z}$ let $\mathcal{C}(m)$ be the ring $\mathcal{C}$ with the *twisted* G-action

$$\gamma \cdot \varphi = (c\zeta + d)^m \left(\varphi \circ \gamma^{-1}\right), \tag{2.32}$$

where (c, d) is the bottom row of γ^{-1}. Then $\varphi \mapsto \varphi(\zeta)(u - \zeta v)^k v^{n-k} \mod \Phi^{k+1}$ is an isomorphism

$$\Psi_k : \mathcal{C}(n - 2k) \simeq \Phi^k/\Phi^{k+1}. \tag{2.33}$$

In a completely analogous way we introduce the filtration $\Phi^k\mathcal{R}$

$$\Phi^k\mathcal{R} = \mathcal{R} \cap \Phi^k\mathcal{C} \tag{2.34}$$

and deduce that

$$\Phi^k(\mathcal{C}/\mathcal{R}) = \Phi^k\mathcal{C}/\Phi^k\mathcal{R} \tag{2.35}$$

is a filtration on $\mathcal{C}/\mathcal{R}$.

Morita duality with values in M.

Consider the perfect pairing

$$(.,.) : M_n \times M_n \to K \tag{2.36}$$

given by $Symm^n(\det)$. Explicitly, fix an identification det of $\bigwedge^2 V^\vee$ with K. For $x_i, y_i \in V^\vee$ the pairing

$$(x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{i=1}^{n} \det(x_{\sigma(i)} \wedge y_i) \tag{2.37}$$

factors through the symmetric power. In particular if $\det(u \wedge v) = 1$ then

$$\left(u^i v^{n-i}, u^{n-i} v^i\right) = \frac{i!(n-i)!}{n!}(-1)^{n-i} \tag{2.38}$$

and $\left(u^i v^{n-i}, u^{i'} v^{n-i'}\right) = 0$ if $i' \neq n - i$. Note that this pairing is symmetric for even n and alternating for odd n.

If $\varphi \in \mathcal{C}(M)$ and $\omega \in \Omega(M)$ we let (φ, ω) be the $\mathbb{C}_p$-valued function obtained from the above pairing of M with itself (defined in the complement of some large enough affinoid in $\mathfrak{X}$), and

$$\langle \varphi, \omega \rangle = \lim_{\mathcal{T}'} \sum_{e \in Ends(\mathcal{T}')} res_e((\varphi, \omega)). \tag{2.39}$$

The usual Morita duality implies that this is a perfect pairing between $\mathcal{C}(M)/\mathcal{R}(M)$ and $\Omega(M)$ with values in $\mathbb{C}_p$. Of course,

$$\mathcal{C}(M)/\mathcal{R}(M) \simeq \mathcal{C}^{an}(M)/\mathbb{C}_p \otimes_K M. \tag{2.40}$$

We now study the filtrations in this pairing.

Lemma 2.4. *Φ^k and $\Omega \otimes F^{n+1-k}$ are orthogonal to each other under the pairing of $\mathcal{C}/\mathcal{R}(M)$ and $\Omega(M)$.*

Proof. This is a computation that reduces to the fact that for $0 \leq d \leq k - 1$

$$\sum_{m=0}^{k} \frac{m^d}{m!(k-m)!}(-1)^m = 0. \tag{2.41}$$

To prove this well-known identity, differentiate d times the function $(1 - e^x)^k$ and evaluate at $x = 0$. ∎

Corollary 2.5. *The isomorphisms Θ'_{n-k} and Ψ_k induce a* duality

$$(\mathcal{C}/\mathcal{R})\,(n-2k)\times\Omega(2k-n)\to\mathbb{C}_p \tag{2.42}$$

identifying each side with the strong dual of the other side. This is nothing else than the pairing

$$\Phi^k(\mathcal{C}/\mathcal{R})/\Phi^{k+1}(\mathcal{C}/\mathcal{R})\times F^{n-k}\Omega/F^{n-k+1}\Omega\to\mathbb{C}_p \tag{2.43}$$

induced from the lemma (where we have written $F^{n-k}\Omega$ for $\Omega\otimes F^{n-k}$).

Corollary 2.6. *$\Phi^k\,(\mathcal{C}/\mathcal{R})$ and $F^{n+1-k}\Omega$ are* exact *annihilators of each other.*

Another interpretation. It is often desirable to have an interpretation for $\mathcal{C}/\mathcal{R}(m)$ as we had when $m=0$:

$$\mathcal{C}/\mathcal{R}=\mathcal{C}^{an}/\mathbb{C}_p. \tag{2.44}$$

Note that $(m\ge 0)$

$$\mathcal{C}/\mathcal{R}=\mathcal{C}^{an}[m\infty]/P_m \tag{2.45}$$

where $\mathcal{C}^{an}[m\infty]$ denotes the functions that are locally analytic, *except* near ∞ where

$$ord_\infty\varphi+m\ge 0, \tag{2.46}$$

and

$$P_m=\mathcal{C}^{an}[m\infty]\cap\mathcal{R}. \tag{2.47}$$

This means that at ∞ the functions are allowed to have a polynomial part of degree at most m, and P_m is of course the polynomials of degree at most m. If $m<0$, then $P_m=0$, and $\mathcal{C}^{an}[m\infty]$ are functions vanishing to order $|m|$ at infinity. The point is that these spaces are invariant under the mth twisted action of G : if φ is of this sort,

$$(c\zeta+d)^m\varphi\left(\frac{a\zeta+b}{c\zeta+d}\right) \tag{2.48}$$

is also there: you only have to check zeroes and poles at $\zeta=\infty$ and $\zeta=-d/c$.

In particular we get, when $k=0$ and $n\ge 0$

$$\Omega(-n)_b'\simeq\left(\mathcal{C}^{an}[n\infty]/P_n\right)(n) \tag{2.49}$$

(the twist reminding us of the action of G). This is [D-T], Theorem 2.2.1 (their k is our $n+2$, and their $\mathcal{O}(k)$ is our $\mathcal{O}(-k)$).

The subspace $d\mathcal{O}(M)$ is closed in $\Omega(M)$. We have seen that its annihilator is the space $\mathcal{C}^\infty(M)/\mathbb{C}_p\otimes_K M$, which is therefore dual to $H^1_{dR}(\mathfrak{X};M)$, or to $C^1_{har}(M)$. Under projection of $\mathcal{C}/\mathcal{R}\otimes M$ modulo Φ^1 onto $\mathcal{C}/\mathcal{R}(n)=\left(\mathcal{C}^{an}[n\infty]/P_n\right)(n)$, the space $\mathcal{C}^\infty(M)/\mathbb{C}_p\otimes_K M$ gets mapped isomorphically onto

$$\mathcal{C}^\infty(M)/\mathbb{C}_p\otimes_K M\simeq\left(\mathcal{C}^{pol,n}[n\infty]/P_n\right)(n). \tag{2.50}$$

Here $\mathcal{C}^{pol,n}$ is the space of locally-polynomial-of-degree-n functions on $\mathcal{A}$ and the $[n\infty]$ is there only to remind us that they are locally analytic everywhere except at ∞ where they have a pole of order n. The notation (n) refers to the G action. The map can be computed. It sends $\chi u^i v^{n-i}$ for a locally constant function χ to $\chi\zeta^i$.

The (strong) dual of the exact sequence

$$0\to\mathcal{O}(n)/P_n(n)\to F^n\Omega\to H^1_{dR}(\mathfrak{X};M)\to 0 \tag{2.51}$$

in which the first arrow takes f to $\frac{1}{n!}f^{(n+1)}(z)(u-zv)^n dz$, is the exact sequence
(2.52)

$$0 \leftarrow \mathcal{C}^{an}[(-n-2)\infty](-n-2) \leftarrow (\mathcal{C}^{an}[n\infty]/P_n)\,(n) \leftarrow \left(\mathcal{C}^{pol,n}[n\infty]/P_n\right)(n) \leftarrow 0$$

where the first (backward) arrow is $\frac{(-1)^n}{n!}\left(\frac{d}{d\zeta}\right)^{n+1}$. Note that this operator kills the principal (polynomial) part at ∞ and increases the order of the zero at infinity to $n+2$.

Summary. On $\mathfrak{X}$ we have
(2.53)

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}(n)/P_n(n) & \stackrel{\frac{1}{n!}\left(\frac{d}{dz}\right)^{n+1}}{\to} & \mathcal{O}(-n-2)=F^n\Omega & \to & H^1_{dR}(\mathfrak{X};M_n) & \to & 0 \\ & & \cap & & \cap & & || & & \\ 0 & \to & d\mathcal{O}(M_n) & \to & \Omega(M_n) & \to & C^1_{har}(M_n) & \to & 0 \end{array}$$

The strong dual of this diagram is
(2.54)

$$\begin{array}{ccccc} \mathcal{C}^{an}[(-n-2)\infty](-n-2) & \twoheadleftarrow & (\mathcal{C}^{an}[n\infty]/P_n)\,(n) & \hookleftarrow & \left(\mathcal{C}^{pol,n}[n\infty]/P_n\right)(n) \\ \uparrow & & \uparrow\ \mathrm{mod}\ \Phi^1 & & ||\ \mathrm{mod}\ \Phi^1 \\ \mathcal{C}^{an}(M_n)/\mathcal{C}^{\infty}(M_n) & \twoheadleftarrow & \mathcal{C}/\mathcal{R}(M_n) & \hookleftarrow & \mathcal{C}^{\infty}(M_n)/\mathbb{C}_p\otimes_K M_n \end{array}$$

The maps I and J (= the Poisson kernel) in [D-T] are the duality isomorphisms

$$\begin{array}{cccc} (I_k) & I & : & \mathcal{O}(-n-2)'_b \simeq (\mathcal{C}^{an}[n\infty]/P_n)\,(n) \\ & \lambda & \mapsto & I(\lambda)(\zeta)=\lambda(\frac{1}{z-\zeta}) \end{array} \tag{2.55}$$

and

$$\begin{array}{cccc} ({}^tI_k & = & J_k) & J:(\mathcal{C}^{an}[n\infty]/P_n)\,(n)'_b \simeq \mathcal{O}(-n-2) \\ \mu & \mapsto & \displaystyle\int_{\mathbb{P}^1(K)} \frac{1}{z-\zeta}d\mu(\zeta). & \end{array} \tag{2.56}$$

3. Bounded forms and bounded cohomology

3.1. **Bounded differential forms.** Let $\mathbf{M}$ be an integral structure on M (as defined in the introduction).

Definition 3.1. *$f\in\mathcal{O}(M)$ is $\mathbf{M}$-integral if for any $\sigma\in\mathcal{T}$ (vertex or an open edge), $f(z)\in\mathbf{M}(\sigma)\otimes_{\mathcal{O}_K}\mathcal{O}_{\mathbb{C}_p}$ for all $z\in\mathfrak{X}_\sigma=r^{-1}(|\sigma|)$, where $r:\mathfrak{X}\to|\mathcal{T}|$ is the reduction map.*

Definition 3.2. *(i) The* canonical integral structure $\mathbf{\Omega}$ *on $\underline{\Omega}$ is the $\mathcal{O}_{\mathbb{C}_p}$-subsheaf of all the sections ω such that the image of ω in the fiber at any $z_0\in\mathfrak{X}$, denoted $\omega|_{z_0}$, belongs to the $\mathcal{O}_{\mathbb{C}_p}$-span of $d\log(z-\zeta)|z_0$, for all $\zeta\in K$. (Note that this submodule is independent of the coordinate z.)*

(ii) Call $\omega\in\Omega(M)$ $\mathbf{M}$-integral if for any σ and any $z_0\in\mathfrak{X}_\sigma$

$$\omega|_{z_0}\in\mathbf{\Omega}|_{z_0}\otimes_{\mathcal{O}_K}\mathbf{M}(\sigma). \tag{3.1}$$

(iii) As before, bounded *means* integral w.r.t. some integral structure.

Proposition 3.1. *The residue map $\omega\mapsto c_\omega$ maps a bounded M-valued differential form to a bounded M-valued harmonic cochain.*

$$\Omega(M)^{bnd}\to C^1_{har}(M)^{bnd}. \tag{3.2}$$

Proof. Let $\mathbf{M}$ be an integral structure such that ω is $\mathbf{M}$-integral. Fix a vertex v and let $\mathfrak{X}(v)$ be the preimage, under the reduction map r, of the vertex v and the $q+1$ edges $e_0, \dots, e_q$ starting at v. We label them so that e_0 is the unbounded edge (i.e. $\infty \in \mathcal{A}_{e_0}$). Let

$$\rho = dist(z, \mathbb{P}^1_K) \tag{3.3}$$

be the distance from any $z \in \mathfrak{X}_v = r^{-1}(v)$ to the boundary of $\mathfrak{X}$. Let $\zeta_i \in \mathcal{A}_{e_i}$ and $\zeta_0 \in \mathcal{A}_{\bar{e}_0}$. Then the Mittag-Leffler decomposition of ω in $\mathfrak{X}(v)$ is

$$\omega = \sum_{i=0}^{q} f_{e_i}(z)dz \tag{3.4}$$

where for $1 \le i \le q$

$$f_{e_i} = \sum_{\nu=1}^{\infty} a_{i,\nu}(z-\zeta_i)^{-\nu} \tag{3.5}$$

and

$$f_{e_0} = \sum_{\nu=0}^{\infty} a_{0,\nu}(z-\zeta_0)^{\nu}. \tag{3.6}$$

For $z \in \mathfrak{X}_v$ and $1 \le j \le q$, $|z-\zeta_j| = \rho$ so $(z-\zeta_j)^{-1}dz$ is a basis of the fiber $\mathbf{\Omega}|_z$ over $\mathcal{O}_{\mathbb{C}_p}$.

Let l be any linear functional $\mathbf{M}(v) \to \mathcal{O}_K$. For any $z \in \mathfrak{X}_v$ $l(\omega) \in \mathbf{\Omega}|_z$ and therefore

$$\left| \sum_{i=0}^{q} l(f_{e_i}(z)) \right| \le \rho^{-1}. \tag{3.7}$$

By the Mittag-Leffler theorem each $|l(f_{e_i}(z))| \le \rho^{-1}$ thoughout $\mathfrak{X}_v$, and by the equality of the Gauss norm and the sup norm, for every $1 \le i \le q$, $\nu \ge 1$

$$|l(a_{i,\nu})| \le \rho^{\nu-1}$$

and in particular $|l(a_{i,1})| \le 1$ for every l, so $a_{i,1} \in \mathcal{O}_{\mathbb{C}_p} \otimes_{\mathcal{O}_K} \mathbf{M}(v)$. But these are the residues of ω along e_i, and the residue along e_0 is, up to a sign, the sum of them. It follows that $c_\omega(e_i) \in \mathbf{M}(v) \subset \mathbf{M}(e_i)$ for every i. Since v was arbitrary, c_ω is $\mathbf{M}$-integral. ∎

Theorem 3.2. *The residue homomorphism induces an isomorphism*

$$\Omega(M_n)^{bnd} \cap F^n\Omega \simeq C^1_{har}(M_n)^{bnd}. \tag{3.8}$$

We begin by checking the injectivity. Let

$$\omega = f(z)(u-zv)^n dz \tag{3.9}$$

be a form in $F^n\Omega$. To proceed we need to introduce some notation. For every vertex v let ρ_v be the diameter of the set $\mathfrak{X}_v$, which is also the distance from it to the boundary of $\mathfrak{X}$. Let $\mathcal{E}(v)$ be the set of the $q+1$ oriented edges starting at v. If e is an edge and $\mathfrak{X}_e = r^{-1}(|e|)$ is the corresponding annulus, then its complement consists of two closed disks, and we denote by ρ_e the diameter of the disk which is bounded, and ζ_e a K-rational point in it, chosen at random. If $e \in \mathcal{E}(v)$ then $\rho_e = |\pi|\rho_v$ if e is bounded, and $\rho_e = \rho_v$ if e is unbounded.

Lemma 3.3. *There exist functions f_e, one for each oriented edge e, with the following properties:*

(i) If e is bounded, f_e is defined in the complement of $\{z|\,|z-\zeta_e|\le\rho_e\}$ by a convergent Laurent series without constant term

$$f_e=\sum_{\nu=1}^{\infty}a_{e,\nu}(z-\zeta_e)^{-\nu}. \tag{3.10}$$

If e is unbounded, f_e is defined in $\left\{z|\,|z-\zeta_e|<|\pi|^{-1}\rho_e\right\}$ by a convergent Taylor series

$$f_e=\sum_{\nu=0}^{\infty}a_{e,\nu}(z-\zeta_e)^{\nu}. \tag{3.11}$$

(ii) If $\mathcal{T}'$ is a finite connected subtree of $\mathcal{T}$ and $\mathcal{E}(\mathcal{T}')$ is the set of edges not in $\mathcal{T}'$ but adjacent to it, oriented away from $\mathcal{T}'$, then if $\mathfrak{X}(\mathcal{T}')=r^{-1}(|\mathcal{T}'|)$ is the affinoid reducing to $\mathcal{T}'$, we have the Mittag-Leffler decomposition

$$f|_{\mathfrak{X}(\mathcal{T}')}=\sum_{e\in\mathcal{E}(\mathcal{T}')}f_e. \tag{3.12}$$

(iii) For every bounded edge e

$$res_e\omega=res_e f_e(z)(u-zv)^n dz. \tag{3.13}$$

Proof. For a given finite subtree $\mathcal{T}'$, the existence of a collection of functions as in (i), indexed by $e\in\mathcal{E}(\mathcal{T}')$, satisfying (ii), is guaranteed by the Mittag-Leffler decomposition for the affinoid subdomain $\mathfrak{X}(\mathcal{T}')$. Let us employ temporarily the notation $f_{e,\mathcal{T}'}$ $(e\in\mathcal{E}(\mathcal{T}'))$ for this collection, to emphasize the dependence on $\mathcal{T}'$. We claim that the f_e do not depend on $\mathcal{T}'$. It is enough to consider the case where $\mathcal{T}''$ is obtained from $\mathcal{T}'$ by adding a vertex u and an edge $e_0=(v,u)$ for somed $v\in\mathcal{T}'$. Let $e_1,\ldots,e_q$ be the edges starting at u different from $\bar{e}_0$. Define

$$f_{e_0,\mathcal{T}'}=\sum_{i=1}^{q}f_{e_i,\mathcal{T}''} \tag{3.14}$$

and $f_{e,\mathcal{T}'}=f_{e,\mathcal{T}''}$ for every $e\in\mathcal{E}(\mathcal{T}')\cap\mathcal{E}(\mathcal{T}'')$. If e_0 is bounded, then so are all the e_i $(1\le i\le q)$, and it is readily checked that $f_{e_0,\mathcal{T}'}$ is holomorphic in the complement of the bounded disk defined by e_0, and vanishes at infinity, so is given by a Laurent series without constant term. If e_0 is unbounded, then precisely one of the e_i is unbounded, and again it follows that $f_{e_0,\mathcal{T}'}$ is holomorphic in the bounded disk defined by e_0 and given by a convergent Taylor series there. Finally the fact that $f|_{\mathfrak{X}(\mathcal{T}')}=\sum_{e\in\mathcal{E}(\mathcal{T}')}f_{e,\mathcal{T}'}$ follows from the analogous decomposition for $\mathcal{T}''$ by restricting the domain.

To prove (iii), look at the Mittag-Leffler decomposition in $\mathfrak{X}(v)$ where v is the origin of e, and note that all the other $f_{e'}$ appearing in that decomposition extend holomorphically accross $\{z||z-\zeta_e|\le\rho_e\}$ so do not contribute to the residue. ∎

Lemma 3.4. *Let $||f_e||$ denote the sup norm of f_e in $\left\{z||z-\zeta_e|\ge|\pi^{-1}|\rho_e\right\}$ if e is bounded, and in $\{z||z-\zeta_e|\le\rho_e\}$ if e is unbounded. Note that*

$$||f_e||=||f_e||_{\mathfrak{X}_v} \tag{3.15}$$

if v is the origin of e. Assume that ω is bounded and that all its residues vanish. Then

$$||f_e||\rho_e^{\frac{n+2}{2}} \tag{3.16}$$

are bounded as e runs over the oriented edges of the tree.

Proof. The vanishing of the residues of ω is equivalent to the vanishing of the residues along every e of $z^i f(z)dz$ for $0 \leq i \leq n$. If e is bounded this simply means that the Laurent series giving f_e starts in degree $-n-2$, and has no terms in higher degrees, or equivalently that it vanishes at infinity to order which is at least $n+2$. Let v_0 be the standard vertex of $\mathcal{T}$ and pick an arbitrary vertex $v = \gamma(v_0)$. Since ω is bounded there exists an $\mathcal{O}_K$-lattice M_0 in M independent of v or γ such that for $w \in \mathfrak{X}_v$

$$\omega|_w \in \mathbf{\Omega}|_w \otimes_{\mathcal{O}_K} \gamma(M_0). \tag{3.17}$$

Write $w = \gamma(z)$ for $z \in \mathfrak{X}_{v_0}$ and note that $\mathbf{\Omega}|_w = (\gamma^{-1})^* \mathbf{\Omega}|_z$. It follows that

$$\begin{aligned} (\gamma^{-1}\omega)|_z &= (1 \otimes \gamma^{-1})\gamma^*(\omega|_w) \\ &\in \mathbf{\Omega}|_z \otimes_{\mathcal{O}_K} M_0. \end{aligned} \tag{3.18}$$

Let us invoke the Mittag-Leffler decomposition of ω at v :

$$\omega|_{\mathfrak{X}_v} = \sum_{e \in \mathcal{E}(v)} f_e(z)(u - zv)^n dz. \tag{3.19}$$

Then

$$\gamma^{-1}\omega|_{\mathfrak{X}_{v_0}} = \sum_{e \in \mathcal{E}(v_0)} f_{\gamma e}(\gamma z)(cz + d)^{-n-2}(u - zv)^n dz \tag{3.20}$$

where (c, d) is the bottom row of γ. The key to the lemma is the observation that since all the residues of $z^i f(z)dz$ for $0 \leq i \leq n$ vanish, this last sum *is* the Mittag Leffler decomposition of $\gamma^{-1}\omega$ at v_0. Indeed, let us distinguish three types of $e \in \mathcal{E}(v_0)$. If γe and e are both bounded, then

$$g_e(z) = f_{\gamma e}(\gamma z)(cz + d)^{-n-2} \tag{3.21}$$

is holomorphic everywhere in $\{z||z - \zeta_e| > \rho_e\}$, including the questionable point $-d/c$, since $f_{\gamma e}$ vanishes at infinity to order $\geq n + 2$. It also clearly vanishes at infinity (in fact, to order $\geq n + 2$). If γe is bounded but e is unbounded, the same analysis applies, noting that γ carries the bounded disk $\{z||z - \zeta_e| \leq \rho_e\}$ onto the unbounded disk $\left\{w||w - \zeta_{\gamma e}| \geq |\pi^{-1}|\rho_{\gamma e}\right\}$. If γe is unbounded then $-d/c$ does not belong to the domain of definition of g_e (whether bounded or not) so again g_e is holomorphic there. It follows that g_e satisfy the conditions characterizing the Mittag-Leffler decomposition of $\gamma^{-1}\omega$ at v_0, so by the uniqueness of the decomposition,

$$\gamma^{-1}\omega = \sum_{e \in \mathcal{E}(v_0)} g_e(z)(u - zv)^n dz \tag{3.22}$$

is the Mittag-Leffler decomposition of $\gamma^{-1}\omega$ at v_0.

Now we have seen above that the boundedness of ω is translated to the uniform boundedness of all the $\gamma^{-1}\omega$ (in the usual sense) on $\mathfrak{X}_{v_0}$. This means that $||g_e|| = ||g_e||_{\mathfrak{X}_{v_0}}$ are uniformly bounded (for every γ). However, if ζ is a point in one of the bounded disks in the complement of $\mathfrak{X}_{v_0}$, for which $\gamma(\zeta)$ also lies in one of the bounded disks in the complement of $\mathfrak{X}_v$ ($v = \gamma(v_0)$), then

$$\gamma(z) - \gamma(\zeta) = \frac{z-\zeta}{(cz+d)(c\zeta+d)} \tag{3.23}$$

and

$$|c\zeta + d| = |c||\zeta - (-d/c)| = |c||z - (-d/c)| = |cz+d| \tag{3.24}$$

(note that ζ and $-d/c$ do not lie in the same disk, as the latter is mapped to ∞ by γ and the first to a bounded disk). This computation shows that

$$|cz+d|^2 = \rho_{v_0}/\rho_v. \tag{3.25}$$

It follows that

$$||g_e|| = ||f_{\gamma e}||(\rho_v/\rho_{v_0})^{\frac{n+2}{2}} \tag{3.26}$$

and the lemma follows because ρ_v and $\rho_{\gamma e}$ at most differ by a factor of $|\pi|$. ∎

Proposition 3.5. *A bounded form in $F^n\Omega$, all of whose residues vanish, is identically 0. In other words, the map*

$$\Omega(M_n)^{bnd} \cap F^n\Omega \to C^1_{har}(M_n)^{bnd} \tag{3.27}$$

is injective.

Proof. (compare [dS], (3.9.5) where the same proof was applied to Γ-invariant forms for a discrete cocompact subgroup Γ). Let ω be as in the proposition. From the last lemma we know that there exists a constant R such that

$$||f_e|| \left(|\pi^{-1}|\rho_e\right)^{\frac{n+2}{2}} \leq R \tag{3.28}$$

for every e. Fix an affinoid $\mathcal{K} = \mathfrak{X}(\mathcal{T}')$ for a finite connected subtree $\mathcal{T}'$ and let $0 < \delta < 1$ be small enough so that if $e \in \mathcal{E}(\mathcal{T}')$ is bounded, $\delta < \rho_e$ and if e is unbounded, $\rho_e < \delta^{-1}$. Fix a second affinoid $\mathcal{K}' = \mathfrak{X}(\mathcal{T}'')$ containing $\mathcal{K}$ so that its ends satisfy the same estimates on ρ_e with δ replaced by $|\pi|\delta^4$. Let $z \in \mathcal{K}$ and consider the Mittag-Leffler decomposition which corresponds to $\mathcal{K}'$:

$$\omega = \sum_{e\in\mathcal{E}(\mathcal{T}'')} f_e(z)(u-zv)^n dz. \tag{3.29}$$

If e is bounded, $|z - \zeta_e| > \delta$ since $z \in \mathcal{K}$, so

$$\begin{aligned} \left|a_{e,\nu}(z-\zeta_e)^{-\nu}\right| &\leq \delta^{-\nu}||f_e||(|\pi^{-1}|\rho_e)^\nu \leq \delta^{-\nu}R(|\pi^{-1}|\rho_e)^{\nu-\frac{n+2}{2}} \\ &\leq R\delta^{3\nu-2(n+2)} \leq R\delta^{n+2}. \end{aligned} \tag{3.30}$$

In the last step we used the fact that only $\nu \geq n+2$ count. On the other hand, if e is the unbounded end, and $\nu \geq 0$,

$$\begin{aligned} |a_{e,\nu}(z-\zeta_e)^\nu| &\leq ||f_e|| \leq R\left(|\pi^{-1}|\rho_e\right)^{-\frac{n+2}{2}} \\ &\leq R|\pi|^{n+2}\delta^{2(n+2)}. \end{aligned} \tag{3.31}$$

In any case, letting $\delta \to 0$, we see that f vanishes on $\mathcal{K}$. Since $\mathcal{K}$ was arbitrary, $\omega = 0$. ∎

3.2. **The theorem of Amice-Velu-Vishik.** Next we deal with surjectivity.

Proposition 3.6. *Every $c \in C^1_{har}(M_n)^{bnd}$ is obtained as c_ω for an $\omega \in \Omega(M_n)^{bnd} \cap F^n\Omega$.*

We follow the sketch of [D-T], 2.3.2, fill in some details, but omit verifications of convergence which may be found at [MTT] and [C].

Let $c \in C^1_{har}(M_n)^{bnd}$ and let

$$\lambda_c \in \left(C^{pol,n}[n\infty]/P_n\right)(n)' \tag{3.32}$$

be the corresponding linear functional defined by

$$\lambda_c(\zeta^i \chi_{U(\varepsilon)}) = \left(c(\varepsilon), u^i v^{n-i}\right) \tag{3.33}$$

for $0 \le i \le n$, ε an oriented edge of $\mathcal{T}$ and $U(\varepsilon)$ the subset of $\mathbb{P}^1(K) = Ends(\mathcal{T})$ to which ε "is pointing" (denoted $\mathcal{A}_\varepsilon$ before). Here χ_U is the characteristic function of U. Let

$$g = \begin{pmatrix} \pi^{-m} & -\pi^{-m}a \\ 0 & \pi^m \end{pmatrix} \in G = SL_2(K) \tag{3.34}$$

so that $g : D(a, |\pi^{2m}|) \simeq D(0,1)$. Then in the G action on $\left(C^{pol,n}[n\infty]/P_n\right)(n)$

$$g^{-1}\left(\zeta^j \chi_{D(0,1)}\right) = \pi^{mn} g(\zeta)^j \chi_{D(a,|\pi^{2m}|)} \tag{3.35}$$

so

$$\begin{aligned} & \lambda_c\left((\zeta - a)^j \chi_{D(a,|\pi^{2m}|)}\right) \\ = \; & \lambda_c\left(\pi^{(2j-n)m} g^{-1}\left(\zeta^j \chi_{D(0,1)}\right)\right) \\ = \; & \pi^{(2j-n)m} g(\lambda_c)\left(\zeta^j \chi_{D(0,1)}\right) \\ = \; & \pi^{(2j-n)m} \left\langle c(g^{-1}\varepsilon_0), g^{-1}(u^j v^{n-j})\right\rangle \end{aligned} \tag{3.36}$$

where ε_0 is the edge corrresponding to the standard annulus such that $U(\varepsilon_0) = D(0,1) \cap K$. Now by boundedness, and by the G-equivariance of the pairing on M_n, the expression in the brackets belongs to some lattice which is independent of g. We conclude that as long as $2j > n$ the whole thing tends to 0 as $m \to \infty$.

A similar computation holds in the unbounded disk: take

$$g = \begin{pmatrix} 0 & -\pi^{-m} \\ \pi^m & 0 \end{pmatrix} \tag{3.37}$$

so that $g : D(\infty, |\pi^{-2m}|) \simeq D(0,1)$ and

$$g^{-1}\left(\zeta^j \chi_{D(0,1)}\right) = (-1)^j \pi^{(n-2j)m} \zeta^{n-j} \chi_{D(\infty,|\pi^{-2m}|)} \tag{3.38}$$

and

$$\begin{aligned} & \lambda_c\left(\zeta^{n-j} \chi_{D(\infty,|\pi^{-2m}|)}\right) \\ = \; & (-1)^j \pi^{(2j-n)m} \left\langle c(g^{-1}\varepsilon_0), g^{-1}(u^j v^{n-j})\right\rangle \end{aligned} \tag{3.39}$$

and the same conclusion holds. Note that this time it is ζ^{n-j} for $2j > n$ whose integrals over smaller and smaller neighborhoods of ∞ tend to 0.

As a corollary of the computations above we can make the following definition. Let $\varphi \in (C^{an}[n\infty]/P_n)(n)$ and c as above. Extend λ_c to φ by taking a large subtree $\mathcal{T}'$, finding for any $\varepsilon \in Ends(\mathcal{T}')$ a Taylor (resp. MacLaurin) expansion of φ in

$U(\varepsilon)$ centered at some $\xi_\varepsilon \in U(\varepsilon)$ (resp. at ∞ if $U(\varepsilon)$ is an unbounded disk), letting $T^n_{\xi_\varepsilon}(\varphi)$ be the *truncation* of this expansion at degrees $\leq n$ (resp. degrees ≥ 0) and setting

$$\tilde{\lambda}_c(\varphi) = \lim_{\mathcal{T}'} \sum_{\varepsilon} \left\langle c(\varepsilon), T^n_{\xi_\varepsilon}(\varphi) \right\rangle \tag{3.40}$$

where you identify $P_n(n)$ with M_n via $\zeta^i \mapsto u^i v^{n-i}$.

Theorem 3.7. *The limit exists and is independent of the choices involved. Furthermore if we put*

$$\omega_c = \left(\int_{\mathbb{P}^1(K)} \frac{d\tilde{\lambda}_c(\zeta)}{z - \zeta} \right) (u - zv)^n dz \tag{3.41}$$

then the Poisson integral (which is defined for every z pointwise since the integrand is locally analytic in ζ) is a holomorphic function of z so $\omega_c \in F^n\Omega$ and it represents the functional $\tilde{\lambda}_c$ under Morita duality. Moreover, ω_c is bounded.

For the verification, based on the growth estimates on λ_c derived above, see [MTT], Section 11 (where the same estimates are used in the construction of p-adic L functions), or [C], Theorem 2.5 (where it is shown that λ_c extends to a distribution "of order n", and not just to locally analytic functions).

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