THETA OPERATORS ON UNITARY SHIMURA VARIETIES

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Abstract. We define a theta operator on p-adic vector-valued modular forms on unitary groups of arbitrary signature, over a quadratic imaginary field in which p is inert. We study its effect on Fourier-Jacobi expansions and prove that it extends holomorphically beyond the μ-ordinary locus, when applied to scalar-valued forms.

Introduction

The purpose of this article is to define a theta operator Θ for p-adic vector-valued modular forms on unitary Shimura varieties of arbitrary signature, and prove some fundamental results concerning it. For reasons explained below, we focus on the case where the prime p is inert in the quadratic imaginary field E associated with the Shimura variety. Specifically, we prove a formula for the action of Θ in terms of Fourier-Jacobi expansions (Theorem 3.2.5). We also prove that Θ extends to a holomorphic operator outside the μ-ordinary locus, when acting on scalar-valued modular forms in characteristic p (Theorems 4.2.3 and 4.3.2).

When the prime p is *split* in E, general points on the special fiber of the Shimura variety parametrize ordinary abelian varieties. A theta operator and a whole array of differential operators derived from it, were defined in this context in Eischen's thesis [Ei]. Her construction was generalized in [E-F-M-V] to unitary Shimura varieties associated with a general CM field, but still under the ordinariness assumption. In their work, these authors circumvent the study of Θ on Fourier-Jacobi expansions by expressing it in Serre-Tate coordinates at CM points.

"Ordinariness" is a strong assumption. Over the ordinary locus, it provides a unit-root splitting of the Hodge filtration in the cohomology of the universal abelian variety. This allows one to extend Katz's approach to Θ [Ka2]. The unit-root splitting serves as a p-adic replacement for the Hodge decomposition over the complex numbers, which underlies the construction of similar C^∞-differential operators of Ramanujan and Maass-Shimura [Sh].

In [dS-G1], we defined a Θ-operator on unitary modular forms of signature $(2, 1)$ and determined its effect on q-expansions, for *p inert* in the quadratic imaginary field E. The main obstacle in this case was that the abelian variety parametrized by a general point is not ordinary anymore, but so-called μ-ordinary, and its cohomology does not admit a unit-root splitting. Our approach there, taken also in the present paper, is to make systematic use of Igusa varieties; we first define the theta operator on them, and show that it descends to the Shimura variety.

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Recently, we have learned of the work of Ellen Eischen and Elena Mantovan [E-M] in which they construct the same differential operators in the μ-ordinary (p inert) case. Their method is closer to the original idea of Katz, but they replace the unit-root splitting by slope filtration splitting of F-crystals. Their construction is more general than ours, as it applies to unitary Shimura varieties associated with a general CM field. They apply their differential operators to the study of p-adic families of modular forms in the spirit of Serre, Katz and Hida. Their work should have applications to questions of over-convergence and to construction of p-adic L-functions. However, the issues addressed in the present paper, the effect of Θ on Fourier-Jacobi expansions and its holomorphic extension beyond the μ-ordinary locus, are not addressed there.

We now provide some background and motivation for the study undertaken in this paper. The theta operator for elliptic modular forms is related to an operator already defined by Ramanujan. In terms of classical modular forms for $SL_2(\mathbb{Z})$ the operator Θ is given on the level of q-expansions by

$$f = \sum_n a_n q^n \mapsto \Theta(f) = \sum_n n a_n q^n.$$

Viewed at the level of q-expansions over the complex numbers, this operator does not preserve the space of holomorphic modular forms. However, viewed at the level of q-expansions for p-adic, or mod p, modular forms, it does, at least when one has reasonable demands: in characteristic p one has to multiply $\Theta(f)$ by h, the Hasse invariant, which is a modular form of weight $p-1$ vanishing outside the ordinary locus; p-adically one has to be content with working merely over the ordinary locus.

These aspects were present from the very start in the work of Swinnerton-Dyer and Serre [Se1, Se2, Sw-D]. In fact, already in [Se1], motivated by relation to Galois representations, Serre investigates the notion of filtration of modular forms. The filtration of a q-expansion of a mod p modular form is the minimal weight in which one may find a modular form with that q-expansion; one is interested in its variation under applications of Θ, which at the level of Galois representations corresponds to a cyclotomic twist. Following closely on the heels of these developments, Katz gave a geometric construction of Θ on (essentially) all modular curves with good reduction at p in [Ka2].

Not much later, Jochnowitz [Joc] studied Θ-cycles. The basic idea is simple. If $g = \Theta(f)$ has filtration w_0, the series of filtrations w_i of $\Theta^i(g)$, $i = 0, 1, \ldots, p-1$, is a collection of weights that is generally increasing, but not always, because $w_{p-1} = w_0$. The question of the variation of the filtration along the cycles is interesting and has important applications. See [Gr, Joc]. Further deep uses of the Θ-operator to over-convergence and classicality of p-adic modular forms were given in [Col, C-G-J].

Shortly after [Ka2], Katz has studied in [Ka3] such an operator for Hilbert modular forms associated to a totally real field L, and in fact enriched the theory by introducing $g = [L : \mathbb{Q}]$ basic theta operators. In that work, as in the case of modular curves, strong use is made of the behaviour of de Rham cohomology and the unit root splitting over the ordinary locus. The study of these operators was further developed by Andreatta and the second author [A-G], who constructed mod p versions of them by means of the Igusa variety, and provided some results on filtrations, Θ-cycles and relations to cyclotomic twists.

It seemed a natural idea at that point to extend the theory of the theta operator to other Shimura varieties. However, two obstacles arise: (i) The general point of a Shimura variety in positive characteristic may not be ordinary anymore. In particular, its de Rham cohomology may not admit a unit root splitting. (ii) The natural definition takes modular forms, even if scalar-valued, to vector-valued modular forms.

Bearing in mind the Kodaira-Spencer isomorphism, which is involved in the definition of Θ, the second problem could be anticipated. In the Hilbert modular case, it is the abundance of endomorphisms that allows one to return to scalar-valued modular forms. In spite of these difficulties, progress has been made on other Shimura varieties: As Eischen had already remarked in her thesis, her construction generalizes almost immediately to the symplectic case. Panchishkin and Courtieu discussed similar operators for Siegel modular forms in [Co-Pa, Pan]. For different aspects in the symplectic case see the papers by Böcherer-Nagaoka [B-N] and Ghitza-McAndrew [G-M], and additional references therein. For other cases, see the work of Johansson [Joh].

The contents of this paper are as follows. Let E be a quadratic imaginary field, p a rational prime that is inert in E and $\kappa = \mathcal{O}_E/(p)$ its residue field. Let $n \geq m$ be positive integers. Fixing additional data, one obtains a scheme over $\mathcal{O}_E$ that parametrizes abelian schemes with $\mathcal{O}_E$-action of signature (n, m), endowed with polarization and level structure. Its complex points are a union of Shimura varieties in Deligne's sense associated to forms of the unitary group $GU(n, m)$. Let $\mathcal{S}$ denote its base change to the completion $\mathcal{O}_{E,p}$ and $S \to \mathrm{Spec}(\kappa)$ its special fiber; let S_s be the base change of $\mathcal{S}$ to $W_s = W_s(\kappa)$.

In §1 we collect background material and definitions, and in particular define the type of vector-valued p-adic modular forms that will be considered in this paper. Automorphic vector bundles over $\mathcal{S}$ correspond to representations of the group $GL_m \times GL_n$, and there are two "basic" vector bundles, $\mathcal{Q}$ and $\mathcal{P}$, corresponding to the standard representations of the two blocks, from which all others are derived. Characteristic p holds its own idiosyncrasies and there are 3 vector bundles, denoted $\mathcal{Q}, \mathcal{P}_0$ and $\mathcal{P}_\mu$, from which all *p-adic* automorphic vector bundles $\mathcal{E}_\rho$ are derived by representation-theoretic constructions; in particular, ρ refers here to a representation of $GL_m \times GL_m \times GL_{n-m}$. We briefly explain the origin of these vector bundles. The relative cotangent bundle of the universal abelian variety $\mathcal{A} \to \mathcal{S}$ decomposes according to signatures, providing us with vector bundles $\mathcal{P}, \mathcal{Q}$ of ranks n, m, respectively. Over the (μ-)ordinary locus S_s^{ord} of S_s, $\mathcal{P}$ has a filtration $0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0$, and this is where $\mathcal{E}_\rho$ "lives". A mod-p^s modular form of weight ρ is defined to be a section of $\mathcal{E}_\rho$ over S_s^{ord}. See §1 for details.

In §2 we define the Igusa tower over S_s^{ord} and study its properties. The key fact about the Igusa tower is that the vector bundles $\mathcal{P}_0, \mathcal{P}_\mu$ and $\mathcal{Q}$ are all *canonically trivialized* over it. To be precise, much as in Katz [Ka1], the Igusa tower is a double limit of schemes $\{T_{t,s} | t, s \geq 1\}$, where $T_{t,s}$ is a scheme over the truncated Witt vectors W_s of length s, and when $t \geq s$ the trivialization mentioned above is obtained. Consequently, we are able to propagate, by linear algebra constructions alone, the trivial connection $d : \mathcal{O}_T \to \Omega_{T/W_s}$ for $T = T_{t,s}, t \geq s$, to a connection

$$\tilde{\Theta} : \mathcal{E}_\rho \to \mathcal{E}_\rho \otimes \Omega_{T/W_s} \cong \mathcal{E}_\rho \otimes \mathcal{P} \otimes \mathcal{Q},$$

the last isomorphism stemming from the Kodaira-Spencer map. When we follow this map by the projection $\mathcal{E}_\rho \otimes \mathcal{P} \otimes \mathcal{Q} \to \mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}$, and combine it with pull back of modular forms under $T \to S_s^{\mathrm{ord}}$, we obtain an operator

$$\Theta : H^0(S_s^{\mathrm{ord}}, \mathcal{E}_\rho) \to H^0(S_s^{\mathrm{ord}}, \mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}).$$

This operator can be iterated and combined with representation-theoretic operations as discussed in the end of §2, to produce an array of differential operators $D_\kappa^{\kappa'}$ as in [E-F-M-V, E-M].

The initial parts of §3 are a review of the theory of toroidal compactifications for the case we need. We follow Faltings-Chai [F-C], that relies on the seminal work of Mumford and his school, Skinner-Urban [S-U], and the definitive volume by Lan [Lan]. In particular, the reader will find a precise explanation of the meaning of the Fourier-Jacobi expansion of a vector-valued modular form

$$f = \sum_{\check{h} \in \check{H}^+} a(\check{h})\, q^{\check{h}}.$$

See §3.1.6. In this notation our first main theorem states the following.

Theorem. *(Theorem 3.2.5) Let ξ be a rank-m cusp. Let f be a global section of $\mathcal{E}_\rho$ and $\sum_{h \in \check{H}^+} a(\check{h})\, q^{\check{h}}$ its Fourier-Jacobi expansion at ξ. Then the section $\Theta(f)$ of $\mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}$ has the Fourier-Jacobi expansion*

$$\Theta(f) = \sum_{\check{h} \in \check{H}^+} a(\check{h}) \otimes \check{h} \cdot q^{\check{h}}.$$

In §4 we consider the extension of the operator Θ to the complement of the μ-ordinary locus. This we are able to do, so far, only for scalar-valued modular forms. The proof requires a partial compactification of a particular Igusa variety as in [dS-G1], and delicate computations with Dieudonné modules in the spirit of our recent work [dS-G2]. Let $\mathcal{L} = \det \mathcal{Q}$ and $k \geq 0$.

Theorem. *(Theorems 4.2.3, 4.3.2) Consider the operator*

$$\Theta : H^0(S^{\mathrm{ord}}, \mathcal{L}^k) \to H^0(S^{\mathrm{ord}}, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Then Θ extends holomorphically to an operator

$$\Theta : H^0(S, \mathcal{L}^k) \to H^0(S, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Finally, in §5 we introduce the notion of Θ-cycles and recall interesting phenomena observed in [dS-G1].

Our paper and the work of Eischen-Mantovan suggest several directions in which the theory can be further developed. In addition to those mentioned in [E-M] we suggest the following problems. (i) Provide a formula for the Fourier-Jacobi expansion and the theta operator Θ at general cusps; this would, likely, involve the compatibility of the theta operator with theta operators on lower rank unitary Shimura varieties. We expect the final formula to be an amalgamation of our results and those obtained in [E-F-M-V, E-M]. (ii) Study the extension of Θ to a holomorphic operator for general vector-valued unitary modular forms. (iii) Develop a theory of mod p operators, such as U and V and characterize the kernel of Θ in terms of V, *cf.* [Ka2]. (iv) Study Θ-cycles in relation to mod p Galois representations.

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1. Background

1.1. **The Shimura variety.**

1.1.1. *Linear algebra.* We review some background and set up standard notation. Let E be a quadratic imaginary field, embedded in $\mathbb{C}$, $0 \leq m \leq n$ and $\Lambda = \mathcal{O}_E^{n+m}$. Let

$$I_{n,m} = \begin{pmatrix} & & I_m \\ & I_{n-m} & \\ I_m & & \end{pmatrix} \tag{1.1.1}$$

where I_l is the unit matrix of size l, and introduce the perfect hermitian pairing

$$(u, v) = {}^t\overline{u} I_{n,m} v \tag{1.1.2}$$

on Λ. Let

$$\boldsymbol{G} = GU(\Lambda, (,))$$

be the group of unitary similitudes of Λ, regarded as a group scheme over $\mathbb{Z}$, and denote by $\nu : \boldsymbol{G} \to \mathbb{G}_m$ the similitude character. For any commutative ring R

$$\boldsymbol{G}(R) = \{g \in GL_{n+m}(\mathcal{O}_E \otimes R) | \, \forall u, v \in \Lambda \otimes R \quad (gu, gv) = \nu(g)(u, v)\} .$$

Then $\boldsymbol{G}(\mathbb{R}) = GU(n, m)$ is the general unitary group of signature (n, m), and $\boldsymbol{G}(\mathbb{C}) \simeq GL_{n+m}(\mathbb{C}) \times \mathbb{C}^\times$.

Let δ_E be the unique generator of the different $\mathfrak{d}_E$ of E with $\mathrm{Im}(\delta_E) > 0$. The *polarization pairing*

$$\langle u, v\rangle = Tr_{E/\mathbb{Q}}(\delta_E^{-1}(u, v)) \tag{1.1.3}$$

is then a perfect alternating pairing $\Lambda \times \Lambda \to \mathbb{Z}$ satisfying $\langle au, v\rangle = \langle u, \overline{a}v\rangle$ $(a \in E)$.

Let p be an odd prime which is inert in E, and fix once and for all an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$. Let E_p be the completion of E and $\mathcal{O}_p$ its ring of integers. As -1 is a norm from E_p to $\mathbb{Q}_p$, one easily checks that $\boldsymbol{G}_{/\mathcal{O}_p}$ is quasi-split. In fact, over $\mathcal{O}_p$ the lattice $\Lambda_p = \mathbb{Z}_p \otimes \Lambda = \mathcal{O}_p^{n+m}$, equipped with the hermitian form (1.1.2), is isomorphic to the same lattice equipped with the pairing ${}^t\overline{u}J_{n+m}v$, where by J_l we denote the matrix with 1's on the anti-diagonal and 0's elsewhere. This will be useful later.

If R is an $\mathcal{O}_{E,(p)}$-algebra then any R-module M endowed with a commuting $\mathcal{O}_E$-action decomposes according to types,

$$M = M(\Sigma) \oplus M(\overline{\Sigma}),$$

where $M(\Sigma)$ is the R-submodule on which $\mathcal{O}_E$ acts via the canonical homomorphism $\Sigma : \mathcal{O}_E \hookrightarrow \mathcal{O}_{E,(p)} \to R$, while $M(\overline{\Sigma})$ is the part on which it acts via the conjugate homomorphism $\overline{\Sigma}$. Indeed, it is enough to decompose $\mathcal{O}_E \otimes R = R(\Sigma) \times R(\overline{\Sigma})$ as an $\mathcal{O}_E$-algebra. The same notation will be applied to coherent sheaves with $\mathcal{O}_E$-action on schemes defined over $\mathcal{O}_{E,(p)}$.

We denote by κ the field $\mathcal{O}_E/p\mathcal{O}_E$ of p^2 elements.

1.1.2. *The Shimura variety and the moduli problem.* Fix an integer $N \geq 3$ relatively prime to p. Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the adèle ring of $\mathbb{Q}$, where $\mathbb{A}_f = \mathbb{Q} \cdot \widehat{\mathbb{Z}}$ are the finite adèles. Let $K_f \subset \boldsymbol{G}(\widehat{\mathbb{Z}})$ be an open subgroup of the form $K_f = K^p K_p$, where $K^p \subset \boldsymbol{G}(\mathbb{A}^p)$ is the principal congruence subgroup of level N, and

$$K_p = \boldsymbol{G}(\mathbb{Z}_p) \subset \boldsymbol{G}(\mathbb{Q}_p)$$

the hyperspecial maximal compact subgroup at p. Let $K_\infty \subset \boldsymbol{G}(\mathbb{R})$ be the stabilizer of the negative definite subspace spanned by $\{-e_i + e_{n+i};\, 1 \leq i \leq m\}$ in $\Lambda_\mathbb{R} = \mathbb{C}^{n+m}$, where $\{e_i\}$ stands for the standard basis. This K_∞ is a maximal compact-modulo-center subgroup, isomorphic to $G(U(m) \times U(n))$. By $G(U(m) \times U(n))$ we mean the pairs of matrices $(g_1, g_2) \in GU(m) \times GU(n)$ having the same similitude factor. Let $K = K_\infty K_f \subset \boldsymbol{G}(\mathbb{A})$ and $\mathfrak{X} = \boldsymbol{G}(\mathbb{R})/K_\infty$.

To the Shimura datum $(\boldsymbol{G}, \mathfrak{X})$ there is associated a Shimura variety Sh_K. It is a quasi-projective non-singular variety of dimension nm defined over E. If $m = n$ the Shimura variety may even be defined over $\mathbb{Q}$, but we still denote by Sh_K its base-change to E. The complex points of Sh_K are identified, as a complex manifold, with

$$Sh_K(\mathbb{C}) = \boldsymbol{G}(\mathbb{Q}) \backslash \boldsymbol{G}(\mathbb{A})/K.$$

Following Kottwitz [Ko] we define a scheme $\mathcal{S}$ over $\mathcal{O}_{E,(p)}$. This $\mathcal{S}$ is a fine moduli space whose R-points, for every $\mathcal{O}_{E,(p)}$-algebra R, classify isomorphism types of tuples $\underline{A} = (A, \iota, \phi, \eta)$ where

- A is an abelian scheme of dimension $n + m$ over R.
- $\iota : \mathcal{O}_E \hookrightarrow \mathrm{End}(A)$ has signature (n, m) on the Lie algebra of A.
- $\phi : A \overset{\sim}{\to} A^t$ is a principal polarization whose Rosati involution induces $\iota(a) \mapsto \iota(\overline{a})$ on the image of ι.
- η is an $\mathcal{O}_E$-linear full level-N structure on A compatible with $(\Lambda, \langle ., . \rangle)$ and ϕ ([Lan] 1.3.6).

See [Lan] for the comparison of the various languages used to define the moduli problem.

The generic fiber $\mathcal{S}_E$ of $\mathcal{S}$ is, in general, a union of *several* Shimura varieties, one of which is Sh_K. This is due to the failure of the Hasse principle for $\boldsymbol{G}$, which can happen when $m + n$ is odd ([Ko] §7). We also remark that the assumption $N \geq 3$ could be avoided if we were willing to use the language of stacks. As this is not essential to the present paper, we keep the scope slightly limited for the sake of clarity.

As shown by Kottwitz, $\mathcal{S}$ is *smooth* of relative dimension nm over $\mathcal{O}_{E,(p)}$.

1.1.3. *The universal abelian variety and its p-divisible group.* By virtue of the moduli problem which it represents, $\mathcal{S}$ carries a universal abelian scheme $\mathcal{A}_{/\mathcal{S}}$ equipped with a PEL structure as above. Let

$$S = \mathcal{S} \times_{\mathrm{Spec}(\mathcal{O}_{E,(p)})} \mathrm{Spec}(\kappa)$$

be the special fiber of $\mathcal{S}$. Recall that for any geometric point $x : \mathrm{Spec}(k) \to S$ the p-divisible group of $A = \mathcal{A}_x$ carries a canonical filtration by p-divisible groups

$$\mathrm{Fil}^0 = A[p^\infty] \supset \mathrm{Fil}^1 = A[p^\infty]^0 \supset \mathrm{Fil}^2 = A[p^\infty]^\mu \supset 0, \tag{1.1.4}$$

where $gr^2 = A[p^\infty]^\mu$ is multiplicative, $gr^1 = A[p^\infty]^0/A[p^\infty]^\mu$ is local-local and $gr^0 = A[p^\infty]/A[p^\infty]^0$ is étale. Over $\mathrm{Spec}(k)$ this filtration is even split, i.e. $A[p^\infty]$ is uniquely expressible as a product of multiplicative, local-local and étale p-divisible groups, but this fact is special for algebraically closed (or perfect) fields, while a filtration like (1.1.4) often exists over more general bases.

The special fiber S contains an open dense subset S^{ord}, called the (μ-)ordinary locus, *cf.* [We1] and [Mo1], Theorem 3.2.7. It is characterized by the fact that for any geometric point x of S, x lies in S^{ord} if and only if the height of $A[p^\infty]^\mu$ is $2m$, which is as large as it can get. Equivalently, the Newton polygon of $A[p^\infty]$ has slopes $0, 1/2$ and 1 with horizontal lengths $2m$, $2(n-m)$ and $2m$ respectively, which is as low as it can get. In fact, Wedhorn and Moonen show that the isomorphism type of $A[p^\infty]$, as a polarized $\mathcal{O}_E$-group, is the same for all $x \in S^{\mathrm{ord}}(k)$:

$$A[p^\infty] \simeq (\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m \times \mathfrak{G}_k^{n-m} \times (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)^m.$$

Here $\mathfrak{G}_k$ is the p-divisible group denoted by $G_{\frac{1}{2},\frac{1}{2}}$ in the Dieudonné-Manin classification. It is the unique height-2 one-dimensional connected p-divisible group over k. It is well-known that the ring $\mathcal{O}_p$ acts as endomorphisms of $\mathfrak{G}_k$. We normalize this action so that the induced action of $\mathcal{O}_p$ on the Lie algebra of $\mathfrak{G}_k$ is via $\Sigma : \mathcal{O}_p \twoheadrightarrow \kappa \subset k$, and this pins down $\mathfrak{G}_k$ as an $\mathcal{O}_E$-group up to isomorphism. We polarize it fixing an isomorphism of $\mathfrak{G}_k$ with its Serre dual. The appearance of the inverse different in the first factor is a matter of choice, and is meant to allow a more natural way to write the Weil pairing between the first and last factors, namely

$$\langle a \otimes x, b \otimes y \rangle = Tr_{E/\mathbb{Q}}(\overline{a}b) \langle x, y \rangle .$$

Over S^{ord}, a filtration like (1.1.4) exists globally, but is far from being split now. Nevertheless, its graded pieces are, locally in the pro-étale topology, isomorphic to the constant p-divisible groups $(\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m$, $\mathfrak{G}_k^{n-m}$ and $(\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)^m$, and the isomorphisms can be taken to respect the endomorphisms and the polarization. This is well-known for gr^0 and gr^2. For gr^1 it follows from the rigidity of isoclinic Barsotti-Tate groups with endomorphisms, namely from the fact that the universal deformation ring of $(\mathfrak{G}_k, \iota)$ where $\iota : \mathcal{O}_p \hookrightarrow \mathrm{End}(\mathfrak{G}_k)$, is $W(k)$ ([Mo1], Corollary 2.1.5, see *loc.cit* §3.3.1 for the polarization). This result implies that for any geometric point $x \in S^{\mathrm{ord}}(k)$, $gr^1\mathcal{A}[p^\infty]$ becomes isomorphic over $\hat{\mathcal{O}}_{S,x}$ to $\mathfrak{G}^{n-m}$, with its additional structures of endomorphisms and polarization. By Artin's approximation theorem they become isomorphic already over the strict henselization $\mathcal{O}_{S,x}^{\mathrm{sh}}$, which means that they are locally isomorphic in the pro-étale topology.

1.1.4. *The basic vector bundles on S.* The Hodge bundle $\omega = \omega_{\mathcal{A}/\mathcal{S}}$ is the pullback via the zero section $e_{\mathcal{A}} : \mathcal{S} \to \mathcal{A}$ of the relative cotangent sheaf $\Omega_{\mathcal{A}/\mathcal{S}}$ of the universal abelian scheme. It decomposes as

$$\omega = \omega(\Sigma) \oplus \omega(\overline{\Sigma}) = \mathcal{P} \oplus \mathcal{Q}$$

according to types. Thus, $\mathrm{rk}(\mathcal{P}) = n$ and $\mathrm{rk}(\mathcal{Q}) = m$.

Lemma 1.1.1. *The line bundles* $\det(\mathcal{P})$ *and* $\det(\mathcal{Q})$ *are isomorphic over* $\mathcal{S}$.

Proof. The proof is similar to [dS-G3], Proposition 1.3. Automorphic vector bundles over the generic fiber $\mathcal{S}_E$ correspond functorially to representations of the group $GL_m \times GL_n$, as discussed below in §1.2.3. The vector bundles $\det(\mathcal{Q})$ and $\det(\mathcal{P})$ correspond to the determinant of GL_m and the *inverse* of the determinant of GL_n.

Their ratio therefore corresponds to the determinant of $GL_m \times GL_n$. If the level subgroup K is small enough, as we always assume, then the arithmetic group by which we divide the symmetric space to get a complex uniformization of every connected component of $\mathcal{S}_{\mathbb{C}}$ is contained in $SU(n,m)$. This means that over $\mathbb{C}$, the automorphic line bundle corresponding to det is trivial, hence $\det(\mathcal{P}) \simeq \det(\mathcal{Q})$. From this it is easy to get the claim even over the base $\mathcal{O}_{E,(p)}$. We stress that we do not know a direct moduli-theoretic proof of the claim in the lemma, and we do not know if the particular isomorphism supplied by the complex analytic uniformization is defined over $\overline{\mathbb{Q}}$. See however Corollary 1.1.3 below. $\square$

Over the special fiber S we have the Verschiebung homomorphism $V : \omega \to \omega^{(p)}$ induced by the Verschiebung isogeny $\mathrm{Ver} : \mathcal{A}^{(p)} \to \mathcal{A}$. As V commutes with the endomorphisms it maps $\mathcal{P}$ to $\mathcal{Q}^{(p)}$ and $\mathcal{Q}$ to $\mathcal{P}^{(p)}$. We denote the restriction of V to $\mathcal{P}$ (resp. $\mathcal{Q}$) by $V_{\mathcal{P}}$ (resp. $V_{\mathcal{Q}}$). The homomorphism

$$H = V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}^{(p^2)}$$

is called the *Hasse matrix.* We let $\mathcal{L} = \det(\mathcal{Q})$, a line bundle. Then

$$h = \det(H) : \mathcal{L} \to \mathcal{L}^{(p^2)} \simeq \mathcal{L}^{p^2} \tag{1.1.5}$$

is a global section of $\mathcal{L}^{p^2-1}$ called the *Hasse invariant* (see [G-N], Appendix B). Here we used the well-known fact that for a line bundle $\mathcal{L}$ over a scheme in characteristic p, there is a canonical isomorphism between $\mathcal{L}^{(p)}$ and $\mathcal{L}^p$, sending the base-change $s^{(p)} = 1 \otimes s$ of the section s under the absolute Frobenius of S to $s \otimes \cdots \otimes s$. It is an important fact that $h \neq 0$ precisely on S^{ord}. If $n > m$ the zero-divisor of h is even *reduced*, so equals $S^{\mathrm{no}} = S \smallsetminus S^{\mathrm{ord}}$ with its reduced subscheme structure. A proof of this fact may be found in [Woo], Proposition 7.2.11, but can also be extracted from the Dieudonné module computations in Theorem 4.1.3 below.

If $n = m$ this is not true; h vanishes then on S^{no} to order $p + 1$. There is a variant, though, that will be useful for us in the study of the holomorphicity of the theta operator.

Lemma 1.1.2. *Let $n = m$. Consider the maps of line bundles*

$$h_{\mathcal{Q}} = \det(V_{\mathcal{Q}}) : \det(\mathcal{Q}) \to \det(\mathcal{P})^{(p)} = \det(\mathcal{P})^p,$$

$$h_{\mathcal{P}} = \det(V_{\mathcal{P}}) : \det(\mathcal{P}) \to \det(\mathcal{Q})^{(p)} = \det(\mathcal{Q})^p.$$

Both $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$ vanish precisely on S^{no} with multiplicity 1 and the following relation holds:

$$h = h_{\mathcal{P}}^p \circ h_{\mathcal{Q}}.$$

Proof. The claim concerning the vanishing of $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$ follows again from [Woo], Proposition 7.2.11, or from the computations in Theorem 4.3.2 below. The relation $h = h_{\mathcal{P}}^p \circ h_{\mathcal{Q}}$ is a direct consequence of the definition. $\square$

Although the following Corollary is weaker than Lemma 1.1.1, it is of interest because its proof is entirely moduli-theoretic.

Corollary 1.1.3. *Let S be of arbitrary signature (n, m). There is an isomorphism*

$$\det(\mathcal{P})^{p+1} \simeq \det(\mathcal{Q})^{p+1}.$$

Proof. Consider first the case of equal signatures (m,m). By comparing divisors of global sections, we obtain from the last Lemma an isomorphism of line bundles $\det(\mathcal{P})^p \otimes \det(\mathcal{Q})^{-1} \simeq \det(\mathcal{Q})^p \otimes \det(\mathcal{P})^{-1}$, implying the Corollary in this case.

For S of signature (n,m), and a geometric point x of S, we can embed S in a suitable Shimura variety $\mathbb{S}$ of signature $(n+m, n+m)$ by a morphism given on objects by $\underline{A} \mapsto \underline{A} \times \underline{B_x}$, where $\underline{B_x}$ is the abelian variety corresponding to x with the twisted $\mathcal{O}_E$ structure. One easily checks that the pull-back of the relation $\det(\mathcal{P})^{p+1} \simeq \det(\mathcal{Q})^{p+1}$ on $\mathbb{S}$ gives the same relation on S. □

Coming back to the case $m = n$ we have the following Lemma.

Lemma 1.1.4. *Over an algebraic closure of κ, we may fix the isomorphism* $\det(\mathcal{P}) \simeq \det(\mathcal{Q}) = \mathcal{L}$ *so that $h_{\mathcal{P}} = h_{\mathcal{Q}}$, hence $h = h_{\mathcal{Q}}^{p+1}$.*

Proof. Fix a smooth toroidal compactification $\overline{S}$ of S. The vector bundles $\mathcal{P}$ and $\mathcal{Q}$, as well as the homomorphisms $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$ extend to $\overline{S}$. In Corollary 3.1.5 below we show that h does not vanish on any irreducible component of the boundary $\overline{S} \smallsetminus S$. The same therefore must be true for $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$. It follows that

$$\mathrm{div}(h_{\mathcal{P}}) = \mathrm{div}(h_{\mathcal{Q}})$$

as divisors on the smooth, complete variety $\overline{S}$. Fix any isomorphism as in Lemma 1.1.1. Having the same divisors, the sections $h_{\mathcal{P}}$ and $h_{\mathcal{Q}}$ of $\mathcal{L}^{p-1}$ are equal up to a multiplication by a nowhere vanishing function on $\overline{S}$, hence equal up to a scalar. By extracting a $p-1$ root from this scalar, we can normalize the isomorphism $\det(\mathcal{P}) \simeq \det(\mathcal{Q})$ so that $h_{\mathcal{P}} = h_{\mathcal{Q}}$. □

We remark that for more general Shimura varieties of PEL type the construction of the Hasse invariant requires substantial work and is due to Goldring and Nicole [G-N].

1.1.5. *The vector bundles $\mathcal{P}_0$ and $\mathcal{P}_\mu$.* The geometric fibers of the subsheaf

$$\mathcal{P}_0 = \ker(V_{\mathcal{P}}) \subset \mathcal{P}$$

have constant rank $n-m$ over an open subset $S_\sharp$ containing the ordinary stratum

$$S^{\mathrm{ord}} \subset S_\sharp \subset S.$$

As the base is non-singular, this implies that *over* $S_\sharp$ this $\mathcal{P}_0$ is a vector-sub-bundle of $\mathcal{P}$, hence so is the quotient

$$\mathcal{P}_\mu = \mathcal{P}/\mathcal{P}_0.$$

In fact, $V_{\mathcal{P}}$ induces there an isomorphism

$$V_{\mathcal{P}} : \mathcal{P}_\mu \simeq \mathcal{Q}^{(p)}, \tag{1.1.6}$$

because as long as its kernel has rank $n-m$, $V_{\mathcal{P}}$ must be surjective. The open subscheme $S_\sharp$ is of much interest, and was analyzed in [dS-G2]. It is the union of Ekedahl-Oort strata ([Oo],[V-W]) that can be determined precisely. When $m = 1$, for example, its complement in S is zero-dimensional (the superspecial points). When $m < n$ this $S_\sharp$ contains a unique Ekedahl-Oort stratum S^{ao} of dimension $mn - 1$. This will be used later on in our work.

The vector bundles $\mathcal{P}_0$, $\mathcal{P}_\mu$ and $\mathcal{Q}$ will turn out to be the building blocks of the mod-p automorphic vector bundles over S^{ord}. See §1.2.3 for a discussion why we need to substitute the two subquotients $\mathcal{P}_0$ and $\mathcal{P}_\mu$ *in lieu* of the classical automorphic vector bundle $\mathcal{P}$.

It is a remarkable fact that $\mathcal{P}_0$ and $\mathcal{P}_\mu$ can be defined on the ordinary stratum also modulo p^s for any $s \geq 1$, although the Verschiebung isogeny is defined only in characteristic p. One way to see it is as follows. Let $R = \mathcal{O}_{E,(p)}$ and

$$W_s = W_s(\kappa) = W(\kappa)/p^s W(\kappa) = R/p^s R$$

(we identify the Witt vectors $W = W(\kappa)$ with the completion $\mathcal{O}_p$ of R). Denote by S_s^{ord} the open subscheme of $S_s = \mathcal{S} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/p^s R)$ whose underlying topological space is S^{ord}. The filtration of the p-divisible group of $\mathcal{A}$ by its connected and multiplicative parts extends uniquely from S^{ord} to S_s^{ord}. This is well-known for the connected part, and by Cartier duality follows also for the multiplicative part. It is crucial for us that the filtered pieces in (1.1.4) have *constant height* along S_s^{ord}. Moreover, by the same result of Moonen quoted above ([Mo1], Corollary 2.1.5) the graded pieces of $\mathcal{A}[p^\infty]$ with their additional structures of endomorphisms and polarization become isomorphic, locally in the pro-étale topology on S_s^{ord}, to the constant p-divisible groups $(\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m$, $\mathfrak{G}_k^{n-m}$ and $(\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)^m$. In other words, not only modulo p but modulo p^s as well, we can trivialize $gr^i\mathcal{A}[p^t]$ with the additional structures after passing to a finite étale covering. This remark will be instrumental in the construction of the big Igusa tower below.

Coming back to the definition of $\mathcal{P}_0$ and $\mathcal{P}_\mu$ over S_s^{ord}, if $t \geq s$ the exact sequence

$$0 \to \mathcal{A}[p^t] \to \mathcal{A} \xrightarrow{p^t} \mathcal{A} \to 0 \tag{1.1.7}$$

shows that $\mathrm{Lie}(\mathcal{A}[p^t]/S_s) \to \mathrm{Lie}(\mathcal{A}/S_s)$ is an isomorphism[1]. The filtration of $\mathcal{A}[p^t]$ induces (over S_s^{ord} only) a filtration of its Lie algebra by $\mathcal{O}_{S_s}$-sub-bundles, hence a similar filtration of $\mathrm{Lie}(\mathcal{A}/S_s)$. By duality we get (again over S_s^{ord}) a filtration of ω by sub-bundles, which on its Σ-part yields the exact sequence

$$0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0. \tag{1.1.8}$$

For future reference we record the fact that

$$\mathcal{P}_0 = \omega_{\mathcal{A}[p^\infty]^0/\mathcal{A}[p^\infty]^\mu}, \quad \mathcal{P}_\mu = \omega_{\mathcal{A}[p^\infty]^\mu}(\Sigma), \quad \mathcal{Q} = \omega_{\mathcal{A}[p^\infty]^\mu}(\overline{\Sigma}).$$

We do not know how to extend (1.1.8) in any intelligible way to the s-th infinitesimal thickening of $S_\sharp$, as we did when $s = 1$ using Verschiebung.

1.2. p-adic automorphic vector bundles.

1.2.1. *Representations of GL_m.* We review some well known facts from the representation theory of GL_m. Let R be any ring, and $\underline{\mathrm{Rep}}_R(GL_m)$ the category of algebraic representations of GL_m on projective R-modules of finite rank. If $\rho \in \underline{\mathrm{Rep}}_R(GL_m)$, we denote by $\rho(R)$ the associated projective R-module, endowed with a left $GL_m(R)$ action. Given an R-scheme S, the functoriality in R allows us to regard $\rho(\mathcal{O}_S) = \mathcal{O}_S \otimes_R \rho(R)$ as a vector bundle with a left $GL_m(\mathcal{O}_S)$ action on S. The category $\underline{\mathrm{Rep}}_R(GL_m)$ is a rigid tensor category, and if R is a field, it is also abelian. Some special objects of the category are the standard representation st, and the symmetric and exterior powers $\mathrm{Sym}^r\mathrm{st}$ and $\wedge^r\mathrm{st}$ of st, defined as suitable *quotients* of $\otimes^r\mathrm{st}$.

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[1]For any group scheme G over T we define $\mathrm{Lie}(G/T)$ to be the kernel of the map "mod ε" from $G(T[\varepsilon])$ to $G(T)$, where $\varepsilon^2 = 0$.

If R is a field of characteristic 0, the category is even semisimple. It is well known that the simple objects are then classified by dominant weights. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m)$ ($\lambda_i \in \mathbb{Z}$) is a dominant weight of GL_m, the corresponding object is

$$\rho_\lambda = \mathrm{Sym}^{\lambda_1-\lambda_2}(\mathrm{st}) \otimes \mathrm{Sym}^{\lambda_2-\lambda_3}(\wedge^2\mathrm{st}) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_m}(\wedge^m\mathrm{st}). \tag{1.2.1}$$

Note that $\wedge^m\mathrm{st}$ is of rank 1, so $\mathrm{Sym}^{\lambda_m}(\wedge^m\mathrm{st}) = \otimes^{\lambda_m}(\wedge^m\mathrm{st})$ makes sense even if λ_m is negative. In Herman Weyl's construction of ρ_λ we assume first that $\lambda_m \geq 0$, view λ as a partition (Young tableau) of size $d = \sum_{i=1}^m \lambda_i$, project $\otimes^d\mathrm{st}$ onto a sub-representation using the Young symmetrizer $c_\lambda = a_\lambda b_\lambda \in \mathbb{Z}[\mathfrak{S}_d]$, and then the resulting quotient is a model for ρ_λ, *cf.* [F-H], Ch.6. When λ is not necessarily positive, one reduces to the positive case by a twist by a power of the determinant $\wedge^m\mathrm{st}$.

Recall, however, that over a field of characteristic p the ρ_λ, defined directly by (1.2.1), are in general reducible (e.g. $m = 2$ and $\lambda = (p \geq 0)$), and the category $\underline{\mathrm{Rep}}_R(GL_m)$ is not semi-simple. As the Young symmetrizers are only quasi-idempotents (i.e. $c_\lambda^2 = n_\lambda c_\lambda$ for some integer n_λ called the hook length of λ, which might be divisible by p) using them to study the representations of GL_m becomes tricky.

A more *geometric* construction of ρ_λ that works *over any ground ring* R, hence produces an element of $\underline{\mathrm{Rep}}_R(GL_m)$ functorially in R, is via the Borel-Weil theorem ([F-H] Claim 23.57). Let $\overline{\lambda} = (\lambda_m, \ldots, \lambda_1)$ be the anti-dominant weight for the standard torus of GL_m, which is the opposite of λ. Let $G = GL_m$ and let B be the standard upper-triangular Borel subgroup. Let $\overline{\lambda}$ denote also the character of B obtained by first projecting modulo the unipotent radical U to the torus and then applying $\overline{\lambda}$. On the flag variety G/B define the line bundle L_λ by

$$L_\lambda = G \times^B \overline{\lambda}.$$

This is the quotient of $G \times \mathbb{A}^1$ under the equivalence relation $(gb, t) \sim (g, \overline{\lambda}(b)t)$ ($b \in B$). A global section of L_λ is identified with a map $\sigma : G \to \mathbb{A}^1$ satisfying $\sigma(gb) = \overline{\lambda}(b)^{-1}\sigma(g)$. In particular, letting w be the element of maximal length in the Weyl group (the matrix with 1's on the anti-diagonal), we may define such a section on the (open dense) big cell $UwB \subset G$ by

$$\sigma_0(uwb) = \overline{\lambda}(b)^{-1}. \tag{1.2.2}$$

The Borel-Weil theorem says that if λ is dominant, then (a) L_λ is ample and $V_\lambda = H^0(G/B, L_\lambda) \neq 0$, (b) if we let G act on V_λ by left translation, i.e. $(g\sigma)(g') = \sigma(g^{-1}g')$, this becomes a model for ρ_λ, and finally (c) the σ_0 of (1.2.2) extends to a regular section on all of G/B, the group $B \subset G$ acts on it via the character λ, and up to a scalar, σ_0 is the unique highest weight vector in ρ_λ.

This geometric formulation makes it evident that ρ_λ so defined is functorial in R. Moreover, the linear functional

$$\Psi_\lambda : \sigma \mapsto \sigma(w) \in \mathbb{A}^1 \tag{1.2.3}$$

is easily seen to be in $\mathrm{Hom}_{\overline{B}}(\rho_\lambda|_{\overline{B}}, \lambda)$ where $\overline{B}$ is the *lower* triangular Borel. What's more, since $L_{\lambda+\mu} = L_\lambda \otimes L_\mu$ there is a canonical map (multiplication of global sections)

$$m_{\lambda,\mu} : \rho_\lambda \otimes \rho_\mu \to \rho_{\lambda+\mu}, \tag{1.2.4}$$

which is compatible with the functionals Ψ_λ, Ψ_μ and $\Psi_{\lambda+\mu}$. From now on, whenever we write ρ_λ or Ψ_λ we shall have this specific model in mind.

We finally remark that if R is an $\mathbb{F}_p$-algebra, and $\phi : R \to R$ is the absolute Frobenius $\phi(x) = x^p$, then every representation $\rho \in \underline{\mathrm{Rep}}_R(GL_m)$ admits a *Frobenius twist* $\rho^{(p)} = \phi^*(\rho)$. In concrete terms, locally on R we may write ρ in matrices, using a basis of the underlying projective module, and $\rho^{(p)}$ is the representation obtained by raising all the entries of the matrices to power p.

1.2.2. *Twisting a representation by a vector bundle.* Let S be a scheme over R. For every vector bundle $\mathcal{F}$ of rank m over S we let $\underline{\mathrm{Isom}}(\mathcal{O}_S^m, \mathcal{F})$ be the right GL_m-torsor of isomorphisms between $\mathcal{O}_S^m$ and $\mathcal{F}$, the group scheme $GL_{m/S}$ acting on the right by pre-composition. If $\rho \in \underline{\mathrm{Rep}}_R(GL_m)$ we consider the vector bundle

$$\mathcal{F}_\rho = \underline{\mathrm{Isom}}(\mathcal{O}_S^m, \mathcal{F}) \times^{GL_m} \rho(\mathcal{O}_S)$$

(contracted product). One should think of $\mathcal{F}_\rho$ as "ρ twisted by $\mathcal{F}$". For example, for a dominant weight λ,

$$\mathcal{F}_{\rho_\lambda} = \mathrm{Sym}^{\lambda_1-\lambda_2}(\mathcal{F}) \otimes \mathrm{Sym}^{\lambda_2-\lambda_3}(\wedge^2\mathcal{F}) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_m}(\wedge^m\mathcal{F}).$$

What we have constructed is a *tensor functor* $\rho \rightsquigarrow \mathcal{F}_\rho$ from $\underline{\mathrm{Rep}}_R(GL_m)$ into the category $\underline{\mathrm{Vec}}_S$ of vector bundles over S. These functors are compatible with base-change of the underlying scheme S, and with isomorphisms $\mathcal{F}_1 \simeq \mathcal{F}_2$ between rank m vector bundles. Thus if over $S' \to S$ the pull-backs of two vector bundles $\mathcal{F}_i$ become isomorphic via an isomorphism ε, this ε induces, over S', functorial isomorphisms $\varepsilon_\rho : \mathcal{F}_{1,\rho} \simeq \mathcal{F}_{2,\rho}$ for every $\rho \in \underline{\mathrm{Rep}}_R(GL_m)$.

Note that if R is an $\mathbb{F}_p$-algebra, then $\mathcal{F}_{\rho^{(p)}} = \mathcal{F}_\rho^{(p)}$, where for any sheaf $\mathcal{F}$ over S we denote by $\mathcal{F}^{(p)} = \Phi_S^*\mathcal{F}$ its pull-back by the absolute Frobenius of S. By $\mathcal{F}_\rho^{(p)}$ we mean either $(\mathcal{F}_\rho)^{(p)}$ or $(\mathcal{F}^{(p)})_\rho$, the two being canonically identified.

The above generalizes to representations of a product of any number of linear groups, say $M = \prod_{i=1}^r GL_{m_i}$. Given $\rho \in \underline{\mathrm{Rep}}_R(M)$ and vector bundles $\mathcal{F}_i$ of ranks m_i we let

$$\mathcal{E}_\rho = \prod_{\mathrm{i}=1}^{\mathrm{r}} \underline{\mathrm{Isom}}(\mathcal{O}_S^{m_i}, \mathcal{F}_i) \times^M \rho(\mathcal{O}_S). \tag{1.2.5}$$

We call it the vector bundle obtained by *twisting ρ by the vector bundles $\mathcal{F}_i$*.

1.2.3. *p-adic automorphic vector bundles over S_s^{ord}.* Classically, automorphic vector bundles on $\mathcal{S}_\mathbb{C}$ are defined in the following way. Every connected component $\mathcal{S}_\mathbb{C}^0$ is of the form $\Gamma \setminus \boldsymbol{G}(\mathbb{R})/K_\infty$ where K_∞ is a maximal compact-modulo-center subgroup, and Γ an arithmetic subgroup of $\boldsymbol{G}(\mathbb{R})$. By a standard procedure due to Harish-Chandra one may embed the symmetric space $\mathfrak{X} = \boldsymbol{G}(\mathbb{R})/K_\infty$ as an open subset of its *compact dual* $\check{\mathfrak{X}}$. In our case the compact dual happens to be the Grassmannian $GL_{n+m}(\mathbb{C})/P_\mathbb{C}$, where $P_\mathbb{C}$ is the standard maximal parabolic of type (m, n). (The change of variables involved in the Harish-Chandra embedding for $U(n, m)$ is called the Cayley transform, as it generalizes the well-known embedding of the upper half plane as the open unit disk in $\mathbb{P}_\mathbb{C}^1$ when $n = m = 1$.) The Levi quotient of $P_\mathbb{C}$ is $M_\mathbb{C} = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$, and the automorphic vector bundles are attached to representations $\rho \in \underline{\mathrm{Rep}}_\mathbb{C}(M)$.

Let such a representation ρ be given. Let $P_{\mathbb{C}}$ act on $\rho(\mathbb{C})$ via its quotient $M_{\mathbb{C}}$, consider the vector bundle

$$GL_{n+m}(\mathbb{C}) \times^{P_{\mathbb{C}}} \rho(\mathbb{C})$$

on $\check{\mathfrak{X}} = GL_{n+m}(\mathbb{C})/P_{\mathbb{C}}$, and denote by $\widetilde{\mathcal{E}}_\rho$ its restriction to $\mathfrak{X}$. Since left multiplication by Γ commutes with right multiplication by $P_{\mathbb{C}}$, this vector bundle descends to a vector bundle $\mathcal{E}_\rho$ on $\mathcal{S}^0_{\mathbb{C}} = \Gamma \setminus \mathfrak{X}$. Using the complex analytic description of the universal abelian variety over $\Gamma \setminus \mathfrak{X}$ one checks that the standard representations of the two blocks in M yield the vector bundles $\mathcal{Q}$ and $\mathcal{P}^\vee$. Easy group theory shows then that this complex analytic construction gives, for any $\rho \in \underline{\mathrm{Rep}}_{\mathbb{C}}(M)$, a vector bundle which may be canonically identified with the $\mathcal{E}_\rho$ obtained by twisting ρ by the pair of vector bundles $\mathcal{Q}$ and $\mathcal{P}^\vee$, as in the preceding paragraph.

This suggests to adopt the construction outlined in §1.2.2 as an algebraic construction of automorphic vector bundles that works equally well over the arithmetic scheme $\mathcal{S}$, hence also over its special fiber S.

For the purpose of studying *p-adic vector-valued modular forms* this is however not always sufficient. In the classical complex setting, a great advantage of the construction is that $\widetilde{\mathcal{E}}_\rho$ becomes trivial on $\mathfrak{X}$, hence may be described by matrix-valued factors of automorphy. In the mod-p or p-adic theory we need an analogous covering of S^{ord} (or S^{ord}_s), over which our basic building blocks, hence all the $\mathcal{E}_\rho$, will be trivialized. This is crucial both for Katz's theory of p-adic modular forms, and for the construction of Maass-Shimura-like differential operators below. This analogue of $\mathfrak{X}$ is th*e (big) Igusa tower,* to be described in §2.1.

At this point the μ-ordinary case becomes fundamentally different from the ordinary one. If p is split in E, or if p is inert but $m = n$, then both $\mathcal{P}$ and $\mathcal{Q}$ are trivialized over the Igusa tower and everything works well with the usual automorphic vector bundles. However, if p is inert and $m < n$ then $\mathcal{P}$ can not be trivialized over the Igusa tower, nor on any other pro-étale cover. The best we can do is to trivialize its subquotients $\mathcal{P}_0$ and $\mathcal{P}_\mu$ separately. This explains why we need to start with *three* basic bundles $\mathcal{Q}, \mathcal{P}_\mu$ and $\mathcal{P}_0$ over S^{ord}_s, and why our ρ will be an element of $\underline{\mathrm{Rep}}_R(M)$ with

$$M = GL_m \times GL_m \times GL_{n-m}$$

rather than $GL_m \times GL_n$ as over $\mathbb{C}$.

After this long discussion, we can finally make the following definition.

Definition 1.2.1. Let $\rho \in \underline{\mathrm{Rep}}_R(M)$ where $M = GL_m \times GL_m \times GL_{n-m}$, and define $\mathcal{E}_\rho$ on S^{ord}_s by (1.2.5) with $(\mathcal{Q}, \mathcal{P}_\mu, \mathcal{P}_0)$ replacing the $\mathcal{F}_i$. We call $\mathcal{E}_\rho$ the *p-adic automorphic vector bundle of weight* ρ (mod p^s), and $\varprojlim_s H^0(S^{\mathrm{ord}}_s, \mathcal{E}_\rho)$ the space of *p-adic (vector-valued)* modular forms *of weight* ρ.

Remark. (i) Note that by our convention the standard representations of the second and third factors of M correspond to $\mathcal{P}_\mu$ and $\mathcal{P}_0$, while the complex analytic standard representation of GL_n corresponded to $\mathcal{P}^\vee$.

(ii) A p-adic modular form need not come from a global section over $\mathcal{S}$. It is a rigid analytic object, defined over the affinoid which is the generic fiber of the formal completion of $\mathcal{S}$ along S^{ord}. In fact, if $\mathcal{P}_\mu$ and $\mathcal{P}_0$ are "involved" in $\mathcal{E}_\rho$ (in the precise sense that ρ does not come from a representation of the first simple factor of M) then it does not even make sense to ask whether the modular form extends to a global section over $\mathcal{S}$, because the p-adic automorphic vector bundle does not extend there. In order to compare classical and p-adic modular forms we make the following definition.

Definition 1.2.2. Let $\rho \in \underline{\mathrm{Rep}}_R(M)$. We say that the p-adic automorphic vector bundle $\mathcal{E}_\rho$ is *of classical type* is ρ factors through the first factor of M.

A p-adic automorphic vector bundle of classical type is the restriction to S_s^{ord} of a classical automorphic vector bundle. Note however that $\mathcal{P}$, an honest automorphic vector bundle on S_s, is not a p-adic automorphic vector bundle on S_s^{ord} (if $m < n$), as it can not be reconstructed from its graded pieces $\mathcal{P}_0$ and $\mathcal{P}_\mu$.

2. Differential operators on p-adic modular forms

2.1. **The big Igusa tower.**

2.1.1. *The p-divisible group* $\mathfrak{G}$. Following a long-standing tradition going back to Katz in the *ordinary* case, we want to describe a certain *tower* of (big) Igusa varieties $T_{t,s}$, for all $t, s \geq 1$. The variety $T_{t,s}$ will be an Igusa variety of level p^t over $\mathcal{O}_{E,(p)}/p^s\mathcal{O}_{E,(p)}$. By "tower" we mean that the reduction of $T_{t,s+1}$ modulo p^s will be identified with $T_{t,s}$, and that for a fixed s there will be compatible morphisms from level $p^{t'}$ to level p^t for all $t' \geq t$. This "big Igusa tower" has been defined and studied, in much greater generality, in E. Mantovan's thesis [Man].

To describe it, we shall have to choose a model $\mathfrak{G}$ over $W = W(\kappa) = \mathcal{O}_p$ of the p-divisible group that becomes, over $\overline{\kappa}$, the group $\mathfrak{G}_{\overline{\kappa}}$ introduced in §1.1.3. This choice results in freedom, which grows with t and s, and prevents the $T_{t,s}$ (unlike the *small* Igusa varieties, see below) from being canonically defined. This problem will nevertheless disappear over $W(\overline{\kappa})$, so the reader interested in the construction over $W(\overline{\kappa})/p^sW(\overline{\kappa})$ only, can happily ignore the issue.

The easiest way to fix our model is to choose an elliptic curve $\mathscr{C}$ defined over W, with complex multiplication by $\mathcal{O}_E$ and CM type Σ. The theory of complex multiplication guarantees that such an elliptic curve exists, and has supersingular reduction. We then let $\mathfrak{G} = \mathscr{C}[p^\infty]$ be its p-divisible group. Its special fiber $\mathfrak{G}_\kappa$ is of local-local type, height 2 and dimension 1. The canonical polarization of the elliptic curve supplies an isomorphism of $\mathfrak{G}$ with its Serre dual, hence a compatible system of perfect alternating Weil pairings

$$\langle , \rangle : \mathfrak{G}[p^t] \times \mathfrak{G}[p^t] \to \mu_{p^t}$$

$(t \geq 1)$.

The completion $\mathcal{O}_p$ of $\mathcal{O}_E$ maps isomorphically onto $\mathrm{End}(\mathfrak{G}_{/W}) \subset \mathrm{End}(\mathfrak{G}_{/\kappa})$. Furthermore, for any W-algebra R

$$\mathrm{End}_{\mathcal{O}_E}(\mathfrak{G}[p^t]_{/R}) = \mathcal{O}_p/p^t\mathcal{O}_p.$$

We have $\langle \iota(a)u, v\rangle = \langle u, \iota(\overline{a})v\rangle$ for every $a \in \mathcal{O}_E$.

2.1.2. *The Igusa moduli problem.* If R is a $W_s(\kappa)$-algebra and $A_{/R}$ is fiber-by-fiber μ-ordinary, then its p-divisible group admits a filtration like (1.1.4) whose graded pieces we label $gr^iA[p^\infty]$. We choose the indices in such a way that locally in the pro-étale topology on $\mathrm{Spec}(R)$ there exist isomorphisms

$$\epsilon^0 : (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)_R^m \simeq gr^0, \quad \epsilon^1 : \mathfrak{G}_R^{n-m} \simeq gr^1, \quad \epsilon^2 : (\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})_R^m \simeq gr^2, \tag{2.1.1}$$

respecting the action of $\mathcal{O}_E$ and the pairings. Note that gr^1 is self-dual, while ϵ^0 and ϵ^2 determine each other. For future reference we want to make the pairings on these "model group schemes" explicit. If

$$\alpha = (x_1, \ldots, x_m, y_1, \ldots, y_{n-m}, z_1, \ldots, z_m) \in (\mathcal{O}_E \otimes \mathbb{Q}_p/\mathbb{Z}_p)^m_R \times \mathfrak{G}^{n-m}_R \times (\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m_R,$$

and similarly $\alpha' = (x'_1, \ldots, x'_m, y'_1, \ldots, y'_{n-m}, z'_1, \ldots, z'_m)$, we define

$$\langle \alpha, \alpha' \rangle = \prod_{i=1}^{m} \langle x_i, z'_{m+1-i} \rangle \prod_{j=1}^{n-m} \langle y_j, y'_{n-m+1-j} \rangle \prod_{i=1}^{m} \langle z_i, x'_{m+1-i} \rangle . \tag{2.1.2}$$

In matrix form, writing μ_{p^∞} additively, we take, ${}^t\alpha J_{n+m} \alpha'$ where J_l is the anti-diagonal matrix of size l, and *not* ${}^t\alpha I_{n,m} \alpha'$ where $I_{n,m}$ is the matrix (1.1.1). As remarked in §1.1.1, these two pairings produce *isomorphic* polarized $\mathcal{O}_E$-groups. Thus, there is no real difference which pairing we take at this point, but for later book-keeping purposes, we prefer the one with J_{n+m}.

We call $\epsilon = (\epsilon^0, \epsilon^1, \epsilon^2)$ a *graded symplectic trivialization* of the p-divisible group. A *graded symplectic trivialization* of $A[p^t]$ is a similar system of isomorphisms of the p^t-torsion in the p-divisible groups, defined over R, which is locally étale liftable to a graded symplectic trivialization of the whole p-divisible group.

Definition 2.1.1. The *big Igusa moduli problem of level* p^t over $W_s(\kappa)$, denoted $T_{t,s}$, classifies tuples

$$(\underline{A}, \epsilon)_{/R/W_s},$$

where $\underline{A} \in S_s^{\mathrm{ord}}(R)$ and ϵ is a graded symplectic trivialization of $A[p^t]$ as in (2.1.1), up to isomorphism.

The representability of this moduli problem by a scheme, denoted also $T_{t,s}$, is standard. One only has to check that it is *relatively representable* over S_s^{ord} (see [K-M], Chapter 4). The maps between the levels are self-evident. The morphism

$$\tau : T_{t,s} \to S_s^{\mathrm{ord}}$$

is a Galois étale covering of S_s^{ord} ([Man], Proposition 4).

The *small Igusa variety* of the same level classifies tuples $(\underline{A}, \epsilon^2)$ of the same nature. There is an obvious morphism form the big tower to the small one: "forget ϵ^1". Since ϵ^0 is determined by ϵ^2 we do not have to forget anything more.

2.1.3. *The Galois group.* The Galois group Δ_t of the covering $\tau : T_{t,s} \to S_s^{\mathrm{ord}}$ is isomorphic to $GL_m(\mathcal{O}_E/p^t\mathcal{O}_E) \times U_{n-m}(\mathcal{O}_E/p^t\mathcal{O}_E)$ under

$$\Delta_t \ni \gamma \mapsto [\gamma] = (\gamma_2, \gamma_1) \in GL_m(\mathcal{O}_E/p^t\mathcal{O}_E) \times U_{n-m}(\mathcal{O}_E/p^t\mathcal{O}_E),$$

where

$$\gamma(\underline{A}, \epsilon) = (\underline{A}, \epsilon \circ [\gamma]^{-1}). \tag{2.1.3}$$

Here by $U_{n-m}(\mathcal{O}_E/p^t\mathcal{O}_E)$ we mean the quasi-split unitary group, consisting of matrices g of size $n-m$ satisfying the relation ${}^t\overline{g} J_{n-m} g = J_{n-m}$. As explained before, it is isomorphic to the group of matrices satisfying ${}^t\overline{g} g = I$. By $\epsilon \circ [\gamma]^{-1}$ we mean that we compose ϵ^1 with γ_1^{-1} and ϵ^2 with γ_2^{-1} (the action on ϵ^0 being determined by the one on ϵ^2). As usual, the group Δ_t acts simply transitively on the fibers of the morphism τ.

2.1.4. *Trivializing the three basic vector bundles over the Igusa tower.* For simplicity write $T = T_{t,s}$, $\Delta = \Delta_t$, and assume that $t \geq s$. There is enough level structure then to "see" the relative Lie algebra of $\mathcal{A}_{/S_s}$ on $\mathcal{A}[p^t]_{/S_s}$, as explained in the paragraph following (1.1.7).

As the cotangent space at the origin of $\mathfrak{d}_E^{-1} \otimes \mu_{p^t/W_s}$ is canonically identified with $\mathcal{O}_E \otimes W_s = W_s(\Sigma) \oplus W_s(\overline{\Sigma})$, the isomorphism ϵ^2 induces canonical trivializations of $\mathcal{O}_E$-vector bundles over T

$$\varepsilon^2 = ((\epsilon^2)^{-1})^* : \mathcal{O}_E \otimes \mathcal{O}_T^m \simeq \mathcal{Q} \oplus \mathcal{P}_\mu$$

(we write $\mathcal{Q}$ for $\tau^*\mathcal{Q}$ etc. as $\tau^*\mathcal{Q}$ is "the" $\mathcal{Q}$ of $\mathcal{A}/T$), or

$$\varepsilon^2(\overline{\Sigma}) : \mathcal{O}_T^m \simeq \mathcal{Q}, \;\; \varepsilon^2(\Sigma) : \mathcal{O}_T^m \simeq \mathcal{P}_\mu.$$

Similarly fix, once and for all, an isomorphism of the cotangent space at the origin of $\mathfrak{G}[p^t]_{/W_s}$ (as an $\mathcal{O}_E$-module) with $W_s(\Sigma)$. The isomorphism ϵ^1 induces then also a canonical trivialization over T

$$\varepsilon^1 : \mathcal{O}_T^{n-m} \simeq \mathcal{P}_0.$$

The action (2.1.3) of $\gamma \in \Delta$ on T induces the following action on the trivializations

$$\gamma(\varepsilon^i) = \varepsilon^i \circ {}^t\gamma_i. \tag{2.1.4}$$

$(i = 1, 2)$. Let us check the last formula, dropping the index i:

$$\gamma(\varepsilon) = (\gamma(\epsilon)^{-1})^* = (\gamma \circ \epsilon^{-1})^* = (\epsilon^{-1})^* \circ \gamma^* = \varepsilon \circ {}^t\gamma,$$

because the matrix representing $[\gamma]^*$ on the cotangent space is the transpose of the matrix representing $[\gamma]_*$ on the Lie algebra, which is simply $[\gamma]$.

2.2. **The theta operator.**

2.2.1. *Pre-theta.* Let ρ be a representation of $GL_m \times GL_m \times GL_{n-m}$ over W_s, and let $\mathcal{E}_\rho$ be the automorphic vector bundle on S_s^{ord} defined above. We define a connection

$$\widetilde{\Theta} : \mathcal{E}_\rho \to \mathcal{E}_\rho \otimes \Omega_{S_s/W_s}$$

over S_s^{ord}.

Let $t \geq s$. Denote by $\mathcal{O}_\rho = \rho(\mathcal{O}_T)$ the vector bundle over $T = T_{t,s}$ obtained by twisting the representation ρ by the trivial vector bundles $\mathcal{O}_T^m$, $\mathcal{O}_T^m$ and $\mathcal{O}_T^{n-m}$ as in Definition 1.2.1. The trivial connection on the structure sheaf $\mathcal{O}_T$ induces, by the usual rules, a connection

$$d_\rho : \mathcal{O}_\rho \to \mathcal{O}_\rho \otimes \Omega_{T/W_s}.$$

For example, if $\rho = \rho_\lambda$ where $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a dominant weight depending only on the first GL_m factor, so that $\mathcal{O}_\rho$ is given by (1.2.1), then d_ρ is given by the usual rules of differentiation of symmetric powers, exterior powers and duals.

On the other hand the trivializations ε^1 and ε^2 constructed above yield a trivialization

$$\varepsilon_\rho : \mathcal{O}_\rho \simeq \tau^*\mathcal{E}_\rho$$

over T. To get the action of

$$\gamma = (\gamma_2, \gamma_1) \in \Delta = GL_m(\mathcal{O}_E/p^t\mathcal{O}_E) \times U_{n-m}(\mathcal{O}_E/p^t\mathcal{O}_E)$$

on ε_ρ we first map γ to $GL_m(W_s)\times GL_m(W_s)\times GL_{n-m}(W_s)$ via

$$\gamma\mapsto\iota(\gamma)=(\overline{\gamma_2},\gamma_2,\gamma_1)$$

(well defined because $t\geq s$) and let ${}^t[\gamma]_\rho=\rho({}^t\iota(\gamma))$. Then from (2.1.4) we get

$$\gamma(\varepsilon_\rho)=\varepsilon_\rho\circ{}^t[\gamma]_\rho. \tag{2.2.1}$$

Let $U\subset S_s^{\mathrm{ord}}$ be Zariski open. For $f\in H^0(U,\mathcal{E}_\rho)$ define

$$\widetilde{\Theta}(f)=(\varepsilon_\rho\otimes 1)\circ d_\rho\circ\varepsilon_\rho^{-1}(\tau^*f)\in H^0(\tau^{-1}(U),\tau^*\mathcal{E}_\rho\otimes_{\mathcal{O}_T}\Omega_{T/W_s}). \tag{2.2.2}$$

Since τ is étale, $\Omega_{T/W_s}=\mathcal{O}_T\otimes_{\mathcal{O}_{S_s}}\Omega_{S_s/W_s}$, so

$$\widetilde{\Theta}(f)\in H^0(\tau^{-1}(U),\tau^*\mathcal{E}_\rho\otimes_{\mathcal{O}_{S_s}}\Omega_{S_s/W_s}).$$

We have to show that $\widetilde{\Theta}(f)\in H^0(U,\mathcal{E}_\rho\otimes_{\mathcal{O}_{S_s}}\Omega_{S_s/W_s})$, and for that it would suffice to show that it is invariant under Δ. Let $\gamma\in\Delta$. Then by (2.2.1)

$$\gamma(\widetilde{\Theta}(f))=(\varepsilon_\rho\otimes 1)\circ{}^t[\gamma]_\rho\circ d_\rho\circ{}^t[\gamma]_\rho^{-1}\circ\varepsilon_\rho^{-1}(\tau^*f)=\widetilde{\Theta}(f).$$

Here we used that (a) τ^*f is Galois invariant, (b) d_ρ is Galois invariant since τ is étale, and (c) d_ρ commutes with the scalar matrices ${}^t[\gamma]_\rho$. We summarize our construction in the following theorem.

Theorem 2.2.1. *Let $U\subset S_s^{\mathrm{ord}}$ be an open set and $f\in H^0(U,\mathcal{E}_\rho)$. Then*

$$\widetilde{\Theta}(f)=(\varepsilon_\rho\otimes 1)\circ d_\rho\circ\varepsilon_\rho^{-1}(\tau^*f)\in H^0(U,\mathcal{E}_\rho\otimes_{\mathcal{O}_{S_s}}\Omega_{S_s/W_s})$$

yields a well-defined connection on $\mathcal{E}_\rho$. The connection defined on $\mathcal{E}\otimes\mathcal{F}$, $\mathcal{E}^\vee$ etc. is the tensor product, dual etc. of the connections defined on the individual sheaves. If $s=1$ (i.e. we are in characteristic p), then the connection defined on $\mathcal{E}^{(p)}$ is trivial. Hence, if f and g are sections of $\mathcal{E}$ and $\mathcal{F}$, respectively, then on $\mathcal{E}^{(p)}\otimes\mathcal{F}$ we have $\widetilde{\Theta}(f^{(p)}\otimes g)=f^{(p)}\otimes\widetilde{\Theta}(g)$.

Proof. The functoriality with respect to linear-algebra operations (including Frobenius twist in characteristic p) is clear. The last remark is a general fact about modules with connection. For any vector bundle $\mathcal{E}$ over a base S in characteristic p there is a canonical connection ∇^{can} on $\mathcal{E}^{(p)}$, characterized by $\nabla^{\mathrm{can}}(f^{(p)})=0$ for any section f of $\mathcal{E}$, and if ∇ is *any* connection on $\mathcal{E}$, then is pull-back $\nabla^{(p)}$ to $\mathcal{E}^{(p)}$ is canonically identified with ∇^{can}. □

2.2.2. *Theta.* Using the inverse of the Kodaira-Spencer isomorphism

$$\mathrm{KS}:\mathcal{P}\otimes\mathcal{Q}\simeq\Omega_{S_s/W_s}$$

we may view $\widetilde{\Theta}$ as a map from $\mathcal{E}_\rho$ to $\mathcal{E}_\rho\otimes\mathcal{P}\otimes\mathcal{Q}$. We emphasize that this map is not a sheaf homomorphism, as it is only κ-linear and not $\mathcal{O}_{S_s}$-linear. It is better, however, to consider the operator

$$\Theta=(1\otimes pr_\mu\otimes 1)\circ(1\otimes\mathrm{KS}^{-1})\circ\widetilde{\Theta}:\mathcal{E}_\rho\to\mathcal{E}_\rho\otimes\mathcal{P}_\mu\otimes\mathcal{Q}. \tag{2.2.3}$$

Here $pr_\mu:\mathcal{P}\to\mathcal{P}/\mathcal{P}_0=\mathcal{P}_\mu$ is the canonical projection.

If $s=1$, in characteristic p over S, we may replace $\mathcal{P}_\mu$ by $\mathcal{Q}^{(p)}$ and pr_μ by V. From the point of view of connections, dividing $\Omega_{S/\kappa}$ by $\ker(V\otimes 1)=\mathcal{P}_0\otimes\mathcal{Q}$ means that we restrict the connection to the foliation $TS^+\subset TS$ which has been introduced and studied in [dS-G2], i.e. use it to differentiate sections of $\mathcal{E}_\rho$ only in the direction of TS^+. Although this voluntarily gives up information encoded in $\widetilde{\Theta}$, when restricted to characteristic p, the operator Θ has *four* advantages over its predecessor:

(1) While $\widetilde{\Theta}$ has poles along the complement of S^{ord} in S, we shall see that Θ may be analytically continued everywhere, at least when applied to scalar modular forms.
(2) The effect of Θ on Fourier-Jacobi expansions is particularly nice, while the formulae for $\widetilde{\Theta}$ contain unpleasant terms.
(3) Restricting the connection to the foliation TS^+ should also result in a nice expansion of Θ at a μ-ordinary point in terms of Moonen's generalized Serre-Tate coordinates [Mo1]. This is the approach taken in [E-M]. For the relation between TS^+ and Moonen's generalized Serre-Tate coordinates, see [dS-G2], §3.3, Theorem 13.
(4) Unlike $\widetilde{\Theta}$, the operator Θ lands back in a sheaf which is obtained "by linear algebra operations" from $\mathcal{Q}, \mathcal{P}_\mu$ and $\mathcal{P}_0$. This will allow us to *iterate* Θ, something which we were prohibited from doing with $\widetilde{\Theta}$ due to the presence of $\mathcal{P}$.

2.3. **Higher order differential operators $D_\kappa^{\kappa'}$.** For the sake of completeness we indicate how one gets, by iterating Θ, a whole array of differential operators $D_\kappa^{\kappa'}$. We follow, with minor modifications, Eischen's thesis [Ei]. If $\kappa = (a, b, c)$ is a dominant weight of $M = GL_m \times GL_m \times GL_{n-m}$ we denote the vector bundle $\mathcal{E}_\rho$ associated with the representation $\rho = \rho_\kappa$ by $\mathcal{E}_\kappa$.

Let st be the standard representation of GL_m over W, let a' is a positive dominant weight $a'_1 \geq \cdots \geq a'_m \geq 0$ and $e = \Sigma_{i=1}^m a'_i$. Then in $\underline{\mathrm{Rep}}_W(GL_m)$ there exists a distinguished projection, unique up to a $W^\times$ multiple,

$$\pi_{a'} : \mathrm{st}^{\otimes e} \to \rho_{a'}.$$

One simply has to normalize the projection resulting from the Young symmetrizer $c_{a'}$ so that it is integral, but not divisible by p. Whether $\pi_{a'}$ can be further normalized to eliminate the $W^\times$-ambiguity depends on which model we take for $\rho_{a'}$, as two such models are canonically isomorphic only up to multiplication by a scalar. Since we agreed to take the models given by the Borel-Weil theorem, we do not know how to normalize $\pi_{a'}$ any further.

Let $\kappa' = (a', b', 0)$ be a dominant weight with a' and b' positive, such that

$$e = \sum_{i=1}^m a'_i = \sum_{i=1}^m b'_i.$$

In [E-F-M-V] such a κ' is called sum-symmetric.

We twist $\rho_{\kappa'} = \rho_{a'} \otimes \rho_{b'} \otimes 1$ by the vector bundles $\mathcal{Q}$ and $\mathcal{P}_\mu$. Recall that $\mathcal{Q}$ is used to twist $\rho_{a'}$ and $\mathcal{P}_\mu$ is used for $\rho_{b'}$, while twisting by $\mathcal{P}_0$ is not needed, as the representation associated with GL_{n-m} is the trivial one. We get

$$\pi_{\kappa'} = \pi_{b'} \otimes \pi_{a'} : (\mathcal{P}_\mu \otimes \mathcal{Q})^{\otimes e} \to \mathcal{E}_{\kappa'}.$$

Let $\kappa = (a, b, c)$ be a dominant weight of M. Consider the e-th iteration of the derivation Θ. It maps the sheaf $\mathcal{E}_\kappa$ to $\mathcal{E}_\kappa \otimes (\mathcal{P}_\mu \otimes \mathcal{Q})^{\otimes e}$. We may now use $\pi_{\kappa'}$ to map $(\mathcal{P}_\mu \otimes \mathcal{Q})^{\otimes e}$ to $\mathcal{E}_{\kappa'}$ and finally apply the homomorphism $m_{\kappa,\kappa'} : \mathcal{E}_\kappa \otimes \mathcal{E}_{\kappa'} \to \mathcal{E}_{\kappa+\kappa'}$ of (1.2.4) to get the differential operator

$$D_\kappa^{\kappa'} = m_{\kappa,\kappa'} \circ (1 \otimes \pi_{\kappa'}) \circ \Theta^e : \mathcal{E}_\kappa \to \mathcal{E}_{\kappa+\kappa'}. \tag{2.3.1}$$

As $m_{\kappa,\kappa'} \circ (1 \otimes \pi_{\kappa'})$ is a sheaf homomorphism this $D_\kappa^{\kappa'}$ is a differential operator of order e. It is well-defined only up to a scalar from $W^\times$. The operators $D_\kappa^{\kappa'}$ allow us to increase the weight by any κ' as long as

$$\kappa' = (a', b', 0), \quad a'_1 \geq \cdots \geq a'_m \geq 0, \;\; b'_1 \geq \cdots \geq b'_m \geq 0, \;\; \sum_{i=1}^m a'_i = \sum_{i=1}^m b'_i.$$

Example. Scalar-valued modular forms. If $\kappa = (k, \dots, k; 0, \dots, 0; 0, \dots, 0)$ then

$$\mathcal{E}_\kappa = \det(\mathcal{Q})^k = \mathcal{L}^k.$$

In this case, global sections of $\mathcal{E}_\kappa$ are scalar-valued modular forms on $\boldsymbol{G}$ of weight k. If we take $\kappa' = (k', \dots, k'; k', \dots, k'; 0, \dots, 0)$ then $D_\kappa^{\kappa'}$ maps $\mathcal{L}^k$ to $\mathcal{L}^{k+k'} \otimes \det(\mathcal{P}_\mu)^{k'}$. If $s = 1$, in characteristic p, we may identify $\det(\mathcal{P}_\mu)$ with $\mathcal{L}^p$ (1.1.6), so $D_\kappa^{\kappa'}$ maps $\mathcal{L}^k$ to $\mathcal{L}^{k+(p+1)k'}$. In these cases $D_\kappa^{\kappa'}$ is obtained by applying Θ iteratively mk times and projecting. If $m = 1$ then $D_\kappa^{\kappa'}$ is simply $\Theta^{k'}$.

3. Toroidal compactifications and Fourier-Jacobi expansions

3.1. **Toroidal compactifications and logarithmic differentials.**

3.1.1. *Generalities.* Our goal in this section is to show that the operator Θ, defined so far on S_s^{ord}, extends to a partial compactification $\overline{S}_s^{\mathrm{ord}}$, obtained by fixing a smooth toroidal compactification $\overline{S}_s$ of S_s, and removing from it the closure of $S_s^{\mathrm{no}} = S_s \smallsetminus S_s^{\mathrm{ord}}$. Thus

$$\overline{S}_s^{\mathrm{ord}} = \overline{S}_s \smallsetminus \{\text{Zariski closure of } S_s^{\mathrm{no}}\}$$

is an open subset of $\overline{S}_s$. Note that in general the closure of S_s^{no} may meet the boundary of $\overline{S}_s$, although in some special cases, e.g. whenever $m = 1$, S_s^{no} is proper and does not reach the cusps. For a characterization of $\overline{S}_s^{\mathrm{ord}}$ as the non-vanishing locus of the Hasse invariant see §3.1.7. Once we extend Θ, we shall calculate its effect on Fourier-Jacobi expansions and show that, as in the classical case of GL_2, it is *morally* given by "$q \cdot d/dq$".

The toroidal compactifications $\overline{\mathcal{S}}$ of $\mathcal{S}$ considered below are smooth over $\mathcal{O}_{E,(p)}$ and their boundary $\partial\mathcal{S} = \overline{\mathcal{S}} \smallsetminus \mathcal{S}$ is a divisor with normal crossing. However, they depend on auxiliary combinatorial data, and are not unique. As such, one can not expect $\overline{\mathcal{S}}$ to solve a moduli problem anymore. The universal abelian scheme $\mathcal{A}$ nevertheless *extends canonically* to a semi-abelian scheme $\mathcal{G}$ over $\overline{\mathcal{S}}$. We say that a geometric point x of $\partial\mathcal{S}$ is of rank $1 \leq r \leq m$ if the toric part of $\mathcal{G}_x$ has dimension $2r$, i.e. $\mathcal{O}_E$-rank r. Skinner and Urban [S-U] call such a point "a point of genus $n + m - 2r$", referring to the dimension of the abelian part of $\mathcal{G}_x$ instead.

Constructing the toroidal compactifications, even if all proofs are omitted, requires several pages of definitions and notation. Lan's book [Lan] is an exhaustive, extremely careful and precise reference. Unfortunately, some notation introduced there is too long to fit in a single line. Following [F-C], Skinner and Urban gave a very readable account of the compactification in §5.4 of [S-U], which we will follow closely. It is set for signature (n, n), but the modifications needed to treat an arbitrary signature (n, m) are minor. Yet, this forces us to review everything from scratch, rather than use [S-U] blindly.

We shall contend ourselves with the arithmetical compactification of $Sh_{K/W}$ (several copies of which comprise $\mathcal{S}_{/W}$). In §3.1 only we will write S for $Sh_{K/W}$ or for its base-change to W_s (rather than to $\kappa = W_1$ as before). As smaller Shimura varieties will show up in the process, we shall write

$$S = S_G = S_{G,K}$$

whenever we need to emphasize the dependence on G or K.

Let $\{e_i\}$ denote the standard basis of $V = E^{n+m}$ and consider, for $1 \le r \le m$,

$$0 \subset V_r = \mathrm{Span}_E\{e_1, \dots, e_r\} \subset V_r^{\perp} = \mathrm{Span}_E\{e_1, \dots, e_n, e_{n+r+1}, \dots, e_{n+m}\} \subset V.$$

Since we regard $V = Res_{\mathbb{Q}}^{E}\mathbb{A}^{n+m}$ as a $\mathbb{Q}$-vector group, whose $\mathbb{Q}$-rational points are E^{n+m}, this is a $\mathbb{Q}$-rational filtration. The quotient $V(r) = V_r^{\perp}/V_r$ becomes a hermitian space of signature $(n-r, m-r)$ at infinity, and $\Lambda \cap V_r^{\perp}$ projects to a self-dual lattice $\Lambda(r) \subset V(r)$, defining a smaller general unitary group $\boldsymbol{G}_r$. If $n = m = r$ we understand by $\boldsymbol{G}_r$ the group $\mathbb{G}_m$ (accounting for the similitude factor, which is present even if $V(r) = 0$).

The subgroup

$$P_r = \mathrm{Stab}_{\boldsymbol{G}}(V_r)$$

stabilizes also $V_r^{\perp}$, and is a maximal $\mathbb{Q}$-rational parabolic subgroup of $\boldsymbol{G}$. Its unipotent radical is

$$U_r = \{g \in P_r |\, g \text{ acts trivially on } V_r,\ V(r), \text{and } V/V_r^{\perp}\}.$$

Its Levi quotient $L_r = P_r/U_r$ is identified with $Res_{\mathbb{Q}}^{E} GL_r \times \boldsymbol{G}_r$ under $g \mapsto (g|_{V_r}, g|_{V(r)})$. The center $Z_r = Z(U_r)$ of U_r turns out to be

$$Z_r = \{g \in U_r |\, (g-1)(V_r^{\perp}) = 0,\ (g-1)(V) \subset V_r\}.$$

In matrix block form

$$P_r = \left\{ g = \begin{pmatrix} A & C & B \\ & D & C' \\ & & \nu\, {}^t\overline{A}^{-1} \end{pmatrix} \in \boldsymbol{G} \right\}, \tag{3.1.1}$$

where A is a square matrix of size r, and D is a square matrix of size $(n+m-2r)$. The group U_r is characterized by $\nu = 1$, $A = 1$, $D = 1$, and Z_r by the additional properties $C = 0$, $C' = 0$. When this is the case, $B = -\,{}^t\overline{B}$. We regard L_r also as a subgroup of P_r, mapping (g, h) to the matrix which in block form is $(g, h, \nu(h)\, {}^t\overline{g}^{-1})$. Thus $P_r = L_r U_r$.

Every maximal $\mathbb{Q}$-rational parabolic subgroup of $\boldsymbol{G}$ is conjugate to P_r for some r.

3.1.2. *Cusp labels, the minimal compactification and the toroidal compactifications.* Let $1 \le r \le m$. The set of cusp labels of level K and rank r ([S-U] §5.4.3) is the finite set

$$\mathscr{C}_r = [GL_r(E) \cdot \boldsymbol{G}_r(\mathbb{A}_f)] \cdot U_r(\mathbb{A}_f) \backslash \boldsymbol{G}(\mathbb{A}_f)/K.$$

As before, the rank r will be the $\mathcal{O}_E$-rank of the toric part of the universal abelian variety over the corresponding cuspidal component. If $g \in \boldsymbol{G}(\mathbb{A}_f)$ we denote by $[g] = [g]_r = [g]_{r,K} \in \mathscr{C}_r$ the corresponding double coset. The minimal (Baily-Borel) compactification S^* of S is constructed in [Lan] §7.2.4 and, when $n = m$, [S-U]

§5.4.5. It is a singular compactification admitting a stratification by finitely many locally closed strata

$$S^* = \bigsqcup_{r=0}^{m} \bigsqcup_{[g]_r \in \mathscr{C}_r} S_{\boldsymbol{G}_r, K_{r,g}},$$

where $K_{r,g} = \boldsymbol{G}_r(\mathbb{A}_f) \cap gKg^{-1}$. Each $S_{\boldsymbol{G}_r, K_{r,g}}$ is an $(n-r)(m-r)$-dimensional Shimura variety, so when r attains its maximal value m, it is 0-dimensional. When $r = 0$ we get one stratum, which is the open dense S. The closure of $S_{\boldsymbol{G}_r, K_{r,g}}$ is the union of $S_{\boldsymbol{G}_{r'}, K_{r',g'}}$ for $r \le r'$ and g' such that the cusp label $[g']_{r'}$ is a specialization of $[g]_r$ in an appropriate sense ([Lan], Definition 5.4.2.13). We call each $S_{\boldsymbol{G}_r, K_{r,g}}$ a *rank r cuspidal component of S^**.

Any toroidal compactification that we consider will be a smooth scheme $\overline{S}_{/W}$ endowed with a proper morphism

$$\pi : \overline{S} \to S^*.$$

Moreover, it will come equipped with a stratification

$$\overline{S} = \bigsqcup_{r=0}^{m} \bigsqcup_{[g]_r \in \mathscr{C}_r} \bigsqcup_{\sigma \in \Sigma_{H^{++}_{g,\mathbb{R}}}/\Gamma} Z([g]_r, \sigma)$$

by finitely many smooth, locally closed W-subschemes $Z([g]_r, \sigma)$. The indexing set $\Sigma_{H^{++}_{g,\mathbb{R}}}/\Gamma$ will become clear shortly. The morphism π will respect the stratifications.

Every $Z([g]_r,\sigma)$ is constructed in three steps, related to the structure of the semi-abelian scheme $\mathcal{G}$ over it, as follows.

- First, $S_{\boldsymbol{G}_r, K_{r,g}}$ is the moduli space of the abelian part of $\mathcal{G}$ (with the associated PEL structure), which is of signature $(n-r, m-r)$, hence is a smooth Shimura variety of dimension $(n-r)(m-r)$. Let $\mathcal{A}_r$ denote the universal abelian scheme over it. In contrast to the abelian part, the toric part of $\mathcal{G}$ is *fixed* by the cusp label $[g]$, and is given by

$$T_X = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m)$$

  where $X = X_g$ is a rank-r projective $\mathcal{O}_E$-module determined by g. Thus $\dim(T_X) = 2r$. For example, if $g = 1$ (the "standard cusp of rank r") then $X = \mathrm{Hom}(\Lambda \cap V_r, \mathbb{Z})$.
- The second step in the construction of $Z([g]_r, \sigma)$ is the construction of an abelian scheme C which classifies the extensions of $\mathcal{A}_r$ by T_X. Let $X^* = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{O}_E)$ and

$$C = C([g]_r) := \mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{A}_r, T_X).$$

  This can be written also as

$$C = X^* \otimes_{\mathcal{O}_E} \mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{A}_r, \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m) = X^* \otimes_{\mathcal{O}_E} \mathcal{A}_r^t = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}_r^t),$$

  using the fact that $Tr_{E/\mathbb{Q}} \otimes 1 : \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m \to \mathbb{G}_m$ induces an isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_E}(\mathcal{A}_r, \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m) \simeq \mathrm{Ext}^1(\mathcal{A}_r, \mathbb{G}_m) = \mathcal{A}_r^t.$$

  The relative dimension of C over $S_{\boldsymbol{G}_r, K_{r,g}}$ is $r(n+m-2r)$ so its total dimension is

$$(n-r)(m-r) + r(n+m-2r) = nm - r^2.$$

- In the last and final step one uses auxiliary combinatorial data and the theory of toroidal embeddings [Fu] to construct the $Z([g]_r,\sigma)$. Each of them is a torus torsor over $C([g]_r)$. For details, see the next subsection.

The stratification by disjoint locally closed strata does not shed any light on the way these strata are glued together, even if the closure relations between them are given. However, each stratum $Z = Z([g],\sigma)$ is actually the underlying reduced scheme (the "support") of a formal scheme $\mathfrak{Z} = \mathfrak{Z}([g],\sigma)$ whose over-all dimension (counting the "formal parameters" too) is mn. The semi-abelian scheme together with the PEL structure extend from Z to $\mathfrak{Z}$ "in the infinitesimal directions" to give a structure called *degeneration data*. As described originally in [Mu] in the totally degenerate setting, and later on in [A-M-R-T], [F-C] and [Lan], this allows one to use *Mumford's construction* to glue all the pieces together. We do not reproduce this by-now-classical construction, but remark that the key to it is the presence of a polarization, which allows, at a crucial step, to use Grothendieck's algebraization theorem.

3.1.3. *The torsor* Ξ. As our purpose is to establish just enough notation to be able to study Θ at the cusps, and as this will be done only at the *standard* cusps, we shall explain now the third and final step in the construction of $\mathfrak{Z}([g]_r,\sigma)$ under the assumption that $g = 1$. The general case can be treated in a similar manner, transporting all structures by g. Even if necessary for applications, it does not add much conceptually.

Assume therefore that the cusp label is $[g]_r = [1]_r$ and drop the g from the notation. Let

$$X = \mathrm{Hom}(\Lambda \cap V_r, \mathbb{Z}), \quad Y = \Lambda/(\Lambda \cap V_r^{\perp}).$$

Let $\phi_X : Y \simeq X$ be the isomorphism given by $\phi_X(u)(v) = \langle u, v\rangle$. It satisfies $\phi_X(au) = \overline{a}\phi_X(u)$. If $c \in C = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}_r^t)$ we denote by $c^t \in \mathrm{Hom}_{\mathcal{O}_E}(Y, \mathcal{A}_r)$ the unique homomorphism satisfying $\phi_r \circ c^t = c \circ \phi_X$, where $\phi_r : \mathcal{A}_r \simeq \mathcal{A}_r^t$ is the tautological principal polarization of the abelian scheme $\mathcal{A}_r$ over $S_{\boldsymbol{G}_r, K_{r,g}}$.

$$\begin{tikzcd} X \arrow[r, "c"] & \mathcal{A}_r^t \\ Y \arrow[u, "\phi_X"] \arrow[r, "c^t"] & \mathcal{A}_r \arrow[u, "\phi_r"'] \end{tikzcd}$$

We construct a torus T_H and use it to define a T_H-torsor Ξ over C which will be basic for the construction of the local charts below. Let

$$H = Z_r(\mathbb{Q}) \cap K$$

where Z_r, as before, is the center of the unipotent radical of P_r, and K the level subgroup. Let $\check{H} = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z})$ and

$$T_H = H \otimes \mathbb{G}_{m/W} = \mathrm{Spec}(W[\check{H}]),$$

the split torus over the Witt vectors with character group[2] $\check{H}$ and cocharacter group H. There is another useful way to think of H, as a rank-r^2 lattice of hermitian

[2][S-U] denote $\check{H}$ by S.

bilinear forms on Y (the lattice shrinking as the level increases), *cf.* [S-U] §5.4.1. Simply attach to $h \in H$ the hermitian form $b_h : Y \times Y \to \mathfrak{d}_E^{-1}$ defined by

$$b_h(y, y') = \delta_E^{-1}((h-1)y, y'). \tag{3.1.2}$$

Here $(,)$ is the pairing on $V_r \times (V/V_r^\perp)$ induced from (1.1.2). Using the description of Z_r in (3.1.1) we may regard $h \mapsto b_h$ as assigning to $h \in H$ the matrix $\delta_E^{-1}B$.

We denote by Ξ the T_H-torsor over $C = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}_r^t)$ constructed in [S-U], smooth of total dimension mn. Recall that given such a torsor, every character $\chi \in \check{H}$ of T_H determines, by push-out, a $\mathbb{G}_m$-torsor Ξ_χ over C, and the resulting map

$$\chi \mapsto [\Xi_\chi]$$

from $\check{H}$ to the group of $\mathbb{G}_m$-torsors over C is a homomorphism. Conversely, Ξ is uniquely determined by giving such a homomorphism. We proceed to describe Ξ in this way.

If $y, y' \in Y$ let $\chi = [y \otimes y']$ denote the element of $\check{H}$ which sends

$$H \ni h \mapsto Tr_{E/\mathbb{Q}} b_h(y, y') = \langle (h-1)y, y' \rangle \in \mathbb{Z}.$$

Then we require $\Xi_\chi|_c$, the fiber at $c \in C$ of Ξ_χ, to be

$$\mathcal{P}|^\times_{c(\phi_X(y)) \times c^t(y')},$$

where $\mathcal{P}$ is the Poincare bundle over $\mathcal{A}_r^t \times \mathcal{A}_r$. The superscript $\times$ means "the associated $\mathbb{G}_m$-bundle", obtained by removing the zero section. It can be checked that this extends to a homomorphism from $\check{H}$ to the group of $\mathbb{G}_m$-torsors over C. For any $\chi \in \check{H}$ we let $\mathcal{L}(\chi)$ be the line bundle on C whose associated $\mathbb{G}_m$-bundle is Ξ_χ. Over the complex numbers, sections of $\mathcal{L}(\chi)$ are classical theta functions on the abelian scheme C. We shall often denote elements of $\check{H}$ also by $\check{h}$. We have a canonical identification $\mathcal{L}(\check{h}_1 + \check{h}_2) = \mathcal{L}(\check{h}_1) \otimes \mathcal{L}(\check{h}_2)$.

Having constructed Ξ we proceed to study its equivariance properties under the group

$$\Gamma = GL(V_r)(\mathbb{Q}) \cap K.$$

Using (3.1.1), this is the group of rational matrices A that also lie in K. Since the action of P_r on Z_r by conjugation factors through $P_r/U_r = L_r$, the group $\Gamma \subset L_r$ acts on Z_r. Using (3.1.1) again, A sends B to $AB\,{}^t\overline{A}$. In particular Γ acts on H, hence it acts on T_H by automorphisms of the torus.

We also have an action of Γ on $C = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}_r^t)$ induced form its action on X. Any $\gamma \in \Gamma$ maps $\mathcal{L}(\check{h})|_c$ to $\mathcal{L}(\gamma(\check{h}))|_{\gamma(c)}$. If $\Gamma(\check{h})$ is the stabilizer of $\check{h} \in \check{H}$ then ([S-U] Lemma 5.4.2) $\Gamma(\check{h})$ acts trivially on the global sections of $\mathcal{L}(\check{h})$ over C.

Finally, as the push-out of $\Xi|_{\gamma(c)}$ by $[\gamma(y) \otimes \gamma(y')]$ is identically the same as the push out of $\Xi|_c$ by $[y \otimes y']$, or equivalently

$$\Xi|_{\gamma(c)} = (\Xi \times^{T_H, \gamma} T_H)|_c$$

the isomorphism $1 \times \gamma : \Xi = \Xi \times^{T_H} T_H \to \Xi \times^{T_H, \gamma} T_H$ of torsors *over* C, yields an action of Γ on Ξ which *covers* its action on C, and is compatible with the Γ-action on T_H. In short, all the constructions so far are equivariant under Γ.

3.1.4. *The local charts.* Now comes the choice of the auxiliary data involved in the toroidal compactification. Let

$$H_{\mathbb{R}}^{+} \subset H_{\mathbb{R}}$$

be the cone of positive semi-definite hermitian bilinear forms on $Y_{\mathbb{R}}$ whose radical is a subspace defined over $\mathbb{Q}$ (i.e. the $\mathbb{R}$-span of a subspace of $Y_{\mathbb{Q}}$). Let $\Sigma = \{\sigma\}$ be a Γ-admissible (infinite) rational polyhedral cone decomposition of $H_{\mathbb{R}}^{+}$ ([Lan], Definition 6.1.1.10). Admissibility means that the action of Γ on $H_{\mathbb{R}}$ permutes the σ's, and that modulo Γ there are only finitely many cones in Σ. By convention, the cones σ do not contain their proper faces, and every face of a cone in Σ also belongs to Σ. In particular, Σ contains the origin as its unique 0-dimensional cone. When we treat all cusp labels, and not only one at a time, an additional assumption has to be imposed about the compatibility of the polyhedral cone decompositions associated with a cusp ξ and with a higher rank cusp to which ξ specializes. It is a non-trivial fact that such polyhedral cone decompositions exist. In the Siegel case, see Chapter 2 of [A-M-R-T]. Moreover, every two Γ-admissible rational polyhedral cone decompositions of $H_{\mathbb{R}}^{+}$ have a common refinement of the same sort. One can even find such a polyhedral cone decomposition in which every σ is spanned by a part of a basis of H. The $T_{H,\sigma}$ defined below will then be smooth over W, and from now on we assume that this is the case. Lan [Lan] calls such a Σ a Γ-*admissible smooth rational polyhedral cone decomposition of* $H_{\mathbb{R}}^{+}$. If K is large enough so that Γ is *neat*, refinements exist such that, in addition, the closures of σ and $\gamma(\sigma)$, for $\sigma \in \Sigma$ and $1 \neq \gamma \in \Gamma$, meet only at the origin.

Each cone $\sigma \in \Sigma$ defines a torus embedding

$$T_H \hookrightarrow T_{H,\sigma} = \mathrm{Spec}(W[\check{H} \cap \sigma^{\vee}])$$

where $\sigma^{\vee} \subset \check{H}_{\mathbb{R}}$ is the dual cone and $W = W(\kappa)$ as before. By definition

$$\sigma^{\vee} = \{v \in \check{H}_{\mathbb{R}} |\, v(u) \geq 0,\, \forall u \in \sigma\},$$

so, unlike σ, $\sigma^{\vee}$ contains its faces. Observe that T_H naturally acts on $T_{H,\sigma}$. Since σ does not contain a line, $\sigma^{\vee}$ has a non-empty interior.

Let

$$\sigma^{\perp} = \{v \in \check{H}_{\mathbb{R}} |\, v(u) = 0,\, \forall u \in \sigma\}.$$

When $d_\sigma = \dim(\sigma) < r^2$, $\sigma^{\vee} \supset \sigma^{\perp} \neq 0$. Then $Z_{H,\sigma} = \mathrm{Spec}(W[\check{H} \cap \sigma^{\perp}])$ is a torus, $\dim Z_{H,\sigma} = r^2 - d_\sigma$. In fact, $Z_{H,\sigma}$ is the unique minimal orbit of T_H in its action on $T_{H,\sigma}$, an orbit which lies in the closure of any other orbit. There is an obvious surjection $T_{H,\sigma} \twoheadrightarrow Z_{H,\sigma}$. This surjection admits a section $Z_{H,\sigma} \hookrightarrow T_{H,\sigma}$, corresponding to $W[\check{H} \cap \sigma^{\perp}] \simeq W[\check{H} \cap \sigma^{\vee}]/I_\sigma$, where I_σ is the ideal generated by $\check{H} \cap \sigma^{\vee} \smallsetminus \check{H} \cap \sigma^{\perp}$. Another way to think of $Z_{H,\sigma}$ is as

$$Z_{H,\sigma} = T_{H,\sigma} \smallsetminus \bigcup_{\tau < \sigma} T_{H,\tau}$$

where τ runs over all the *proper* faces of σ.

The $T_{H,\sigma}$ glue to form a toric variety (locally of finite type, but not of finite type in general) $T_{H,\Sigma}$, in which each $T_{H,\sigma}$ is open and dense:

$$T_{H,\Sigma} = \bigcup_{\sigma \in \Sigma} T_{H,\sigma}.$$

This $T_{H,\Sigma}$ is stratified by the disjoint union of the $Z_{H,\sigma}$. The actions of Γ on H and Σ induce an action of Γ on T_H and a compatible action on $T_{H,\Sigma}$. By our assumption on Σ, $T_{H,\Sigma}$ is smooth over W.

We "spread" this construction over $C = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}_r^t)$, twisting it by the torsor Ξ, namely we consider

$$\overline{\Xi}_\Sigma = \Xi \times^{T_H} T_{H,\Sigma}. \tag{3.1.3}$$

The group Γ acts on each of the three symbols on the right in a compatible way, so we get an action of Γ on $\overline{\Xi}_\Sigma$.

Let us bring back the reference to the cusp label $[g]_r$, although in the above we tacitly assumed $[g]_r = 1$ and dropped g from the notation. See [S-U], §5.4.1 for the precise definition of H_g, Σ_g etc. Denote by $H^{++}_{g,\mathbb{R}}$ the set of *positive-definite* hermitian bilinear forms in $H^+_{g,\mathbb{R}}$. For $\sigma \in \Sigma_g$ such that $\sigma \subset H^{++}_{g,\mathbb{R}}$ we let

$$Z([g]_r, \sigma) = \Xi \times^{T_H} Z_{H,\sigma},$$

and let

$$\mathfrak{Z}([g]_r, \sigma)$$

be the formal completion of $\overline{\Xi}_\Sigma$ or, what amounts to be the same, of its open subset $\Xi \times^{T_H} T_{H,\sigma}$, along $Z([g]_r, \sigma)$. These are the local charts at the cuspidal component labeled by $[g]_r$. There is a smooth morphism

$$\mathfrak{Z}([g]_r, \sigma) \to C([g]_r)$$

whose fibers are isomorphic to the completion of $T_{H,\sigma}$ along $Z_{H.\sigma}$. The $\mathfrak{Z}([g]_r, \sigma)$ are nm-dimensional and smooth over W. Each such local chart has $nm - d_\sigma$ "algebraic dimensions" and d_σ "formal dimensions". Specializing the formal variables to 0, one gets the support $Z([g]_r, \sigma)$ of $\mathfrak{Z}([g]_r, \sigma)$, whose dimension is $nm - d_\sigma$. The action of $\gamma \in \Gamma$ on $\overline{\Xi}_\Sigma$ induces an isomorphism γ_* between $\mathfrak{Z}([g]_r, \sigma)$ and $\mathfrak{Z}([g]_r, \gamma(\sigma))$. For comparison, we remark that in [Lan] §6.2.5 the $\mathfrak{Z}([g]_r, \sigma)$ are denoted $\mathfrak{X}_{\Phi_\mathcal{H}, \delta_\mathcal{H}, \sigma}$ and $Z([g]_r, \sigma)$ are denoted $\Xi_{\Phi_\mathcal{H}, \delta_\mathcal{H}, \sigma}$. Also, under our assumptions the stabilizers denoted in [Lan] by $\Gamma_{\Phi_\mathcal{H}, \sigma}$ are trivial.

Once we have described the local charts, it remains to construct on each of them the *degeneration data* which allows one to carry on the *Mumford construction*. This results in gluing the various charts together, and at the same time constructing $\mathcal{G}$ with the accompanying PEL structure over the glued scheme. Care has to be taken not only to glue pieces labeled by the same cusp label $[g]_r$, but also to respect the way cusp labels specialize. In the process of gluing, one has to divide by the action of Γ on the formal completion of (3.1.3) along the complement of $\Xi = \Xi \times^{T_H} T_H$. Note that it does not make sense to divide $\overline{\Xi}_\Sigma$ by Γ, just as it did not make sense to divide Ξ, or the abelian scheme C over which it lies, by the action of Γ. For the gluing of the local charts, that we do not review here, see [Lan] §6.3. The final result is [Lan] Theorem 6.4.1.1.

3.1.5. *Logarithmic differentials.* We construct certain formal differentials on the local chart $\mathfrak{Z}([g]_r, \sigma)$, relative to $C([g]_r)$, with logarithmic poles along $Z([g]_r, \sigma)$. We shall denote the module of these differentials

$$\Omega_{\mathfrak{Z}/C}[d \log \infty].$$

They will play an important role in our formulae for Θ.

Notation as above, consider a cone $\sigma \subset H^{++}_{g,\mathbb{R}}$ and let $h_1, \dots, h_{d_\sigma}$ be positive semi-definite, part of a basis of $H = H_g$, such that

$$\sigma = Cone(h_1, \dots, h_{d_\sigma}).$$

Complete the h_i to a basis $h_1, \dots, h_{r^2}$ of H, let $\{\check{h}_i\}$ be the dual basis of $\check{H} = \mathrm{Hom}(H, \mathbb{Z})$ and introduce formal variables $q_i = q^{\check{h}_i}$ (to be able to write the group structure on $\check{H}$ multiplicatively rather than additively). Then

$$T_{H,\sigma} = \mathrm{Spec}(W[q_1, \dots, q_{d_\sigma}, q^{\pm 1}_{d_\sigma+1}, \dots, q^{\pm 1}_{r^2}])$$

and

$$Z_{H,\sigma} = \mathrm{Spec}(W[q^{\pm 1}_{d_\sigma+1}, \dots, q^{\pm 1}_{r^2}]).$$

Locally on $\mathfrak{Z}([g]_r, \sigma)$ we use as coordinates the pull-back of any system of $nm - r^2$ local coordinates on the base $C = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}^t_r)$, together with the "algebraic" coordinates $q_{d_\sigma+1}, \dots, q_{r^2}$, and the "formal" coordinates $q_1, \dots, q_{d_\sigma}$. We emphasize that because of the twist by the torsor Ξ in the construction of the local charts, the q_i are not global coordinates. The correct way to think of them is as local sections of the line bundles $\mathcal{L}(-\check{h}_i)$ on C. If the h_i are positive definite, these line bundles will be anti-ample, and the q_i will not globalize.

If $\check{h} \in \check{H}$ is of the form $\check{h} = \sum n_i \check{h}_i$ we write $q^{\check{h}} = \prod q_i^{n_i}$ and define

$$\omega(\check{h}) = \frac{dq^{\check{h}}}{q^{\check{h}}} = \sum_{i=1}^{r^2} n_i \frac{dq_i}{q_i} \in \Omega_{\mathfrak{Z}/C}[d\log\infty].$$

This $\omega(\check{h})$ is invariant under the action of T_H, essentially since $d\log(q_0 q) = d\log q$. Hence, despite the fact that the q_i were only *local* coordinates, $\omega(\check{h})$ defines a relative differential on all of $\Xi \times^{T_H} T_{H,\sigma}$, as well as on its completion $\mathfrak{Z}([g]_r, \sigma)$ along $Z([g]_r, \sigma) = \Xi \times^{T_H} Z_{H,\sigma}$, with logarithmic poles along $Z([g]_r, \sigma)$. The following proposition is an immediate by-product of the theory of toroidal compactifications.

Proposition 3.1.1. *(i) The differentials $\omega(\check{h})$ are well-defined formal differentials on $\mathfrak{Z}([g]_r, \sigma)$, relative to $C([g]_r)$, with logarithmic poles along $Z([g]_r, \sigma)$. They are independent of the choice of bases and depend only on $\check{h}$.*

(ii) $\omega(\check{h}_1 + \check{h}_2) = \omega(\check{h}_1) + \omega(\check{h}_2)$.

(iii) The differentials $\omega(\check{h})$ are compatible with gluing of the local charts. If $\gamma \in \Gamma$ then the induced isomorphism between the local charts $\mathfrak{Z}([g]_r, \sigma)$ and $\mathfrak{Z}([g]_r, \gamma(\sigma))$ carries $\omega(\check{h})$ to $\omega(\gamma(\check{h}))$.

(iv) The differentials $\omega(\check{h})$ are compatible with the maps between toroidal compactifications obtained from refinements of the admissible smooth rational polyhedral cone decompositions, as in [Lan] *§6.4.2.*

3.1.6. *Fourier-Jacobi expansions.* Let $\overline{S}$ be a fixed smooth toroidal compactification of S over W_s $(1 \le s)$ as a base ring. Let $\mathcal{G}$ be the universal semi-abelian scheme over $\overline{S}$ and $e_{\mathcal{G}} : \overline{S} \to \mathcal{G}$ its zero section. Then $\omega = e^*_{\mathcal{G}} \Omega^1_{\mathcal{G}/\overline{S}}$ defines an extension of the Hodge bundle to a rank $n + m$ vector bundle with $\mathcal{O}_E$-action on $\overline{S}$. We continue to denote by $\mathcal{P}$ and $\mathcal{Q}$ its sub-bundles of type Σ and $\overline{\Sigma}$, of ranks n and m respectively.

Let $\overline{S}^{\mathrm{ord}}$ denote the complement in $\overline{S}$ of the Zariski closure of $S \smallsetminus S^{\mathrm{ord}}$. Over this open subset of $\overline{S}$ the semi-abelian variety $\mathcal{G}$ is μ-ordinary in the sense that the connected part of its p-divisible group at every geometric point $x : \mathrm{Spec}(k) \to \overline{S}^{\mathrm{ord}}$ satisfies

$$\mathcal{G}_x[p^\infty]^0 \simeq (\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m \times \mathfrak{G}_k^{n-m}.$$

To see this, assume that x lies on a rank r cuspidal component, but that the abelian part $\mathcal{A}_x$ of $\mathcal{G}_x$ is not μ-ordinary, i.e. the multiplicative part of $\mathcal{A}_x[p^\infty]$ has height strictly less than $2(m-r)$. Mumford's construction shows that we may deform $\mathcal{G}$ into an abelian variety $\mathcal{A}_y$ (y signifying a point on the base of the deformation "near" x) so that the multiplicative part of $\mathcal{A}_y[p^\infty]$ has height strictly less than $2m$. But such a point y being non-μ-ordinary, we conclude that x lies in the closure of $S \smallsetminus S^{\mathrm{ord}}$, contrary to our assumption.

It follows that the filtration

$$0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0$$

extends to a filtration by a sub-vector bundle over $\overline{S}^{\mathrm{ord}}$. Thus the automorphic vector bundles $\mathcal{E}_\rho$ defined in §1.2.3 extend to $\overline{S}^{\mathrm{ord}}$ too. In the following discussion fix the representation ρ.

Let $[g]_r \in \mathscr{C}_r$ be a cusp label of rank $0 < r \leq m$ and let $Z = Z([g]_r)$ be the corresponding cuspidal component of ∂S obtained by "gluing" the $Z([g]_r, \sigma)$ for $\sigma \in \Sigma_g$, $\sigma \subset H_{g,\mathbb{R}}^{++}$, and dividing by Γ. Let $\mathfrak{Z}([g]_r)$ be the formal completion of $\overline{S}$ along $Z([g]_r)$. Let $\xi \in S_{\boldsymbol{G}_r, K_{r,g}}$ be a geometric point, and let Z_ξ be the pre-image of ξ in Z. Then Z_ξ is obtained by "gluing" the pre-image $Z([g]_r, \sigma)_\xi$ of ξ in $Z([g]_r, \sigma)$ for all σ as above, dividing by the action of Γ. Observe that the toric part T_X and the abelian part $\mathcal{A}_{r,\xi}$ of $\mathcal{G}$ are constant over each $Z([g]_r, \sigma)_\xi$. Thus $\mathcal{P}_0, \mathcal{P}_\mu$ and $\mathcal{Q}$ are trivialized over the pre-image $\mathfrak{Z}([g]_r, \sigma)_\xi$ of ξ in the local chart $\mathfrak{Z}([g]_r, \sigma)$, hence so is $\mathcal{E}_\rho$. In general, however, $\mathcal{E}_\rho$ will not be trivial over $\mathfrak{Z}([g]_r)_\xi$.

Our ξ is a point of the minimal compactification S^* (over W_s). The completed local ring $\widehat{\mathcal{O}}_{S^*,\xi}$ is described in [S-U] Theorem 5.4.7 and [Lan] Proposition 7.2.3.16. In the following, let $\check{H}^+$ be the set of elements of $\check{H}$ which are non-negative on $H_{\mathbb{R}}^+$.

Proposition 3.1.2. *There is a canonical isomorphism between $\widehat{\mathcal{O}}_{S^*,\xi}$ and the ring $\mathcal{FJ}_\xi$ of all formal power series*

$$f = \sum_{\check{h} \in \check{H}^+} a(\check{h}) q^{\check{h}}$$

which are invariant under Γ. Here $a(\check{h}) \in H^0(C_\xi, \mathcal{L}(\check{h}))$ where $C_\xi = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}_{r,\xi}^t)$ is the abelian variety which is the fiber of C over ξ.

Recall that $\pi : \overline{S} \to S^*$ was the map between the toroidal compactification and the minimal one. There is a similar description of the completion of the stalk of $\pi_* \mathcal{E}_\rho$ at ξ ([S-U] Proposition 5.5.4).

Proposition 3.1.3. *The completion of $(\pi_* \mathcal{E}_\rho)_\xi$ is canonically isomorphic to the $\widehat{\mathcal{O}}_{S^*,\xi}$-module of formal power series*

$$f = \sum_{\check{h} \in \check{H}^+} a(\check{h}) q^{\check{h}}$$

which are invariant under Γ. Here $a(\check{h}) \in H^0(C_\xi, \mathcal{L}(\check{h})) \otimes_W \rho(W)$.

The action of Γ on $a(\check{h})$ demands an explanation[3], and for that we must bring back the dependence on $[g]_r \in \mathscr{C}_r$ and even on g itself. Still assuming that we are at the standard cusp, i.e. $[g]_r = [1]_r$, we may replace the representative $g = 1$ by $g = \gamma \in \Gamma = GL(V_r) \cap K = GL_r(E) \cap K$. The following changes then take place. The lattice $\Lambda \cap V_r$ is replaced by $\gamma(\Lambda \cap V_r) = \Lambda \cap V_r$, so does not change. The subgroups X and Y therefore remain the same, but γ acts on them non-trivially. This induces an action of γ on the abelian variety $C_\xi = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathcal{A}^t_{r,\xi})$ classifying the extensions $\mathcal{G}$ of $\mathcal{A}_{r,\xi}$ by T_X, as well as an action on the torus T_X. Thus γ induces an isomorphism

$$\gamma_* : \mathcal{G}_c \simeq \mathcal{G}_{\gamma(c)}$$

($c \in C_\xi$), which on the toric part is the given automorphism of T_X, and on the abelian part induces the identity. This induces isomorphisms $\gamma_* = (\gamma^*)^{-1}$ from the fibers of $\mathcal{P}_0, \mathcal{P}_\mu$ and $\mathcal{Q}$ at c to the corresponding fibers at $\gamma(c)$. As $\mathcal{P}_0$ depends only on the abelian part, the action of γ_* on it is trivial. Assume, for simplicity, that $\mathcal{E}_\rho = \mathcal{P}_\mu$. Then $a(\check{h})$ is a section (over C_ξ) of $\mathcal{L}(\check{h}) \otimes \mathcal{P}_\mu$ and $\gamma(a(\check{h}))$ will be the section of $\mathcal{L}(\gamma\check{h}) \otimes \mathcal{P}_\mu$ satisfying

$$\gamma(a(\check{h}))|_{\gamma(c)} = \gamma_*(a(\check{h})|_c).$$

In the sequel we shall only need the case of the *maximally degenerate* cusps, i.e. $r = m$. Now the Shimura variety $S_{\boldsymbol{G}_m, K_{m,g}}$ is 0-dimensional, and ξ is one of its (schematic) points. The abelian variety C_ξ is $m(n-m)$-dimensional. In this case $\mathcal{P}_\mu$ is the Σ-part of the cotangent space at the origin to

$$T_X = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m)$$

(if $r < m$ it also captures part of the cotangent space at the origin of $\mathcal{A}_{r,\xi}$). In other words, we may identify

$$\mathcal{P}_\mu \simeq \mathcal{O}_{C_\xi} \otimes_{\Sigma, \mathcal{O}_E} X$$

and $\gamma_* : \mathcal{P}_\mu|_c \to \mathcal{P}_\mu|_{\gamma(c)}$ with $\gamma_* \otimes \gamma_*$. Similarly we may identify $\mathcal{Q} \simeq \mathcal{O}_{C_\xi} \otimes_{\overline{\Sigma}, \mathcal{O}_E} X$. As the action of γ on X is via the contragredient $\mathrm{st}^\vee$of the standard representation, it follows that to obtain the action of $\gamma \in \Gamma$ on $\rho(W)$ in general, we have first to embed γ as $\iota^\vee(\gamma) := ({}^t\overline{\gamma}^{-1}, {}^t\gamma^{-1}, 1)$ in $GL_m \times GL_m \times GL_{n-m}$. (Recall that these three factors correspond to $\mathcal{Q}, \mathcal{P}_\mu$ and $\mathcal{P}_0$ in this order, see §1.2.3.) The action of γ on $a(\check{h}) \in H^0(C_\xi, \mathcal{L}(\check{h})) \otimes_W \rho(W)$ will then be via $\gamma_* \otimes \rho(\iota^\vee(\gamma))$.

We remark that when $n = m$ the Fourier-Jacobi expansions are in fact Fourier expansions in the naive sense, and the $a(\check{h})$ are scalars.

3.1.7. *The Fourier-Jacobi expansion of the Hasse invariant.* Assume now that $s = 1$, i.e. we are again over the special fiber in characteristic p, and the automorphic vector bundle is the line bundle $\mathcal{L}^{p^2-1}$ where $\mathcal{L} = \det(\mathcal{Q})$. Let $h \in H^0(S, \mathcal{L}^{p^2-1})$ be the Hasse invariant, previously denoted h (1.1.5).

Proposition 3.1.4. *The Fourier-Jabobi expansion of h at a rank-m cusp is* 1.

Proof. Let us check the claim at the standard cusp. Fix a local chart $\mathfrak{Z}([1]_m, \sigma)$ as above. As we have seen, $\mathcal{Q}$, hence also $\mathcal{L}$, are trivialized there. The trivialization is obtained from a similar trivialization of the p-divisible group of the toric part T_X of the semi-abelian variety $\mathcal{G}$. As the isogeny Ver acts like the identity on $T_X[p^\infty]$, the Hasse invariant maps a trivializing section ℓ of $\mathcal{L}$ over $\mathfrak{Z}([1]_m, \sigma)$ to $\ell^{(p^2)}$. It

[3]Missing in [S-U]; [Lan §7.1] only treats scalar-valued modular forms.

follows that in terms of the basis ℓ^{p^2-1} of $\mathcal{L}^{p^2-1}$, its FJ expansion is simply 1. Note that a choice of another κ-rational section ℓ will result in the same value for h. $\square$

Corollary 3.1.5. *The open set $\overline{S}^{\mathrm{ord}} \subset \overline{S}$ is the non-vanishing locus of h.*

Proof. By definition, $\overline{S}^{\mathrm{ord}}$ is the complement of the Zariski closure of S^{no}, which is the vanishing locus of h in S^{ord}. It is therefore clear that h vanishes on its complement, and to prove the corollary it is enough to check that h does not vanish on any irreducible component of $\partial S = \overline{S} \setminus S$. But any such irreducible component contains a rank m cusp, so the claim follows from the previous Proposition. $\square$

3.2. **Analytic continuation of Θ to the boundary and its effect on Fourier-Jacobi expansions.**

3.2.1. *The partial toroidal compactification of the Igusa scheme.* Fix $s \geq 1$ and work over W_s as a base ring. Since the semi-abelian scheme $\mathcal{G}$ over $\overline{S}_s^{\mathrm{ord}}$ is μ-ordinary, the *relative* moduli problem defining the big Igusa scheme of level p^t makes sense over $\overline{S}_s^{\mathrm{ord}}$. More precisely, for an R-valued point of $\overline{S}_s^{\mathrm{ord}}$ denote by $\mathcal{G}_R$ the pull-back of $\mathcal{G}$ to $\mathrm{Spec}(R)$. Then $\mathcal{G}_R[p^\infty]^0$ is still isomorphic, locally in the pro-étale topology on $\mathrm{Spec}(R)$, to an extension of $\mathfrak{G}^{n-m}$ by $(\mathfrak{d}_E^{-1} \otimes \mu_{p^\infty})^m$. The relative moduli problem $\overline{T}_{t,s}$ classifies Igusa structures (ϵ^1, ϵ^2) on $\mathcal{G}_R$ as in (2.1.1). The compatibility with Weil pairings is imposed on ϵ^1 only, as there is no ϵ^0 to pair with ϵ^2. This makes sense even if $\mathcal{G}_R$ is not an abelian scheme, while when it is, ϵ^0 is determined by ϵ^2. We call the resulting scheme $\overline{T}_{t,s}$. The following proposition is then obvious.

Proposition 3.2.1. *(i) The partially compactified Igusa scheme $\overline{T}_{t,s}$ is a finite étale Galois cover of $\overline{S}_s^{\mathrm{ord}}$ with Galois group Δ_t.*

(ii) If $t \geq s$ then the basic vector bundles $\mathcal{P}_0, \mathcal{P}_\mu$ and $\mathcal{Q}$ are canonically trivialized over $\overline{T}_{t,s}$.

We continue to denote by $\tau : \overline{T}_{t,s} \to \overline{S}_s^{\mathrm{ord}}$ the covering map and by $\varepsilon^1, \varepsilon^2$ the resulting trivializations over $\overline{T}_{t,s}$. The definition of $\widetilde{\Theta}$ over $\overline{S}_s^{\mathrm{ord}}$ is then precisely the same as over the open ordinary stratum S_s^{ord}, *cf.* (2.2.2).

3.2.2. *The extended Θ operator.* To extend the definition of Θ we need to recall how the Kodaira-Spencer isomorphism extends to the toroidal compactification. The answer is given by [Lan], Theorem 6.4.1.1, part 4. See also [F-C], Ch. III, Corollary 9.8. In our case (see [Lan], Definition 6.3.1) it translates into the following.

Proposition 3.2.2. *The Kodaira Spencer isomorphism extends to an isomorphism*

$$\mathrm{KS} : \mathcal{P} \otimes \mathcal{Q} \simeq \Omega^1_{\overline{S}/W}[d\log\infty]$$

over $\overline{S}$.

The inverse isomorphism KS^{-1} therefore maps $\Omega^1_{\overline{S}/W}$ to sections of $\mathcal{P} \otimes Q$ *vanishing along the boundary ∂S*. We deduce the following.

Proposition 3.2.3. *The formula*

$$\Theta = (1 \otimes pr_\mu \otimes 1) \circ (1 \otimes \mathrm{KS}^{-1}) \circ \widetilde{\Theta} : \mathcal{E}_\rho \to \mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}$$

defines an extension of Θ over $\overline{S}^{\mathrm{ord}}$. For any section f of $\mathcal{E}_\rho$, $\Theta(f)$ vanishes along $\partial\overline{S}^{\mathrm{ord}}$.

3.2.3. *The isomorphism between $\mathcal{P}_\mu \otimes \mathcal{Q}$ and $\check{H} \otimes \mathcal{O}_{\overline{S}}$ when $r = m$.* We now turn to determining the effect of Θ on Fourier-Jacobi expansions. This will be done at *maximally degenerate* cusps only. We therefore take $r = m$ and denote by $\xi \in S^*$ a cusp of rank m. Note that there are only finitely many such cusps. Nevertheless, there are sufficiently many of them to lie in every irreducible component of $\overline{S}$. This will allow us to apply the q-expansion principle with these cusps only, not having to worry about expansions at lower rank cusps, where the formulae are not as nice.

Lemma 3.2.4. *Let $x \in \overline{S}$ be any point lying above ξ. Let g be a representative of the cusp label $[g]_m$ to which ξ belongs, $H = H_g$ the rank-m^2 lattice of hermitian bilinear forms on $Y = Y_g$ as in §3.1.4, and $\check{H}$ its $\mathbb{Z}$-dual. Then there is a canonical identification of the completed stalk $(\mathcal{P}_\mu \otimes \mathcal{Q})^\wedge_x$ with $\check{H} \otimes \widehat{\mathcal{O}}_{\overline{S},x}$,*

$$(\mathcal{P}_\mu \otimes \mathcal{Q})^\wedge_x \simeq \check{H} \otimes \widehat{\mathcal{O}}_{\overline{S},x}. \tag{3.2.1}$$

This identification is compatible with the natural action of Γ on both sides.

Proof. Let $R = \widehat{\mathcal{O}}_{\overline{S},x}$. It is enough to deal with the standard cusp. When $r = m$ the stalks of the vector bundles $\mathcal{P}_\mu$ and $\mathcal{Q}$ are the Σ and $\overline{\Sigma}$-parts of ω_{T_X}, the cotangent bundle of the toric part of $\mathcal{G}$. Since $T_X = \mathrm{Hom}_{\mathcal{O}_E}(X, \mathfrak{d}_E^{-1} \otimes \mathbb{G}_m)$, it follows that there are canonical identifications

$$\mathcal{P}^\wedge_{\mu,x} = X \otimes_{\mathcal{O}_E,\Sigma} R, \ \mathcal{Q}^\wedge_x = X \otimes_{\mathcal{O}_E,\overline{\Sigma}} R.$$

The map $Y \otimes Y \to \check{H}$ described in the course of the construction of the torsor Ξ in §3.1.4 yields an isomorphism

$$(Y \otimes_{\mathcal{O}_E,\Sigma} R) \otimes_R (Y \otimes_{\mathcal{O}_E,\overline{\Sigma}} R) \simeq \check{H} \otimes R = \mathrm{Hom}(H, R).$$

Explicitly, $(y \otimes 1) \otimes (y' \otimes 1)$ goes to the map sending $h \in H$ to $((h-1)y', y)$. Using the isomorphism $\phi_X : Y \simeq X$ we get the isomorphism (3.2.1).

Let us verify that the isomorphism given in the lemma is compatible with the natural actions of our group Γ on $\mathcal{P}_\mu \otimes \mathcal{Q}$ and $\check{H}$. At the end of §3.1.6 we computed the action of $\gamma \in \Gamma$ on $(\mathcal{P}_\mu \otimes \mathcal{Q})^\wedge_x$ to be through ${}^t\gamma^{-1} \times {}^t\overline{\gamma}^{-1} \in GL_m \times GL_m$. On the other hand, γ acts on $h \in H$ via $h \mapsto \gamma h\, {}^t\overline{\gamma}$. As $\check{H}$ is the $\mathbb{Z}$-dual of H, these actions match each other. $\square$

3.2.4. *The main theorem.*

Theorem 3.2.5. *Let ξ be a rank-m cusp. Let f be a section of $\mathcal{E}_\rho$ and*

$$f = \sum_{\check{h} \in \check{H}^+} a(\check{h}) \cdot q^{\check{h}}$$

its Fourier-Jacobi expansion at ξ, as in Proposition 3.1.3. Then the section $\Theta(f)$ of $\mathcal{E}_\rho \otimes \mathcal{P}_\mu \otimes \mathcal{Q}$ has the Fourier-Jacobi expansion

$$\Theta(f) = \sum_{\check{h} \in \check{H}^+} a(\check{h}) \otimes \check{h} \cdot q^{\check{h}},$$

using the identification from Lemma 3.2.4.

Proof. We may work over W_s as a base ring. Fix any $t \geq s$ and let $\overline{T} = \overline{T}_{t,s}$. We assume that ξ is the standard cusp of rank m (regarded as a k-valued point of S^*, where k is algebraically closed and contains κ), and fix a geometric point x in the toroidal compactification lying above it. Fix a local chart $\mathfrak{Z}([g]_m, \sigma)$ containing x

(where $[g]_m = [1]_m$ by our assumptions) and let $\mathfrak{Z}([g]_m, \sigma)_\xi$ be the pre-image of ξ in it. As the abelian and toric parts of $\mathcal{G}$ are constant over $\mathfrak{Z}([g]_m, \sigma)_\xi$ we may fix admissible trivializations ϵ^1 and ϵ^2 of the graded pieces gr^1 and gr^2 of $\mathcal{G}[p^t]^0$, over the complete local ring at x. Indeed, the point ξ on the 0-dimensional Shimura variety $S_{\boldsymbol{G}_m, K_{m,g}}$ corresponds to an $n-m$ dimensional abelian variety $\mathcal{A}_m$ over the algebraically closed field k, with associated PEL structure of signature $(n-m, 0)$. Fix a symplectic trivialization

$$\epsilon^1 : \mathfrak{G}[p^t]^{n-m} \simeq \mathcal{A}_m[p^t] = gr^1.$$

Similarly, using the standard basis of $\Lambda \cap V_m$ we get a standard basis on X, which gives us a trivialization

$$\epsilon^2 : (\mathfrak{d}_E^{-1} \otimes \mu_{p^t})^m \simeq T_X[p^t] = gr^2.$$

As usual, since $t \geq s$, these trivializations induce trivializations of $\mathcal{P}_0, \mathcal{P}_\mu$ and $\mathcal{Q}$, hence of $\mathcal{E}_\rho$. They also determine a choice of a point $\overline{x}$ in $\overline{T}$ above x. (If σ is replaced by a Γ-equivalent cone $\gamma(\sigma)$ the trivialization ϵ^2 is twisted by the action of γ on X, and this results in a different $\overline{x}$. The choice of ϵ^1 was also arbitrary, and effects the point $\overline{x}$ in a similar way.)

We use $R = \widehat{\mathcal{O}}_{\overline{T}, \overline{x}} \simeq \widehat{\mathcal{O}}_{\overline{S}, x}$ as the ring in which we compute Θ. Recall that the Fourier-Jacobi coefficient $a(\check{h})$ is a section of the line bundle $\mathcal{L}(\check{h}) \otimes \mathcal{E}_\rho$ over the $m(n-m)$-dimensional abelian scheme C, and that $\mathcal{E}_\rho$ has already been trivialized by our choices. Trivializing also the pull-back of the line bundles $\mathcal{L}(\check{h})$ to $\mathrm{Spec}(R)$, we may write the ring R as

$$R = W_s(\overline{\kappa})[[u_1, \ldots, u_{m(n-m)}, q_1, \ldots, q_{m^2}]],$$

where the u_i are pull-backs of local coordinates on C at the image of x, and we may assume that the $a(\check{h})$ are (vector-valued) functions of the u_i. We now have

$$d(\tau^* f) = \sum_{\check{h} \in \check{H}^+} da(\check{h}) \cdot q^{\check{h}} + \sum_{h \in \check{H}^+} a(\check{h}) \cdot \frac{dq^{\check{h}}}{q^{\check{h}}} \cdot q^{\check{h}}.$$

Recall that the image of $dq^{\check{h}}/q^{\check{h}}$ modulo Ω_{C/W_s} is $\omega(\check{h}) \in \Omega_{\mathfrak{Z}/C}[d\log\infty]$. To complete the proof of the theorem we shall show the following two claims.

(1) For any $\eta \in \Omega_{C/W}$, we have $\eta \in \mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{Q})$.

(2) The resulting isomorphism $\mathrm{KS}^{-1} : \Omega_{\mathfrak{Z}/C}[d\log\infty] \simeq \mathcal{P}_\mu \otimes \mathcal{Q} \simeq \check{H} \otimes R$ (see Lemma 3.2.4) carries $\omega(\check{h})$ to $\check{h} \otimes 1$.

Indeed, by (1), when we follow the definition of Θ and mod out by $\mathcal{P}_0 \otimes \mathcal{Q}$, the first sum, containing the $da(\check{h})$'s, disappears. The second sum provides the desired formula, by (2).

Proof of (1): This follows from the discussion of the Kodaira-Spencer map for semi-abelian schemes in [Lan] §4.6.1. Let C assume the role of the base-scheme denoted there by S, and $\mathcal{G}$ the semi-abelian scheme denoted there by $G^\natural$. Then Lan constructs a Kodaira-Spencer map for semi-abelian schemes $\mathrm{KS}_{\mathcal{G}/C}$, which in our case is an isomorphism

$$\mathrm{KS}_{\mathcal{G}/C} : \mathcal{P}_0 \otimes \mathcal{Q} \simeq \Omega_{C/W_s}.$$

Note that Lan allows the abelian part to deform as well, but in our case $\mathcal{A} = \mathcal{A}_m$ is constant. This implies that the Kodaira-Spencer map, which is a-priori defined on $\omega_{\mathcal{A}} \otimes \omega_{\mathcal{G}}$, factors through its quotient $\omega_{\mathcal{A}} \otimes \omega_T$. In addition, because of the constraints imposed by the endomorphisms, we may restrict it to $\omega_{\mathcal{A}}(\Sigma)\otimes\omega_T(\overline{\Sigma}) = \mathcal{P}_0\otimes\mathcal{Q}$ without losing any information. Finally, [Lan] Remark 4.6.2.7 and Theorem 4.6.3.16 imply that the diagram

$$\begin{array}{ccc} \mathcal{P}_0\otimes\mathcal{Q} & \stackrel{\mathrm{KS}_{\mathcal{G}/C}}{\longrightarrow} & \Omega_{C/W_s} \\ \cap & & \cap \\ \mathcal{P}\otimes\mathcal{Q} & \stackrel{\mathrm{KS}}{\longrightarrow} & \Omega_{\overline{S}/W_s}[d\log\infty] \end{array}$$

is commutative, and this proves (1).

Proof of (2): The second claim goes to the root of how KS is defined on $\overline{S}$. See [Lan] §4.6.2, especially the discussion on p. 269, preceding Definition 4.6.2.10. Fix a basis $y_1,\dots,y_m$ of Y and let $\chi_i=\phi_X(y_i)$ be the corresponding basis of X. Then as a basis of $\check{H}$ we may take the elements $\check{h}_{ij}=[y_i\otimes y_j]$ (*cf.* the proof of Lemma 3.2.4). The corresponding element of the stalk of $\mathcal{P}_\mu\otimes\mathcal{Q}$ at x is $\chi_i\otimes\chi_j$. The variable $q_{ij}=q^{\check{h}_{ij}}$ is then a generator of the invertible R-module denoted in [Lan] by $I(y_i,\chi_j)$, and the extended Kodaira-Spencer homomorphism is defined in [Lan], Definition 4.6.2.12 so that it takes $\chi_i\otimes\chi_j$ to $d\log(q_{ij})=\omega(\check{h}_{ij})$. The base schemes S and S_1 in [Lan] are in our case $\mathrm{Spec}(R)$ and its generic point. □

Corollary 3.2.6. *Let $f\in H^0(S_s^{\mathrm{ord}},\mathcal{E}_\rho)$ and let h be the Hasse invariant (1.1.5). Then $\Theta(hf)=h\Theta(f)$.*

Proof. Obvious. □

Corollary 3.2.7. *(i) Let $f\in H^0(\mathcal{S}^{\mathrm{ord}},\mathcal{E}_\rho)$. Then $\Theta(f)=0$ if and only if the FJ expansion of f at every rank m cusp is constant.*

(ii) Over S, $f\in\ker(\Theta)$ if and only if its Fourier-Jacobi expansion at every rank m cusp is supported on $\check{h}\in p\check{H}^+$.

Proof. (i) This follows from our theorem and the FJ-*expansion principle*: a p-adic modular form vanishes if and only if its FJ expansion at every rank m cusp vanishes. This principle was proved in [Lan], Proposition 7.1.2.14, under the assumption that every irreducible component of $\overline{S}$ contains at least one rank m cuspidal stratum $Z([g]_m,\sigma)$. This assumption was later verified, for our Shimura variety among others, in Corollary A.2.3 of [Lan2]. (ii) Follows by the same argument, noticing that for $a(\check{h})\otimes\check{h}$ to vanish, it is necessary and sufficient that either $a(\check{h})=0$ or $\check{h}\in p\check{H}^+$. □

4. Analytic continuation of Θ to the non-ordinary locus

4.1. The almost ordinary locus.

4.1.1. *The stratum S^{ao}.* In this section we assume that $n>m$, as the question we are about to discuss requires different considerations when $n=m$, which will be handled separately. Let S denote, as in the beginning, the special fiber of the Shimura variety $\mathcal{S}$. Thus S is nm-dimensional, smooth over $\kappa=\mathbb{F}_{p^2}$, and is stratified by the Ekedahl-Oort strata [Oo], [Mo2], [V-W]. The (μ-)ordinary stratum S^{ord} is open and dense, and the operator Θ acts on sections of the automorphic vector bundle $\mathcal{E}_\rho$ over it, sending them to sections of $\mathcal{E}_\rho\otimes\mathcal{P}_\mu\otimes\mathcal{Q}\simeq\mathcal{E}_\rho\otimes\mathcal{Q}^{(p)}\otimes\mathcal{Q}$,

$$\Theta: H^0(S^{\mathrm{ord}},\mathcal{E}_\rho)\to H^0(S^{\mathrm{ord}},\mathcal{E}_\rho\otimes\mathcal{Q}^{(p)}\otimes\mathcal{Q}).$$

Here we have used the fact that in characteristic p the vector bundle homomorphism $V_{\mathcal{P}} : \mathcal{P} \to \mathcal{Q}^{(p)}$ is surjective with kernel $\mathcal{P}_0$, so induces an isomorphism $\mathcal{P}_\mu = \mathcal{P}/\mathcal{P}_0 \simeq \mathcal{Q}^{(p)}$. Our goal in this section is to study the analytic continuation of Θ to all of S. This is reminiscent of the fact that the theta operator on GL_2 (denoted by $A\theta$ in [Ka2]) extends holomorphically across the supersingular points of the modular curve.

Proposition 4.1.1. *There exists a unique EO stratum S^{ao} of dimension $nm-1$. The homomorphism $V_{\mathcal{P}}$ is still surjective in every geometric fiber over S^{ao}, so $\mathcal{P}_0 = \mathcal{P}[V_{\mathcal{P}}]$ extends to a rank $n-m$ vector bundle over $S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$. The same applies to $\mathcal{P}_\mu$ and of course to $\mathcal{Q}$, hence every p-adic automorphic vector bundle $\mathcal{E}_\rho$ extends canonically to the open set $S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$.*

We call S^{ao} the *almost-ordinary* locus. It is the divisor of the Hasse invariant h on $S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$, and, like any other EO stratum, is non-singular.

Proof. The uniqueness of the EO stratum in codimension 1 is proved in [Woo], Corollary 3.4.5, where it is deduced from the classification of the EO strata by Weyl group elements and the calculation of their dimensions in [Mo2]. The assertion on $V_{\mathcal{P}}$ being surjective in every geometric fiber follows from the computation of the Dieudonné space at geometric points of S^{no} ([Woo], Proposition 3.5.6), reviewed below. Since the base scheme is a non-singular variety, constancy of the fibral rank of $V_{\mathcal{P}}$ suffices to conclude that $\mathcal{P}_0$ and $\mathcal{P}_\mu$ are locally free sheaves. Finally, $\mathcal{E}_\rho$ is constructed by twisting the representation ρ of $GL_m \times GL_m \times GL_{n-m}$ (with values in κ) by the vector bundles $\mathcal{Q}, \mathcal{P}_\mu$ and $\mathcal{P}_0$ as in §1.2.2. □

4.1.2. *Dieudonné spaces.* Let k be a perfect field of characteristic p. For the following see [Oda], [B-W] and [We2] (5.3). A polarized Dieudonné space over k is a finite dimensional k-vector space D equipped with a non-degenerate skew-symmetric pairing $\langle , \rangle$ and two linear maps $F : D^{(p)} \to D$ and $V : D \to D^{(p)}$ such that $\mathrm{Im}(F) = \ker(V)$ and $\mathrm{Im}(V) = \ker(F)$, and such that $\langle Fx, y\rangle = \langle x, Vy\rangle^{(p)}$ for every $x \in D^{(p)}$ and $y \in D$. It follows immediately from the definition that $\dim D = 2g$ and F and V have rank g. If M is a principally polarized Dieudonné module over $W(k)$ then $D = M/pM$ is a polarized Dieudonné space. If A is a principally polarized abelian variety over k then its de Rham cohomology $D = H^1_{dR}(A/k)$ is equipped with a canonical structure of a Dieudonné space, which may also be identified with the (contravariant) Dieudonné module of $A[p]$. The Hodge filtration is then related to F via

$$\omega = H^0(A, \Omega^1) = (D^{(p)}[F])^{(p^{-1})}.$$

It is essential for this that we work over a perfect base.

A polarized $\mathcal{O}_E$-Dieudonné space is a polarized Dieudonné space admitting, in addition, endomorphisms by $\mathcal{O}_E$, for which F and V are $\mathcal{O}_E$-linear and $\langle ax, y\rangle = \langle x, \overline{a}y\rangle$ $(a \in \mathcal{O}_E)$. Assume that k contains κ. Then $D(\Sigma)$ and $D(\overline{\Sigma})$ are set in duality by the pairing, hence are each of dimension g, V maps $D(\Sigma)$ to $D(\overline{\Sigma})^{(p)}$ and $D(\overline{\Sigma})$ to $D(\Sigma)^{(p)}$ and a similar statement, going backwards, holds for F. The type (n, m) of ω ($n = \dim \omega(\Sigma)$, $m = \dim \omega(\overline{\Sigma})$) is called the *type*, or *signature*, of D.

Over a non-perfect base $\mathrm{Spec}(R)$ (in characteristic p, say, as this is all that we need) one can still associate to a principally polarized abelian scheme A/R, or to its p-divisible group, a Dieudonné *crystal* as in [Gro], and when *evaluated* at $(\mathrm{Spec}(R) \subset \mathrm{Spec}(R))$ it yields a polarized R-module $D(A/R)$ with an F and a V as before, which may be identified with $H^1_{dR}(A/R)$. If R is an equi-characteristic PD-thickening of k then in fact $D(A/R) = R \otimes_k D(A_k/k)$ with the polarization, F and V extended R-linearly. The Hodge filtration can not be read from $D(A/R)$ any more. In fact, Grothendieck's theorem asserts that giving $(D(A/R), \omega)$ is tantamount to giving the deformation of A from $\mathrm{Spec}(k)$ to $\mathrm{Spec}(R)$. We shall apply these remarks later on when k is algebraically closed, $x \in S(k)$ is a geometric point, and $R = \mathcal{O}_{S,x}/\mathfrak{m}^2_{S,x}$ is its first infinitesimal neighborhood.

Let k be an algebraically closed field containing κ. Consider the following polarized $\mathcal{O}_E$-Dieudonné spaces. We use the convention that $\mathcal{O}_E$ acts on the e_i via Σ and on the f_i via $\overline{\Sigma}$.

(i) $D(\mathfrak{G}_\Sigma[p]) = \mathrm{Span}_k\{e_1, f_1\}$, $\langle e_1, f_1\rangle = 1$, $Ff_1^{(p)} = e_1$, $Fe_1^{(p)} = 0$, $Vf_1 = e_1^{(p)}$, $Ve_1 = 0$. Here $\omega = ke_1$ and the signature is $(1, 0)$.

(i)′ $D(\mathfrak{G}_{\overline{\Sigma}}[p]) = \mathrm{Span}_k\{e_2, f_2\}$, $\langle f_2, e_2\rangle = 1$, $Fe_2^{(p)} = f_2$, $Ff_2^{(p)} = 0$, $Ve_2 = f_2^{(p)}$, $Vf_2 = 0$. Note that $D(\mathfrak{G}_{\overline{\Sigma}}[p]) = D(\mathfrak{G}_\Sigma[p])^{(p)}$, $\omega = kf_2$ and the signature is $(0, 1)$.

(ii) $AO(2,1) = \mathrm{Span}_k\{e_i, f_i \mid 1 \le i \le 3\}$, $\langle e_1, f_3\rangle = \langle f_2, e_2\rangle = \langle f_1, e_3\rangle = 1$ (and $\langle e_i, f_j\rangle = 0$ if $i + j \ne 4$); F and V are given by the following table, where to ease notation the ${}^{(p)}$ is left out.

	e_1	e_2	e_3	f_1	f_2	f_3
F	0	f_1	0	e_1	0	e_2
V	0	0	f_2	0	e_1	e_3

This is the Dieudonné space denoted by $\overline{B}(3)$ in [B-W]. Here $\omega = \mathrm{Span}_k\{e_1, e_3, f_2\}$ and $\mathcal{P}_0 = \omega(\Sigma)[V] = ke_1$.

(iii) $AO(3,1) = \mathrm{Span}_k\{e_i, f_i \mid 1 \le i \le 4\}$, $\langle e_1, f_4\rangle = \langle e_2, f_3\rangle = \langle f_2, e_3\rangle = \langle f_1, e_4\rangle = 1$ (and $\langle e_i, f_j\rangle = 0$ if $i + j \ne 5$); F and V are given by the following table, where to ease notation the ${}^{(p)}$ is left out.

	e_1	e_2	e_3	e_4	f_1	f_2	f_3	f_4
F	0	f_1	0	0	e_1	e_2	0	e_3
V	0	0	0	f_3	0	e_1	e_3	e_4

This is the Dieudonné space denoted by $\overline{B}(4)$ in [B-W]. Here $\omega = \mathrm{Span}_k\{e_1, e_3, e_4, f_3\}$ and $\mathcal{P}_0 = \omega(\Sigma)[V] = \mathrm{Span}_k\{e_1, e_3\}$.

Proposition 4.1.2. *(*[Woo]*, Proposition 3.5.6) Let $x \in S^{\mathrm{ao}}(k)$ be an almost-ordinary geometric point. Then $D_x = D(\mathcal{A}_x/k)$ is isomorphic to the following.*

(i) $n = m + 1$:

$$D = AO(2,1) \oplus \bigoplus_{i=1}^{m-1} \mathrm{Span}_k\{e_i^\mu, e_i^{et}, f_i^\mu, f_i^{et}\}$$

where $\langle e_i^\mu, f_i^{et}\rangle = \langle f_i^\mu, e_i^{et}\rangle = 1$ (and $\langle e_i^\mu, f_i^\mu\rangle = \langle e_i^{et}, f_i^{et}\rangle = 0$), and F and V are given by the following table

	e_i^μ	e_i^{et}	f_i^μ	f_i^{et}
F	0	f_i^{et}	0	e_i^{et}
V	f_i^μ	0	e_i^μ	0

(ii) $n \geq m+2:$

$$D = AO(3,1) \oplus \bigoplus_{i=1}^{m-1} \mathrm{Span}_k\{e_i^\mu, e_i^{et}, f_i^\mu, f_i^{et}\} \oplus D(\mathfrak{G}_\Sigma[p])^{n-m-2}.$$

4.1.3. *The Kodaira-Spencer isomorphism along the almost ordinary stratum.* The following result is the key to the analytic continuation of the theta operator, which will be proved in the next section.

Theorem 4.1.3. *Let*

$$\psi = (pr_\mu \otimes 1) \circ \mathrm{KS}^{-1} : \Omega_{S/\kappa} \to \mathcal{P}_\mu \otimes \mathcal{Q}$$

be the composition of the inverse of the Kodaira-Spencer isomorphism and the projection from $\mathcal{P}$ to $\mathcal{P}_\mu = \mathcal{P}/\mathcal{P}_0$ (well-defined over $S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$). Let $u = 0$ be a local equation of the divisor S^{ao} in a Zariski open set $U \subset S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$. Then $\psi(du)$ vanishes along $S^{\mathrm{ao}} \cap U$.

Remark. Compare with [dS-G1] Proposition 3.11. In terms of the foliation $\mathcal{T}S^+$ introduced in [dS-G2] the theorem asserts that at any point $x \in S^{\mathrm{ao}}$ this foliation is tangential to S^{ao}, i.e $\mathcal{T}S^+|_x \subset \mathcal{T}S^{\mathrm{ao}}|_x$. In [dS-G2] we studied a certain open subset $S_\sharp \subset S$ which was a union of Ekedahl-Oort strata, including S^{ord}, S^{ao} and a unique minimal EO stratum denoted there S^{fol}, of dimension m^2. The subset $S_\sharp$ and the foliation TS^+ are related to the geometry of auxiliary Shimura varieties of parahoric level structure at p, and seem to play an important role. In *loc. cit.* Theorem 25, it was proved that $\mathcal{T}S^+$ is tangential to S^{fol}. In view of these two results, claiming tangentiality to S^{ao} and S^{fol}, it is reasonable to expect that $\mathcal{T}S^+$ is tangential to *every* EO strata in $S_\sharp$. The proofs of the known cases, whether in *loc. cit.* or here, invoke delicate computations with Dieudonné modules, and at present we see no conceptual reason justifying our expectation, which could avoid such computations.

Proof. Let k be an algebraically closed field containing κ, $x \in S^{\mathrm{ao}}(k)$ a geometric point and $D_x = D(\mathcal{A}_x/k)$. Let $R = \mathcal{O}_{S,x}/\mathfrak{m}_{S,x}^2$ and $d : R \to \Omega_{S/k}|_x = \mathfrak{m}_{S,x}/\mathfrak{m}_{S,x}^2$ the canonical derivation $df = (f - f(x)) \mod \mathfrak{m}_{S,x}^2$. Let $D = H^1_{dR}(\mathcal{A}/R)$. The Gauss-Manin connection on $H^1_{dR}(\mathcal{A}/S)$ induces a map

$$\nabla : D \to D_x \otimes_k \Omega_{S/k}|_x$$

satisfying $\nabla(r\alpha) = r(x)\nabla(\alpha) + \alpha \otimes dr$, which by abuse of language we call the Gauss-Manin connection on D. It is easy to see that every $\alpha \in D_x$ has a unique extension to a horizontal section $\alpha \in D$, i.e. a section satisfying $\nabla(\alpha) = 0$. Thus, we may identify D with $R \otimes_k D_x$, the horizontal sections being D_x. Since the Gauss-Manin connection commutes with isogenies, $V : D \to D^{(p)}$ and $F : D^{(p)} \to D$ map horizontal sections to horizontal sections. For the same reason, if x, y are horizontal sections of D, their pairing $\langle x, y\rangle$ is horizontal for d, i.e. lies in k.

We now distinguish between two cases.

I. Assume that $n = m+1$. Then

$$D_x = \mathrm{Span}_k\{\underline{e_1}, e_2, \underline{e_3}, f_1, \underline{f_2}, f_3, \underline{e_i^\mu}, e_i^{et}, \underline{f_i^\mu}, f_i^{et}\}_{1 \leq i \leq m-1}$$

where the first six vectors span $AO(2,1)$, as in Proposition 4.1.2(i). For the convenience of the reader we have underlined the vectors spanning ω_x. The module D is spanned by the same vectors over R, and the pairings and the tables giving F and V remain the same over R.

We now write the most general deformation of ω_x to a projective submodule of D which is invariant under the endomorphisms and isotropic. An easy check yields that it is given by

$$\omega = \mathrm{Span}_R\{\widetilde{e}_1, \widetilde{e}_3, \widetilde{f}_2, \widetilde{e}_i^\mu, \widetilde{f}_i^\mu\}_{1\le i\le m-1},$$

where

- $\widetilde{e}_1 = e_1 + ue_2 + \sum_{i=1}^{m-1} u_i e_i^{et}$
- $\widetilde{e}_3 = e_3 + ve_2 + \sum_{i=1}^{m-1} v_i e_i^{et}$
- $\widetilde{f}_2 = f_2 - vf_1 + uf_3 + \sum_{i=1}^{m-1} w_i f_i^{et}$
- $\widetilde{e}_i^\mu = e_i^\mu + w_i e_2 + \sum_{j=1}^{m-1} w_{ij} e_j^{et}$
- $\widetilde{f}_i^\mu = f_i^\mu - v_i f_1 + u_i f_3 + \sum_{j=1}^{m-1} w_{ji} f_j^{et}$.

The mn parameters $u, u_i, v, v_i, w_i, w_{ij}$ are, according to Grothendieck, the local parameters of R, serving as a basis of $\mathfrak{m}_R$ over k. It follows that $\mathcal{P}_0$ is indeed of rank 1, as claimed before, spanned over R by $\widetilde{e}_1$, while $\mathcal{Q}$ is spanned over R by the m vectors $\widetilde{f}_2, \widetilde{f}_i^\mu$. Furthermore, computing the Hasse matrix $H = V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{Q}}$ in the bases $(\widetilde{f}_2, \widetilde{f}_i^\mu)$ and $(\widetilde{f}_2^{(p^2)}, \widetilde{f}_i^{\mu(p^2)})$ of $\mathcal{Q}$ and $\mathcal{Q}^{(p^2)}$ we get

$$H = \begin{pmatrix} u & u_1 & u_2 & \cdots & u_{m-1} \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & 1 \end{pmatrix},$$

so the (trivialized) Hasse invariant h is simply u. Since we know that S^{ao} is the zero divisor of h, $S^{\mathrm{ao}} \cap \mathrm{Spec}(R)$ is given by the equation $u = 0$. Note, in passing, that this proves that the zero divisor of the Hasse invariant is reduced and equal to the non-ordinary locus.

To compute $\mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{Q})$ recall how it is defined. From the Gauss-Manin connection we get a homomorphism of sheaves

$$\overline{\nabla} : \omega \to \Omega_{S/\kappa} \otimes (H^1_{dR}(\mathcal{A}/S)/\omega)$$

which induces a homomorphism $\mathcal{P} \to \Omega_{S/\kappa} \otimes \mathcal{Q}^\vee$. This induces the map

$$\mathrm{KS} : \mathcal{P} \otimes \mathcal{Q} \to \Omega_{S/\kappa}$$

which happens to be an isomorphism. We begin by computing the Gauss Manin connection on $\mathcal{P}_0$:

$$\nabla(\widetilde{e}_1) = e_2 \otimes du + \sum_{i=1}^{m-1} e_i^{et} \otimes du_i.$$

Projecting D_x to $D_x/\omega_x = H^1(\mathcal{A}_x, \mathcal{O})$ and noting that e_2, e_i^{et} modulo ω_x are a basis for the dual $\mathcal{Q}^\vee$ of $\mathcal{Q}$, equipped with the conjugate action of $\mathcal{O}_E$ via Σ, we find out that

$$\mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{Q})|_x = \mathrm{Span}_k\{du, du_i\}.$$

From the definition of ψ it follows that $\psi(du)|_x = 0$. Now assume that u is a *global* generator of the ideal of S^{ao} in a Zariski open set U. Then we conclude that $\psi(du) = 0$ along $S^{\mathrm{ao}} \cap U$ as claimed.

II. The proof of the theorem in the case $n - m \geq 2$ is similar, using Proposition 4.1.2(ii). Here

$$D_x = \mathrm{Span}_k\{\underline{e_1}, e_2, \underline{e_3}, \underline{e_4}, f_1, f_2, \underline{f_3}, f_4, \underline{e_i^\mu}, e_i^{et}, \underline{f_i^\mu}, f_i^{et}, \underline{e_j^\sharp}, f_j^\sharp\}_{1\leq i\leq m-1,\, 1\leq j\leq n-m-2}$$

where the first eight vectors span $AO(3,1)$ and for every j the vectors $e_j^\sharp, f_j^\sharp$ span a copy of $D(\mathfrak{G}_\Sigma[p])$. For convenience we have again underlined the vectors spanning ω_x. The most general deformation of ω_x in $D = R \otimes_k D_x$ is spanned by the following vectors:

- $\widetilde{e}_1 = e_1 - ue_2 + \sum_{i=1}^{m-1} u_i e_i^{et}$
- $\widetilde{e}_3 = e_3 + ve_2 + \sum_{i=1}^{m-1} v_i e_i^{et}$
- $\widetilde{e}_4 = e_4 + we_2 + \sum_{i=1}^{m-1} w_i e_i^{et}$
- $\widetilde{f}_3 = f_3 + wf_1 + vf_2 + uf_4 - \sum_{l=1}^{m-1} x_l f_l^{et} - \sum_{k=1}^{n-m-2} y_k f_k^\sharp$
- $\widetilde{e}_i^\mu = e_i^\mu + x_i e_2 + \sum_{l=1}^{m-1} x_{il} e_l^{et}$
- $\widetilde{f}_i^\mu = f_i^\mu - w_i f_1 - v_i f_2 + u_i f_4 + \sum_{l=1}^{m-1} x_{li} f_l^{et} + \sum_{k=1}^{n-m-2} y_{ki} f_k^\sharp$
- $\widetilde{e}_j^\sharp = e_j^\sharp + y_j e_2 + \sum_{l=1}^{m-1} y_{jl} e_l^{et}$.

The nm parameters $u, v, w, u_i, v_i, w_i, x_i, x_{il}, y_k, y_{kj}$ form a basis of $\mathfrak{m}_R$ over k. Calculating the Hasse matrix H yields exactly the same $m \times m$ matrix as above, hence $u = 0$ is again the local infinitesimal equation of S^{ao}. The submodule $\mathcal{P}_0$ is $n - m$ dimensional, and is spanned by $\widetilde{e}_1, \widetilde{e}_3$ and the $\widetilde{e}_j^\sharp$. Calculating KS we find that

$$\mathrm{KS}(\mathcal{P}_0 \otimes \mathcal{Q})|_x = \mathrm{Span}_k\{du, du_i, dv, dv_i, dy_j, dy_{jl}\}$$

$(1 \leq i, l \leq m - 1,\ 1 \leq j \leq n - m - 2)$ so as before $\psi(du)|_x = 0$. We conclude the proof as in the first case. $\square$

4.2. **Analytic continuation of Θ $(m < n)$.**

4.2.1. *Compactification of a certain intermediate Igusa cover.* Recall the Igusa tower $T_{t,s}$ over S_s^{ord} that has been constructed in §2.1. Let

$$\Delta_t^1 = SL_m(\mathcal{O}_E/p^t\mathcal{O}_E) \times U_{n-m}(\mathcal{O}_E/p^t\mathcal{O}_E) \lhd \Delta_t$$

and denote by $T_{t,s}^1$ the intermediate covering of S_s^{ord} fixed by Δ_t^1. It is a Galois étale cover of S_s^{ord} with Galois group $(\mathcal{O}_E/p^t\mathcal{O}_E)^\times$. In this section let $T = T_{1,1}^1$, and let $\tau : T \to S^{\mathrm{ord}}$ be the covering map, whose Galois group is identified with $\kappa^\times$.

Let $\mathcal{L} = \det(\mathcal{Q})$ and recall that the Hasse invariant $h \in H^0(S, \mathcal{L}^{p^2-1})$ (1.1.5).

Lemma 4.2.1. *(i) The line bundle $\mathcal{L}$ is canonically trivialized over T, i.e. there is a canonical isomorphism $\varepsilon : \mathcal{O}_T \simeq \tau^*\mathcal{L}$.*

(ii) Denoting by a the global section of $\tau^*\mathcal{L}$ corresponding to the section "1" under the trivialization, we have $a^{p^2-1} = \tau^* h$.

Proof. (i) The canonical trivialization $\varepsilon^2(\overline{\Sigma}) : \mathcal{O}_{T_{1,1}}^m \simeq \tau_{1,1}^* \mathcal{Q}$ over the big Igusa variety $T_{1,1}$ induces a canonical trivialization on the determinants $\varepsilon : \mathcal{O}_{T_{1,1}} \simeq \tau_{1,1}^* \mathcal{L}$. The latter descends to T because it is invariant under Δ_1^1.

(ii) Since Ver is the identity on μ_p, the trivialization ϵ^2 of $gr^2 \mathcal{A}[p]$ satisfies

$$\mathrm{Ver} \circ \mathrm{Ver}^{(p)} \circ (\epsilon^2)^{(p^2)} = \epsilon^2.$$

Passing to cohomology (recall $\varepsilon^2 = ((\epsilon^2)^{-1})^*$) yields the relation $(\varepsilon^2(\overline{\Sigma}))^{(p^2)} = H \circ \varepsilon^2(\overline{\Sigma})$ where H, recall, is $V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{Q}}$. Taking determinants we get

$$\varepsilon^{(p^2)} = h \circ \varepsilon$$

and evaluating at "1" gives the desired relation. □

The following Kummer-type result was proved in [dS-G1] §2.4.2 for signature (2,1) and the proof easily generalizes. See also [Go]. Let

$$S' = S^{\mathrm{ord}} \cup S^{\mathrm{ao}}.$$

Consider the fiber product

$$T' = \mathcal{L} \times_{\mathcal{L}^{p^2-1}} S' \tag{4.2.1}$$

where the two maps to $\mathcal{L}^{p^2-1}$ are $\lambda \mapsto \lambda^{p^2-1}$ and h. Note that the pull-back of $\mathcal{L}$ from S' to T' admits a tautological p^2-1 root of h extending a, which we still call a. Then $T' \to S'$ is finite flat of degree p^2-1, is Galois étale with Galois group $\kappa^\times$ over S^{ord}, and totally (tamely) ramified along S^{ao}. It satisfies a universal property with respect to extracting a p^2-1 root from the section h; *cf.* loc. cit. From part (ii) of the last proposition it follows that there is a canonical map

$$T \to T'.$$

Since both source and target are $\kappa^\times$-torsors over S^{ord} and the map respects the $\kappa^\times$ action, this map is an isomorphism of T with the pre-image of S^{ord} in T'. In this way we may identify T' with a (partial) compactification of T. We then have the following.

Proposition 4.2.2. *(i) The morphism $\tau' : T' \to S'$ is finite flat of degree p^2-1, Galois étale with Galois group $\kappa^\times$ over S^{ord}, and totally (tamely) ramified along S^{ao}.*

(ii) T' is everywhere non-singular.

(iii) Let $x \in S^{\mathrm{ao}}(k)$ be a geometric point, and $y \in T'(k)$ the unique geometric point mapping to it. Then there are formal parameters u, v_i $(1 \le i \le nm-1)$ at x such that $u = 0$ is the infinitesimal equation of S^{ao}, and such that as formal parameters on T' at y we can take w, v_i where $w^{p^2-1} = u$.

(iv) T' and $T = T_{1,1}^1$ are irreducible.

Proof. The proof is the same as in the case of signature $(2,1)$, *cf.* [dS-G1], §2.4.3, Proposition 2.16. □

4.2.2. *The main theorem for scalar-valued modular forms.* We can now prove the analytic continuation of Θ in characteristic p, when applied to *scalar-valued* p-adic modular forms. Recall that $\mathcal{L} = \det(\mathcal{Q})$.

Theorem 4.2.3. *Assume that $m < n$. Consider the operator*

$$\Theta : H^0(S^{\mathrm{ord}}, \mathcal{L}^k) \to H^0(S^{\mathrm{ord}}, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Then Θ extends holomorphically to an operator

$$\Theta : H^0(S, \mathcal{L}^k) \to H^0(S, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Remark. The analytic continuation of Θ to global modular forms is a characteristic p phenomenon and does not seem to extend to S_s (i.e. modulo p^s) for $s > 1$. Had it extended for all s, we would have obtained, for any algebraic modular form f of weight k, a well-defined rigid analytic "$\Theta(f)$", of weight $k+p+1$, on the whole rigid analytic space associated to $\mathcal{S}$. By GAGA (and the Köcher principle) this $\Theta(f)$ would have been algebraic. However, the Maass-Shimura operators in characteristic 0 do not preserve the space of classical modular forms.

Proof. Let $f \in H^0(S, \mathcal{L}^k)$. Then $\Theta(f)$ is a section of $\mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}$ over S^{ord} and we have to show that it extends holomorphically to S. Since S is non-singular, it is enough to show that it extends holomorphically to $S' = S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$, an open set whose complement is of codimension 2. Indeed, locally Zariski we may trivialize the vector bundles, and then any coordinate of $\Theta(f)$ becomes a meromorphic function, whose polar set, if non-empty, should have codimension 1.

Let $\tau' : T' \to S'$ be the intermediate Igusa variety constructed above. Over T (the pre-image of S^{ord}) we can write the trivialization ε of $\mathcal{L}$ as $f \mapsto f/a^k$. This introduces a pole of order k, at most, along $T^{\mathrm{ao}} = \tau'^{-1}(S^{\mathrm{ao}})$. Let $y \in T^{\mathrm{ao}}$ be a geometric point and $x = \tau'(y)$. Let u, v_i be formal parameters at x and w, v_i formal parameters at y as in Proposition 4.2.2. Let

$$f/a^k = \sum_{r=-k}^{\infty} g_r(v) w^r$$

be the Taylor expansion of f/a^k in $\widehat{\mathcal{O}}_{T',y}$, where the $g_r(v)$ are power series in the v_i. Note that $du = d(w^{p^2-1}) = -w^{p^2-2} dw$. Thus,

$$\begin{aligned} d(f/a^k) &= \sum_{r=-k}^{\infty} w^r dg_r(v) + \sum_{r=-k}^{\infty} r g_r(v) w^{r-1} dw \\ &= \sum_{r=-k}^{\infty} w^r dg_r(v) - \sum_{r=-k}^{\infty} r g_r(v) w^{r-(p^2-1)} du \\ &= \sum_{r=-k}^{\infty} w^r dg_r(v) - \sum_{r=-k}^{\infty} r g_r(v) w^r u^{-1} du. \end{aligned}$$

When we compute

$$\widetilde{\Theta}(f) = a^k \left(\sum_{r=-k}^{\infty} w^r dg_r(v) - \sum_{r=-k}^{\infty} r g_r(v) w^r u^{-1} du \right),$$

which we know descends to S', we see that the first sum becomes holomorphic (a vanishes along T^{ao}), while the second sum retains a simple pole along S^{ao}. However, to get $\Theta(f)$ we must still apply the vector-bundle homomorphism ψ. Theorem 4.1.3 says that $\psi(du)$ vanishes along S^{ao}, hence the simple pole disappears and $\Theta(f)$ is holomorphic at x. This being true at every $x \in S^{\mathrm{ao}}$, we conclude that $\Theta(f)$ is everywhere holomorphic. □

4.3. **Analytic continuation of Θ $(m = n)$.** We briefly indicate the modifications in the proof which are necessary to deal with the case $m = n$. In this case $\mathrm{rk}(\ker(V_{\mathcal{P}}))$ changes when we move from S^{ord} to S^{ao}, so $\mathcal{P}_0$ and $\mathcal{P}_\mu$ do not extend, with the same definitions, to vector bundles over $S' = S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$. As such, we cannot extend Θ beyond S^{ord} using pr_μ. Instead, we apply $(V_{\mathcal{P}} \otimes 1) \circ \mathrm{KS}^{-1}$ to $\Omega_{S/\kappa}$, a map that gives the same result as $(pr_\mu \otimes 1) \circ \mathrm{KS}^{-1}$ over S^{ord} in characteristic p, but does not make sense over S_s^{ord} for $s > 1$. Let $\mathcal{L} = \det(\mathcal{Q})$ as before.

4.3.1. *Preliminary results on the Igusa variety when $m = n$.* Let $T = T^1_{1,1}$ as before. Let T' be defined by (4.2.1). As before, it is a partial compactification of T. Since the divisor of the Hasse invariant is not reduced when $n = m$ (see §1.1.4 and Lemma 1.1.4), the proof of the irreducibility of T as in [dS-G1], Proposition 2.16, breaks down.

Proposition 4.3.1. *(i) The morphism $T' \to S'$ is finite flat of degree $p^2 - 1$, with Galois group $\kappa^\times$.*

(ii) T' is non-singular.

(iii) T' and the Igusa variety T decompose into $p+1$ irreducible components T'_ζ (resp. T_ζ) labeled by ζ such that $\zeta^{p+1} = 1$. More canonically,

$$\pi_0(T) = \pi_0(T') \simeq \kappa^\times/\mathbb{F}_p^\times.$$

(iv) The map $T'_\zeta \to S'$ is totally (tamely) ramified over S^{ao} of degree $p - 1$.

Proof. The proof of (i) is the same as when $n > m$. Our T' is still obtained from S' by extracting a $p^2 - 1$ root of h. However, this time $h = h_{\mathcal{Q}}^{p+1}$ where $h_{\mathcal{Q}}$ is in $H^0(S', \mathcal{L}^{p-1})$ so $T' = \bigsqcup T'_\zeta$ where $T'_\zeta = S'[\sqrt[p-1]{\zeta h_{\mathcal{Q}}}]$. As the divisor of $h_{\mathcal{Q}}$ is reduced and equal to S^{ao}, the rest of the proof is similar to the case $n > m$. □

4.3.2. *The main theorem when $m = n$.*

Theorem 4.3.2. *Assume that $m = n$. The operator*

$$\Theta = (1 \otimes V_{\mathcal{P}} \otimes 1) \circ (1 \otimes \mathrm{KS}^{-1}) \circ \widetilde{\Theta} : H^0(S^{\mathrm{ord}}, \mathcal{L}^k) \to H^0(S^{\mathrm{ord}}, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q})$$

extends holomorphically to an operator

$$\Theta : H^0(S, \mathcal{L}^k) \to H^0(S, \mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}).$$

Proof. As before, let

$$\psi = (V_{\mathcal{P}} \otimes 1) \circ \mathrm{KS}^{-1} : \Omega_{S/\kappa} \to \mathcal{Q}^{(p)} \otimes \mathcal{Q}.$$

Let $x \in S^{\mathrm{ao}}(k)$ be a geometric point. The Dieudonné module D_x is now given by

$$D_x = \mathrm{Span}_k\{\underline{e}_i^\mu, \underline{f}_i^\mu, e_i^{et}, f_i^{et}, \underline{e}^\sharp, f^\sharp, e^\flat, \underline{f}^\flat\}_{1 \le i \le m-1}$$

where $\mathrm{Span}_k\{e^\sharp, f^\sharp\}$ is isomorphic to $D(\mathfrak{G}_\Sigma[p])$ and $\mathrm{Span}_k\{e^\flat, f^\flat\}$ to $D(\mathfrak{G}_{\overline{\Sigma}}[p])$ (see [Woo], Proposition 3.5.6). The underlined vectors span ω_x. As before, we let $R = \mathcal{O}_{S,x}/\mathfrak{m}_{S,x}^2$ and $D = R \otimes_k D_x$. The most general deformation of ω_x to $\omega \subset D$ compatible with the endomorphisms and the polarization is spanned by

- $\widetilde{e}_i^\mu = e_i^\mu + w_i e^\flat + \sum_{j=1}^{m-1} w_{ij} e_j^{et}$

- $\widetilde{e}^\sharp = e^\sharp + ue^\flat + \sum_{j=1}^{m-1} u_j e_j^{et}$
- $\widetilde{f}_i^\mu = f_i^\mu + u_i f^\sharp + \sum_{j=1}^{m-1} w_{ji} f_j^{et}$
- $\widetilde{f}^\flat = f^\flat + uf^\sharp + \sum_{j=1}^{m-1} w_j f_j^{et}$.

The m^2 quantities w_i, w_{ij}, u, u_i then form a system of local (infinitesimal) parameters at x. The matrix of $V_{\mathcal{Q}}$ in the bases $\{\widetilde{f}^\flat, \widetilde{f}_i^\mu\}$ of $\mathcal{Q}$ and $\{(\widetilde{e}^\sharp)^{(p)}, (\widetilde{e}_i^\mu)^{(p)}\}$ of $\mathcal{P}^{(p)}$ is

$$\begin{pmatrix} u & u_1 & \dots & u_{m-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

The infinitesimal equation of $S^{\mathrm{ao}} \cap \mathrm{Spec}(R)$ is $u = 0$. As before we compute the Kodaira-Spencer homomorphism and find out that

$$\mathrm{KS}(e^\sharp) = du \wedge e^\flat + \sum_{j=1}^{m-1} du_j \wedge e_j^{et},$$

which means that

$$\mathrm{KS}(e^\sharp \otimes \mathcal{Q}|_x) = \mathrm{Span}_k\{du, du_j\} \subset \Omega_S|_x.$$

This implies that $\mathrm{KS}^{-1}(du) \in e^\sharp \otimes \mathcal{Q}|_x$. However, $V_{\mathcal{P}}$ is expressible in the same bases as above by the matrix

$$\begin{pmatrix} u & w_1 & \dots & w_{m-1} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix},$$

which means that $\ker(V_{\mathcal{P}})$ is 1-dimensional at x, and spanned by $e^\sharp$. Thus if $u = 0$ is a local equation of S^{ao}, $\psi(du)$ vanishes along S^{ao}. This yields Theorem 4.1.3 when $m = n$.

Let $f \in H^0(S, \mathcal{L}^k)$. Then $\Theta(f)$ is a section of $\mathcal{L}^k \otimes \mathcal{Q}^{(p)} \otimes \mathcal{Q}$ over S^{ord} and we have to show that it extends holomorphically to S. Since S is non-singular, as in the case $n > m$, it is enough to show that it extends holomorphically to $S' = S^{\mathrm{ord}} \cup S^{\mathrm{ao}}$.

Let $\tau' : T' \to S'$ be the intermediate Igusa variety constructed above. Let a be, as before, the tautological $p^2 - 1$ root of h over T'; it vanishes to order 1 along $T^{\mathrm{ao}} = \tau'^{-1}(S^{\mathrm{ao}})$. Over T (the pre-image of S^{ord}), where a does not vanish, we can write the trivialization ε of $\mathcal{L}$ as $f \mapsto f/a^k$. This introduces a pole of order k, at most, along T^{ao}. Let $y \in T^{\mathrm{ao}}$ be a geometric point and $x = \tau'(y)$. Let ζ be the $p+1$ root of 1 such that $y \in T'_\zeta$. Let u, v_i be formal parameters at x and w, v_i formal parameters at y so that $u = 0$ is a local equation of S^{ao} and

$$u = w^{p-1}.$$

Let

$$f/a^k = \sum_{r=-k}^{\infty} g_r(v) w^r$$

be the Taylor expansion of f/a^k in $\widehat{\mathcal{O}}_{T',y}$, where the $g_r(v)$ are power series in the v_i. Note that $du = d(w^{p-1}) = -w^{p-2}dw$. Thus, similarly to the case $n > m$

$$\begin{aligned} d(f/a^k) &= \sum_{r=-k}^{\infty} w^r dg_r(v) + \sum_{r=-k}^{\infty} r g_r(v) w^{r-1} dw \\ &= \sum_{r=-k}^{\infty} w^r dg_r(v) - \sum_{r=-k}^{\infty} r g_r(v) w^{r-(p-1)} du \\ &= \sum_{r=-k}^{\infty} w^r dg_r(v) - \sum_{r=-k}^{\infty} r g_r(v) w^r u^{-1} du. \end{aligned}$$

We conclude the proof as in the case $n > m$. $\square$

5. Theta cycles

For the group GL_2, the application of the theta operator to mod p modular forms was linked to twisting Galois representations by the cyclotomic character (see [Se1] over $\mathbb{Q}$ and [A-G] over a totally real base field). The variation of the weight filtration upon iteration of Θ was of much interest in this context. While the connection to Galois representations in the unitary case requires further study, our goal here is to present a similar behavior on the level of q-expansions. We consider only signature $(n, 1)$, $n > 1$, as signature $(1, 1)$ is essentially the case of modular curves.

In this section, let S be a *connected component* of the special fiber of a unitary Shimura variety of signature $(n, 1)$, so that $\mathcal{P}_\mu \otimes \mathcal{Q} \simeq \mathcal{Q}^{(p)} \otimes \mathcal{Q} \simeq \mathcal{L}^{p+1}$. The theta operator maps $\mathcal{L}^k$ to $\mathcal{L}^{k+p+1}$ and may be iterated. The index set $\check{H}$ for the FJ expansions at a given level and a given rank-1 cusp may be identified with $\mathbb{Z}$ so that $\check{H}^+$ is identified with the non-negative integers. The effect of Θ on FJ expansions is

$$\Theta\left(\sum_{n\geq 0} a(n)\cdot q^n\right) = \sum_{n\geq 0} a(n)n\cdot q^n. \tag{5.0.1}$$

Given the q-expansion principle and the irreducibility of the Igusa variety $T^1_{1,1}$ (see Proposition 4.2.2), the proofs of the following results are verbatim as for signature $(2, 1)$, see [dS-G1] §§3.1-3.3.

Lemma 5.0.1. *Let ξ be a rank 1 cusp on S^*. Let ℓ be the non-zero section of $\mathcal{L}$ used to trivialize $\mathcal{L}$ at a formal neighborhood of ξ as before. Consider the homomorphism*

$$FJ_\xi : \bigoplus_{k=0}^{\infty} H^0(S, \mathcal{L}^k) \to \mathcal{FJ}_\xi,$$

where $\mathcal{FJ}_\xi$ is as in Proposition 3.1.2 and where we have identified formal sections of $\mathcal{L}^k$ near ξ with elements of $\widehat{\mathcal{O}}_{S^*,\xi}$ by dividing the sections by ℓ^k. Then the kernel of FJ_ξ is given by the ideal

$$\ker(FJ_\xi) = (h-1),$$

where h is the Hasse invariant.

Given an element $f = f(q) \in \mathcal{FJ}_\xi$ of the form $FJ_\xi(g)$, $g \in H^0(S, \mathcal{L}^k)$, we denote by $\omega(f)$ the minimal $k \geq 0$ for which there exists such a g. We call $\omega(f)$ the *filtration* of f. By the previous Lemma, if f arises from g of weight k then $\omega(f) \equiv k \mod (p^2 - 1)$.

Proposition 5.0.2. *Let $f \in H^0(S, \mathcal{L}^k)$ be in the image of Θ, i.e. $f = \Theta(g)$.*

(i) We have $\Theta^{p-1}(f) = fh$ where h is the Hasse invariant.

(ii) The sequence $\omega(\Theta^i(f))$ $i = 0, 1, 2, \ldots, p-1$ increases by $p+1$ at each step, except for a single $i = i_0(f) < p-1$ for which $\omega(\Theta^{i+1}(f)) = \omega(\Theta^i(f)) - p^2 + p + 2$.

The combinatorics of weights has some peculiarities not present in the case of elliptic modular forms, see [dS-G1].

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