PERFECTOID SPACES AND GALOIS REPRESENTATIONS

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Abstract. These are notes for a seminar conducted with David Kazhdan on Scholze's work. See the bibilography for references. The notes will be updated periodically, as the seminar progresses.

Standard notation.

- $\mathbb{A}$- the adele ring of $\mathbb{Q}$, $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$
- S - a finite set of primes containing ∞ and the finite prime p
- G_S - the Galois group of the maximal extension of $\mathbb{Q}$ which is unramified outside S
- Fr_l - the conjugacy class in G_S of a geometric Frobenius at a prime $l \notin S$
- $\mathbf{G} = Sp_{2n/\mathbb{Z}}, \quad \mathbf{M} = GL_{n/\mathbb{Z}}$
- $\mathbf{A}_H$ - a maximal torus in the center of a connected split reductive group $\mathbf{H}$, $A_H = \mathbf{A}_H(\mathbb{R})^0$
- $\mathbb{T} = \mathbb{T}^G$, resp. $\mathbb{T}^M$ - the unramified Hecke algebra outside S of $\mathbf{G}$, resp. $\mathbf{M}$
- $K = K_\infty K_f$ where $K_\infty = SO(n)$, $K_f = K_S K^S \subset \mathbf{M}(\mathbb{A}_f)$ compact open and similarly for other groups such as $\mathbf{G}$
- $X_K = X_K^G$ - the locally symmetric space $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/A_G K$, similarly X_K^M etc.
- S, S_M^G - the Satake transform, partial Satake transform (depending on circumstances)

1. Introduction [1 week]

1.1. Galois representations.

1.1.1. *Geometric Galois representations.*

Definition. A geometric Galois representation is a continuous representation

$$\rho : G_\mathbb{Q} \to GL_n(\overline{\mathbb{Q}}_p)$$

satisfying:

(i) ρ is unramified outside a finite set S of (finite or infinite) primes of $\mathbb{Q}$. The smallest such S is denoted $Bad(\rho)$.

(ii) ρ is *de Rham* - a technical condition on $\rho|G_{\mathbb{Q}_p}$, to be discussed later.

Remark. (a) Such a ρ is equivalent to a representation $\rho : G_\mathbb{Q} \to GL_n(E)$ where E is a finite extension of $\mathbb{Q}_p$. Prove this as an exercise. Show first that it does not simply follow from the compactness of $G_\mathbb{Q}$. Indeed, show that there exist compact subgroups of $GL_n(\overline{\mathbb{Q}}_p)$ that can not be conjugated into any $GL_n(E)$. Observe, that if we knew that $G_{\mathbb{Q},S}$ were topologically finitely generated, the claim would follow easily. This however, is not known to be true (and according to Neukirch, maybe

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not expected to be true). On the other hand, it *is* known that $G_{F,S}(p)$ (the Galois group of the maximal pro-p extension of F unramified outside S) is topologically finitely generated for any number field F. Prove the claim as follows. Let R be the ring of integers in $\overline{\mathbb{Q}}_p$. Prove that the R-span of the image of ρ is bounded, and in fact finitely generated as an R-module, hence that ρ can be conjugated to lie in $GL_n(R)$. Using compactness show that the image in $GL_n(\overline{\mathbb{F}}_p)$ is finite. Using the fact that $G_{F,S}(p)$ is finitely generated, prove the claim.

(b) Since $G_{\mathbb{Q}}$ is compact, we may even assume that the image of ρ is contained in $GL_n(\mathcal{O}_E)$, hence we may reduce to get a representation into $GL_n(\kappa_E)$. Different integral models may result, in general, in non-equivalent representations over the residuel field.

(c) There are only countably many geometric Galois representations. This relies on the notion of being "de Rham".

(d) If $n = 1$, a Galois character into $\mathcal{O}_E^\times$ is de Rham if and only if, for a suitable embedding of $\overline{\mathbb{Q}}$ in both $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$, ρ comes from a Hecke character of type A_0 (i.e. whose infinity type is $z \mapsto z^k$ for some integer k) via Weil's recipe of attaching Galois characters to such Hecke characters.

Example. Let $X_{/\mathbb{Q}}$ be a proper and smooth variety. Then $(V, \rho) = H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$ is geometric. This is a deep theorem of Tsuji (1999, Inv. Math.). Of course, point (i), more precisely, being unramified at every prime not equal to p where X has good reduction, is old, and due to Grothendieck. Tsuji's contribution was to verify point (ii), that such an X is de Rham. Kisin generalized it to non-proper varieties. Important related work was done by Fontaine and Messing, Faltings and Kato-Hyodo.

The *Fontaine-Mazur conjecture* is a partial converse when $n = 2$. It states that any 2-dimensional geometric ρ which is odd (in the sense that $\det \rho(c) = -1$ where c is complex conjugation) "comes from geometry". In fact, it predicts that the associated geometric object is a *modular form* for a congruence subgroup of $SL_2(\mathbb{Z})$. The precise way in which a modular form may be regarded as a geometric object involves the notion of *motives.* However, to point at a modular form as the source for the representation ρ is much more than to say that ρ "comes from geometry". Thus the Fontaine-Mazur conjecture encompasses all of the modularity results of Taylor-Wiles. Thanks to work of Kisin, Khare and Wintenberger, this conjecture is essentially proven today.

1.1.2. *The L-function.* Let (ρ, V) be as in the example. Let Fr_l be a geometric Frobenius at l. Then Deligne's proof of the Weil conjectures implies that for a good prime $l \neq p$

$$P_{l,\rho}(X) = \det(1 - \rho(Fr_l)X) \in \mathbb{Z}[X] \subset \overline{\mathbb{Q}}_p[X]$$

and is independent of p. It furthermore implies that for S the set of finite primes where X has bad reduction

$$L_S(\rho, s) = \prod_{l \notin S} P_{l,\rho}(l^{-s})^{-1}$$

converges absolutely and uniformly on compact sets in the half-plane $Re(s) > 1 + \frac{i}{2}$.

Neither independence of p, nor convergence of the L-series are known for a general geometric Galois representation ρ.

1.1.3. *A remark on torsion in the cohomology.* Recall that

$$H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p) := H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and that

$$H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_p) := \lim_{\leftarrow} H^i_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^k\mathbb{Z})$$

may contain a lot of torsion classes. These classes give rise to Galois representations over rings like $\mathbb{Z}/p^k\mathbb{Z}$ but both the classes and the associated representations get lost when we tensor by $\mathbb{Q}$, and do not show up in the L function.

1.1.4. *Pseudo-representations and Chenevier's theory of determinants.* The notion of a pseudorepresentation was introduced by Wiles in his work on the Main Conjecture of Iwasawa theory, and developed further by Taylor. It is a technical notion which is needed when one attempts to construct representations by p-adic interpolation. In [Chen] Chenevier generalized it to work over arbitrary p-adic base rings, and gave an elegant treatment of the subject. Roughly speaking a pseudorepresentation of a group G with values in a ring A is a function from G to A that "looks like a character of a representation". Chenevier introduced the notion of a "d-dimensional determinant" which is something that "looks like the characteristic polynomial of d-dimensional representation". [*This subsection will be completed if I need this part in the seminar*].

1.2. The locally symmetric spaces associated with $GL_{n/\mathbb{Q}}$ and their cohomologies.

1.2.1. *The space X_K.* Fix $n \geq 1$, a prime p, and a finite set S of (finite or infinite) primes, containing p and ∞. Let G_S be the Galois group of the maximal extension of $\mathbb{Q}$ which is unramified outside S.

Consider the group scheme $\mathbf{M} = GL_{n/\mathbb{Z}}$. Let $K^0_\infty = SO(n)$ and let $K_f \subset \mathbf{M}(\mathbb{A}_f)$ be a small enough compact, open subgroup (e.g. the full congruence group of level ≥ 3). We assume that

$$K_f = K_S K^S$$

where $K_S \subset \prod_{l\in S} \mathbf{M}(\mathbb{Z}_l)$ is open and $K^S = \prod_{l \notin S} \mathbf{M}(\mathbb{Z}_l)$. Let $K = K^0_\infty K_f$, and

$$X_K = \mathbf{M}(\mathbb{Q}) \setminus \mathbf{M}(\mathbb{A})/\mathbb{R}^\times_+ K = \mathbf{M}(\mathbb{Q}) \setminus [\mathbf{M}(\mathbb{R})/\mathbb{R}^\times_+ K^0_\infty \times \mathbf{M}(\mathbb{A}_f)/K_f].$$

This is the locally symmetric space of level K_f associated with $\mathbf{M}$. It is a finite union of real manifolds of the form $\Gamma \setminus \mathcal{H}_n$ where[1] $\mathcal{H}_n = SL_n(\mathbb{R})/SO(n)$ and Γ is conjugate in $SL_n(\mathbb{R})$ to a congruence subgroup of $SL_n(\mathbb{Z})$.

If $n = 1$ then X_K is a finite group, which is identified, by class field theory, with the Galois group of a finite abelian extension of $\mathbb{Q}$.

If $n = 2$ there is a natural complex structure on X_K induced from the identification of $\mathcal{H}_2$ with the usual upper half plane. In this case X_K is the complex points of a (disconnected, in general, and open) modular curve.

If $n \geq 3$ X_K is only a real manifold. If $K' \subset K$ then $X_{K'} \to X_K$ is a finite unramified cover of degree $[K : K']$.

[1]The map $A \mapsto A \cdot {}^t A$ identifies $\mathcal{H}_n$ with the space of positive definite symmetric real matrices of determinant 1, hence its real dimension is $n(n+1)/2 - 1$.

Fact. *For any i, n, the (singular) cohomology groups $H^i(X_K, \mathbb{Z})$ are* finitely generated. This follows from work of Borel and Harish-Chandra who constructed fundamental domains of finite type for these locally symmetric space (following earlier work of Siegel). See A. Borel: *Introduction aux groupes arithmetiques.* It also follows from the Borel-Serre compactification of X_K into a manifold with corners, which we discuss later.]

We shall consider singular cohomology groups with coefficients in an abelian (additive) group R, such as $\mathbb{F}_p$. Recall the relation between these groups and the homology (with $\mathbb{Z}$-coefficients). For any topological space X there is an exact sequence

$$0 \to Ext(H_{i-1}(X), R) \to H^i(X, R) \to Hom(H_i(X), R) \to 0.$$

Later on we shall have to consider similar locally symmetric spaces attached to other reductive groups. To stress the dependence on $\mathbf{M}$ we shall denote then X_K by X_K^M. In fact, this is the reason for our unusual choice of the letter $\mathbf{M}$ to denote GL_n. The letter $\mathbf{G}$ is reserved for a larger group (when the ground field is $\mathbb{Q}$ as in these notes, it will be Sp_{2n}, if the ground field were quadratic imaginary, it would be an appropriate unitary group). The group $\mathbf{M}$ will show up as the Levi factor of a parabolic subgroup of $\mathbf{G}$ and this will have consequences for X_K^M, as it will "appear" in the boundary of a similar locally symmetric space for $\mathbf{G}$. This set-up is important for all that follows.

1.2.2. *The Hecke algebra.* We work with $\mathbb{Z}_p$ coefficients throughout. For every prime $l \notin S$ (in particular $l \neq p$) the local Hecke algebra at l is

$$\mathbb{T}_l = \mathcal{H}(\mathbf{M}(\mathbb{Q}_l), \mathbf{M}(\mathbb{Z}_l)) = \mathbb{Z}_p[\mathbf{M}(\mathbb{Z}_l) \setminus \mathbf{M}(\mathbb{Q}_l)/\mathbf{M}(\mathbb{Z}_l)].$$

Its elements are compactly supported, $\mathbf{M}(\mathbb{Z}_l)$-bi-invariant, $\mathbb{Z}_p$-valued functions on $\mathbf{M}(\mathbb{Q}_l)$. The product is convolution, where the Haar distribution is normalized so that it gives $\mathbf{M}(\mathbb{Z}_l)$ the measure 1. Thus

$$\phi * \psi(g) = \int_{\mathbf{M}(\mathbb{Q}_l)} \phi(gh^{-1})\psi(h)dh.$$

The theorem on *elementary divisors* says that each double coset in $\mathbf{M}(\mathbb{Z}_l) \setminus \mathbf{M}(\mathbb{Q}_l)/\mathbf{M}(\mathbb{Z}_l)$ is represented by a unique matrix of the form

$$diag[l^{e_1}, l^{e_2}, \ldots, l^{e_n}]$$

with integers $e_1 \geq e_2 \geq \cdots \geq e_n$. This gives the structure of $\mathbb{T}_l$ as a module. The *ring structure* is commutative and is given, over $\mathbb{Z}_p[\sqrt{l}]$, by the following theorem (note that l may or may not be a square in $\mathbb{Z}_p$ so we may need to enlarge the ground ring, but a more subtle point, often overlooked, is that in either case, the Satake isomorphism *depends* on the choice of a square root of l).

Theorem. *(Satake isomorphism) There is an isomorphism*

$$S : \mathbb{Z}_p[\sqrt{l}] \otimes_{\mathbb{Z}_p} \mathbb{T}_l \simeq \mathbb{Z}_p[\sqrt{l}][X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n}.$$

The ring on the right is the ring of symmetric Laurent polynomials.

Recall the construction of S. Let A be the torus of diagonal matrices in $\mathbf{M}(\mathbb{Q}_l)$ and A^0 its intersection with $\mathbf{M}(\mathbb{Z}_l)$. Then

$$ord_l : A \to \Lambda = \mathbb{Z}^d$$

is surjective, has A^0 for its kernel, and the ring $\mathcal{H}(A, A^0)$ (defined in the same way as the Hecke algebra for GL_n) is identified with $\mathbb{Z}_p[\Lambda] = \mathbb{Z}_p[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Indeed, if we denote by $ch(\lambda)$ the characteristic function of $ord_l^{-1}(\lambda)$ then $ch(\lambda) * ch(\lambda') = ch(\lambda + \lambda')$ and the $ch(\lambda)$ make up a basis for $\mathcal{H}(A, A^0)$. Now

$$S : \mathbb{Z}_p[\sqrt{l}] \otimes_{\mathbb{Z}_p} \mathcal{H}(\mathbf{M}(\mathbb{Q}_l), \mathbf{M}(\mathbb{Z}_l)) \to \mathbb{Z}_p[\sqrt{l}] \otimes_{\mathbb{Z}_p} \mathcal{H}(A, A^0)$$

is defined by

$$S\phi(a) = \delta(a)^{1/2} \int_U \phi(au) du.$$

Here U is the unipotent radical of the standard Borel subgroup AU of upper triangular matrices in $\mathbf{M}(\mathbb{Q}_l)$ and du is normalized to give $U(\mathbb{Z}_l)$ the measure 1. The unimodular character δ is given by

$$\delta(diag[a_1, \dots, a_n]) = |a_1^{n-1} a_2^{n-3} \cdots a_n^{1-n}|_l.$$

The theorem asserts that S is an isomorphism onto the invariants of the Weyl group on $\mathbb{Z}_p[\sqrt{l}] \otimes_{\mathbb{Z}_p} \mathcal{H}(A, A^0)$.

Let $T_{l,i} \in \mathbb{T}_l$ $(1 \leq i \leq n)$ be equal to $l^{-i(n-i)/2}$ times the characteristic function of the double coset of the matrix

$$\begin{pmatrix} l & & & & \\ & l & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

where there are i l's and $n - i$ $1's$ on the diagonal. Then one computes that $T_{l,i}$ gets mapped under the Satake isomorphism to the i-th elementary symmetric polynomial. We shall identify $T_{l,i}$ with $S(T_{l,i})$ when necessary. Note that $T_{l,n}$ is the characteristic function of $l\mathbf{M}(\mathbb{Z}_l)$, that it is invertible (its inverse being the characteristic function of $l^{-1}\mathbf{M}(\mathbb{Z}_l)$) and corresponds, under the Satake isomorphism, to $X_1 X_2 \dots X_n$. Every symmetric Laurent polynomial may be multiplied by a high enough power of this element to make it a symmetric polynomial. Thus the Hecke algebra is generated over $\mathbb{Z}_p[\sqrt{l}]$ by $T_{l,i}$ $(1 \leq i \leq n)$ and $T_{l,n}^{-1}$.

Let $P_l(X) \in \mathbb{Z}_p[\sqrt{l}] \otimes_{\mathbb{Z}_p} \mathbb{T}_l[X]$ be given by

$$P_l(X) = \sum_{i=0}^{n} (-1)^i l^{i(n-1)/2} T_{l,i} X^i = \prod_{i=0}^{n} (1 - l^{(n-1)/2} X_i X).$$

This is the *Hecke polynomial* at the prime l.

The global (prime to S) Hecke algebra is

$$\mathbb{T} = \otimes'_{l \notin S} \mathbb{T}_l.$$

By the restricted tensor product we mean the direct limit of the finite tensor products, where we use the unit elements in the rings to embed one finite tensor product in a larger one. The involution $g \mapsto g^{-1}$ induces an involution $\phi \mapsto \dot{\phi}$ on $\mathbb{T}_l$, namely $\dot{\phi}(g) = \phi(g^{-1})$. Since $\dot{T}_{l,i} = T_{l,n}^{-1} T_{l,n-i}$ under the Satake isomorphism this involution carries over to the involution of the symmetric Laurent polynomials induced by $X_i \mapsto X_i^{-1}$.

Remark. (on normalizations). Scholze writes $i(n+1)/2$ instead of $i(n-1)/2$ for the power of l preceeding $T_{l,i}$. My expression agrees with [Rec]. Of course, changing X to lX transforms one into the other. When $n=2$ one usually writes T_l for $\sqrt{l}T_{l,1}$ and $\langle l\rangle$ for $T_{l,2}$. The Hecke polynomial becomes then $1-T_lX+l\langle l\rangle X^2$.

Another common normalization is to consider instead of $P_l(X)$ the polynomial

$$P_l(l^{-(n-1)/2}X)=\sum_{i=0}^{n}(-1)^iT_{l,i}X^i.$$

While the first normalization agrees with a functional equation relating s to $n-s$ the second agrees with $s\mapsto 1-s$. The first normalization is generally preferred by number theorists, and the second by representation theorists.

1.2.3. *The action of the Hecke algebra on the cohomology of X_K.* Write $X=\mathbf{M}(\mathbb{Q})\setminus\mathbf{M}(\mathbb{A})/\mathbb{R}_+^\times$ so that $X_K=X/K$. Let $g\in\mathbf{M}(\mathbb{Q}_l)$. Then the map $kgK\mapsto k(K\cap gKg^{-1})$ $(k\in K)$ is a bijection

$$KgK/K\simeq g^{-1}KgK/K\simeq g^{-1}Kg/(g^{-1}Kg\cap K)\simeq K/(K\cap gKg^{-1})$$

so the degree of $pr_K:X_{K\cap gKg^{-1}}\to X_K$ is $[KgK/K]$. Consider also the maps $pr_{gKg^{-1}}:X_{K\cap gKg^{-1}}\to X_{gKg^{-1}}$ and $R_g:X_{gKg^{-1}}\simeq X_K$ given by $x(gKg^{-1})\mapsto (xg)K$. Let

$$T_g=R_g\circ(pr_{gKg^{-1}})_*\circ(pr_K)^*:X_K\dashrightarrow X_K$$

be the correspondence whose degree is $[KgK/K]$.

If we let the double coset K_lgK_l act as T_g and extend linearly to $\mathbb{T}_l$ this defines a *right action* of $\mathbb{T}_l$ (taken now with $\mathbb{Z}$-coefficients) as a ring of correspondences on X_K. By "right action" we mean that the convolution product $K_lgK_l\cdot K_lhK_l$ in $\mathbb{T}_l$ corresponds to $T_h\circ T_g$. Now quite generally, any correspondence $T:X\dashrightarrow Y$ between two manifolds induces, by pull-back, a homomorphism on the singular cohomology

$$T^*:H^i(Y,\mathbb{Z}_p)\to H^i(X,\mathbb{Z}_p)$$

and $(T\circ R)^*=R^*\circ T^*$, so if we let the double coset K_lgK_l act as T_g^* on $H^i(X_K,\mathbb{Z}_p)$ we get a *left action* of the ring $\mathbb{T}_l$ on $H^i(X_K,\mathbb{Z}_p)$. These actions, for various $l\notin S$, commute, so can be combined to give an action of the global Hecke algebra $\mathbb{T}$. In particular, we denote the operator $T_{l,j}^*|H^i(X_K,\mathbb{Z}_p)$ still by $T_{l,j}$ and call it the jth Hecke operator at l (on $H^i(X_K,\mathbb{Z}_p)$).

In the same way we get actions of $\mathbb{T}$ on the finite-dimensional vector spaces $H^i(X_K,\mathbb{Q}_p)$ and $H^i(X_K,\mathbb{F}_p)$.

Remark. On correspondences. Let X and Y be as above, two smooth oriented manifolds of dimension d. A correspondence between them is a d-dimensional closed submanifold $Z\subset X\times Y$ such that both projections $p_X:Z\to X$ and $p_Y:Z\to Y$ are "nice". If X and Y are complex algebraic varieties and Z is a closed algebraic subvariety of the product, then by "nice" we mean finite and flat (but possibly ramified). For smooth real manifolds one may develop a similar notion, but we shall not do it, as all our correspondences will eventually be given by explicit group theoretic formulae obtained by decomposing double cosets to a finite union of one-sided cosets. In any case, we should at least require "nice" to be "proper". This suffices to define a map

$$[Z]_Y^X=(p_Y)_*\circ p_X^*:\ H^i(X,R)\to H^i(Y,R)$$

between singular cohomology, and a similar map interchanging the roles of X and Y . Here R is any ring of coefficients. Note that p_Y is proper so the push-forward on cohomology is defined. One way to see it is to use the version of Poincaré duality for (arbitrary topological) manifolds asserting that $H^i(X)$ is naturally isomorphic to the Borel-Moore homology $H^{BM}_{d-i}(X)$. The latter is defined as singular homology, when one replaces finite chains by *locally* finite chains (chains whose intersection with every compact subset is finite). Borel-Moore homology is not functorial in X, but for locally compact spaces it *is* covariant with respect to proper maps. If we use $\mathbb{R}$ coefficients we can use instead the duality between $H^i(X)$ and $H^{d-i}_c(X)$ and the fact that compactly supported cohomology is contravariant with respect to proper maps. However, we are specifically interested in torsion classes, so this is not enough! In our example we take $X = X_K$,$Y = X_{gKg^{-1}}$ (which is isomorphic to X under R_g) and $Z = X_{K\cap gKg^{-1}}$.

1.3. **The main theorem and some corollaries.**

1.3.1. *Galois representations attached to eigen-classes in* $H^i(X_K, \overline{\mathbb{F}}_p)$. Let X_K, p and S be as before. By a *system of Hecke eigenvalues occuring in* $H^i(X_K, \overline{\mathbb{F}}_p)$ we mean a homomorphism $\psi : \mathbb{T} \to \overline{\mathbb{F}}_p$ so that $H^i(X_K, \overline{\mathbb{F}}_p)[\psi] \neq 0$.

Theorem 1. *[Tors, Corollary V.4.3] Let ψ be a system of Hecke eigenvalues occuring in $H^i(X_K, \overline{\mathbb{F}}_p)$. Then there exists a unique continuous semisimple representation*

$$\sigma_\psi : G_S \to GL_n(\overline{\mathbb{F}}_p)$$

such that for any $l \notin S$, if we denote by $Fr_l \in G_S$ any geometric *Frobenius at l,*

$$\det(1 - X \cdot \sigma_\psi(Fr_l)) = \psi(P_l(X)).$$

Remark. (i) Uniqueness is obvious since the Fr_l are dense in G_S and their characteristic polynomials determine their traces, hence the character of σ_ψ. But a semi-simple representation is determined by its character. Note that σ_ψ factors through a finite quotient of G_S.

(ii) When $n = 1$ X_K is the finite group $\mathbb{Q}^\times \setminus \mathbb{A}^\times/\mathbb{R}^\times_+ K$, the Hecke operator T_l ($l \notin S$) is the coset lK, its action is the action of l by (right) multiplication, only $i = 0$ counts, $H^0(X_K, \overline{\mathbb{F}}_p)$ is the space of functions from X_K to $\overline{\mathbb{F}}_p$, $P_l(X) = 1 - X$ and the theorem asserts the existence of a Galois character satisfying

$$\sigma_\psi(Fr_l) = \psi(l).$$

The theorem captures class field theory (over $\mathbb{Q}$). Already here we see that it was necessary to include ∞ in S, because in K^0_∞ we included only the *connected component* of the identity.

(iii) When $n = 2$, $X_K = Y_K(\mathbb{C})$ where Y_K is an open (i.e. without the cusps), disconnected (in general) modular curve defined over $\mathbb{Q}$. Let $i = 1$ (this is the only interesting index). We then have a canonical isomorphism

$$H^1(X_K, \mathbb{F}_p) \simeq H^1_{et}(Y_{K/\overline{\mathbb{Q}}}, \mathbb{F}_p)$$

and the group on the right hand side carries a natural action of G_S. One has to distinguish two cases: (a) ψ is "Eisenstein", i.e. it occurs in the cokernel of the injective map

$$H^1(\overline{X}_K, \mathbb{F}_p) \hookrightarrow H^1(X_K, \mathbb{F}_p)$$

where $\overline{X}_K$ is the compactified modular curve. This case is easy, since one can analyze directly the action of the Hecke algebra on the cusps. One can construct σ_ψ "by hand". It is reducible, the sum of two one-dimensional characters. (b) Suppose that the system of Hecke eigenvalues comes from the cohomology of the *complete* modular curve, i.e. $H^1(\overline{X}_K, \overline{\mathbb{F}}_p)[\psi] \neq 0$. The theorem asserts the existence of a 2-dimensional semisimple representation σ_ψ such that

$$\det(X - \sigma_\psi(Fr_l)) = X^2 - \psi(T_l)X + l\psi(\langle l \rangle)$$

for every $l \notin S$. This follows from the Eichler-Shimura congruence relation. It is classicaly proved for cohomology (and Galois representations) with $\overline{\mathbb{Q}}_p$-coefficients, and the corresponding characteristic 0 Galois representation comes out to be irreducible and odd. But curves do not have torsion in their cohomology, so the result as stated follows from it. Let us also remark that cases (a) and (b) are not mutually exclusive when one deals with $\mathbb{F}_p$-coefficients, as there could be congruences modulo p between Eisenstein series and cusp forms.

1.3.2. *Working with $\mathbb{Z}/p^m\mathbb{Z}$ coefficients.* One would like to state a similar theorem for $\mathbb{Z}/p^m\mathbb{Z}$ coefficients. There are two technical difficulties. The first is that Scholze does not quite construct, in this generality, a representation, but only a pseudorepresentation, or more precisely, a "determinant" in the sense of Chenevier. The other difficulty is that the relation between the corresponding "determinant of Frobenius" and "Hecke polynomial" only holds modulo a certain defect. Luckily, this defect is independent of m (and in a certain sense, independent of K too).

Theorem 2. *[Tors, Theorem V.4.1] There exists an integer N depending only on n, such that for any K as above, and any indices $i \geq 0$, $m \geq 1$, if we let $\mathbb{T}(K, i, m)$ be the image of the Hecke algebra $\mathbb{T}$ in the endomorphism algebra of $H^i(X_K, \mathbb{Z}/p^m\mathbb{Z})$, then there is an ideal $I \subset \mathbb{T}(K, i, m)$ with $I^N = 0$ and an n-dimensional determinant D of G_S over the ring $\mathbb{T}(K, i, m)/I$, which satisfies the relation*

$$D(1 - X \cdot Fr_l) = P_l(X) \mod I$$

for every $l \notin S$.

1.3.3. *Working with a local system $\mathcal{M}_\xi/p^m\mathcal{M}_\xi$ as coefficients.* With a little extra effort, it is possible to extend Theorems 1 and 2 to include certain local systems as coefficients. See [Tors, V].

1.3.4. *An application to the Langlands correspondence for GL_n over $\mathbb{Q}$.* Fix an isomorphism of $\mathbb{C}$ with $\overline{\mathbb{Q}}_p$. Let π be a cuspidal automorphic representation of $GL_n(\mathbb{A})$. We assume that π_∞ is regular L-algebraic [*spell out what this means, and note that by results of Clozel, this means that a certain Tate twist of π is cohomological*]. Assume also that π_l is unramified if $l \notin S$, and let $\psi : \mathbb{T} \to \overline{\mathbb{Q}}_p$ be the hecke character ("system of Hecke eigenvalues") attached to π.

Theorem 3. *Under the above assumptions, there exists a unique continuous semisimple representation $\sigma_\pi : G_S \to GL_n(\overline{\mathbb{Q}}_p)$ such that for any $l \notin S$*

$$\det(1 - X \cdot \sigma_\pi(Fr_l)) = \psi(P_l(X)).$$

This theorem is the main theorem of [HLTT], and was proved there by more traditional methods. For self-dual π it has been known long before [Cloz].

Remark. Venkatesh and Bergeron [V-B] have shown that the torsion in the cohomology of $\Gamma\backslash\mathbf{G}(\mathbb{R})/K_\infty^0$ grows exponentially with the covolume of Γ, while the Betti numbers only grow linearly. Thus Theorem 1 and 2 are significantly stronger than Theorem 3 and can not be reduced to it. There are many torison Galois representations attached to torsion cohomology classes that do not come from characteristic 0 representations by reduction modulo p^m.

2. A survey of the proof of Theorem 1 [3 weeks]

2.1. **Galois representations attached to Siegel modular forms.**

2.1.1. *Preliminaries on the symplectic group and its symmetric space.* If one is to attach Galois representations to cohomology classes on X_K, algebraic geometry should show up somewhere. The key idea is due to Clozel. The space X_K "appears" in a Hecke-compatible way in the boundary of a certain compactification of a similar space associated with Sp_{2n}. But the locally symmetric spaces of Sp_{2n}, the Siegel modular varieties, are algebraic. This is still a long way from attaching Galois representations to Hecke eigenforms on Siegel modular varieties, let alone to torsion classes in the cohomology of these varieties, but at least is a good starting point.

Let $\mathbf{G} = Sp_{2n/\mathbb{Z}}$ be the group scheme of all $2n \times 2n$ matrices g satisfying

$$^t g \cdot J \cdot g = J$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the matrix of the standard symplectic form $\langle u, v\rangle = {}^t uJv$. The subgroup $\mathbf{P}$ consisting of all matrices

$$\{g = \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} \mid A^{-1}B \text{ is symmetric}\}$$

is a maximal parabolic subgroup whose Levi factor is the group $\mathbf{M} = GL_n$, the projection from $\mathbf{P}$ to $\mathbf{M}$ being $g \mapsto A$. We regard $\mathbf{M}$ both as a quotient and a subgroup. Thus

$$\mathbf{P} = \mathbf{MU}$$

is a semi-direct product, and the unipotent radical $\mathbf{U}$ in our case, is commutative, and isomorphic to $\mathbb{G}_a^{n(n+1)/2}$.

Let $G_\infty = \mathbf{G}(\mathbb{R})$. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via

$$(z_1, \ldots, z_n) \mapsto (Re(z_1), \ldots, Re(z_n), Im(z_1), \ldots, Im(z_n))$$

and let $(z, w) = \sum \overline{z}_k w_k$ be the standard hermitian inner product on $\mathbb{C}^n$. Then multiplicatoin by i corresponds to $-J$ and

$$(z, w) = \langle z, iw\rangle + i\,\langle z, w\rangle\,.$$

The group $K_\infty = U(n)$ is therefore embedded in G_∞ (check that it is equal to the intersection of G_∞ with $SO(2n)$). It is a maximal compact subgroup. Observe that

$$K_\infty^P = K_\infty \cap P_\infty$$

is the subgrop of all matrices in P_∞ where $A = {}^tA^{-1}$ and $B = 0$, hence is the standard $K_\infty^M = O(n)$ inside $GL_n(\mathbb{R})$. Since, by the Iwasawa decomposition,

$$G_\infty = P_\infty K_\infty,$$

we get for the symmetric spaces

$$\mathfrak{H} = G_\infty/K_\infty = P_\infty/K_\infty^P.$$

Recall that this is the Siegel space of all the complex symmetric matrices $Z = X+iY$ with $Y > 0$ (positive definite). Indeed, G_∞ acts transitively on $\mathfrak{H}$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto (AZ+B)(CZ+D)^{-1}$$

and if $Z_0 = iI$ then its stabilizer is K_∞. When we restrict to P_∞ we get the action

$$Z = X+iY \mapsto AZ\,{}^tA + B\,{}^tA = (AX\,{}^tA + B\,{}^tA) + i(AY\,{}^tA).$$

Now mod out also by the left action of the unipotent radical U_∞ of P_∞, i.e. project $\mathfrak{H}$ to M_∞/K_∞^M where $M_\infty = U_\infty \setminus P_\infty$ and $K_\infty^P \simeq K_\infty^M$ under the projection. Since the action of U_∞ on $\mathfrak{H}$ amounts to adding an arbitrary symmetric matrix to the real part of Z, the quotient is the cone of positive definite symmetric matrices. When we further divide by positive homotheties ($\mathbb{R}_+^\times$) we get the symmetric space of GL_n

$$\mathcal{H} = M_\infty/\mathbb{R}_+^\times K_\infty^M = SL_n(\mathbb{R})/SO(n).$$

The standard split (non compact) torus $\mathbf{T}$ is the set of matrices of the form $diag[a_1,\dots,a_n,a_1^{-1},\dots,a_n^{-1}]$. In our standard $U(n)$ we have a maximal compact torus T_c consisting (under the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$) of all the maps $(z_1,\dots,z_n) \mapsto (\zeta_1 z_1,\dots,\zeta_n z_n)$ where $|\zeta_i| = 1$.

The Weyl group of $\mathbf{G}$ ($W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$) is the group $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ of signed permutations. If we decompose $\mathbb{Q}^{2n}$ into a direct sum of n hyperbolic planes $H_k = \langle e_k, e_{n+k}\rangle$ in the standard way, the group S_n permutes the planes and the kth copy of $\mathbb{Z}/2\mathbb{Z}$ has the effect $e_k \mapsto e_{n+k}$, $e_{n+k} \mapsto -e_k$ in H_k (modulo the action of the standard torus).

2.1.2. *The symplectic Hecke algebra.* Let l be a prime. The local (spherical) Hecke algebra is

$$\mathbb{T}_l = \mathbb{Z}_p[\mathbf{G}(\mathbb{Z}_l) \setminus \mathbf{G}(\mathbb{Q}_l)/\mathbf{G}(\mathbb{Z}_l)] = \mathcal{H}(G_l, K_l).$$

As in the case of GL_n it is commutative and we have the Satake isomorphism

$$\mathbb{T}_l[\sqrt{l}] \simeq \mathbb{Z}_p[\sqrt{l}][X_1^{\pm 1},\dots,X_n^{\pm 1}]^{S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n}$$

whose image consists of all the symmetric Laurent polynomials which are symmetric also under $X_i \mapsto X_i^{-1}$ for each i. The formula for the Satake isomorphism and the proof of the isomorphism is the same as in the case of GL_n. The only thing that changes is that the Weyl group is now larger. An excellent survey of the Satake isomorphism is [Sat].

2.1.3. *The local unramified transfer from Sp_{2n} to GL_{2n+1} and the Hecke polynomial.* [*This subsection and in particular the next one are about things where my understanding is limited. It is also where Scholze's work still depends on work in progress of Arthur, on the stabilization of the twisted trace formula. Surprisingly here, the situation is better if one replaces the ground field by a quadratic imaginary field, because then there are unconditional results of Shin which allow one to get the desired representations of $U(n,n)$, the group which replaces Sp_{2n}.*]

Recall some definitions and conjectures from the theory of automorphic representations, specialized to our context. Let $\mathbf{G}$ be a split reductive group. Fix an integral model $\mathbf{G}_{/\mathbb{Z}}$ and assume that it has good reduction if $l \notin S$. Thus $K_l = \mathbf{G}(\mathbb{Z}_l)$ is a hyperspecial maximal compact subgroup of $G_l = \mathbf{G}(\mathbb{Q}_l)$.

The Langlands dual $\hat{G}$ of $\mathbf{G}$ is a complex reductive group.

Let π be a *discrete automorphic representation* of $\mathbf{G}(\mathbb{A})$, i.e. a closed subspace

$$\pi \subset L^2(\mathbf{G}(\mathbb{Q}) \setminus \mathbf{G}(\mathbb{A}))_\chi$$

where χ is a unitary character of the center of $\mathbf{G}(\mathbb{A})$, which is (topologically) irreducible under right translation by $\mathbf{G}(\mathbb{A})$. Then

$$\pi = \hat{\otimes}\pi_v$$

(a completed tensor product) where v runs over the primes of $\mathbb{Q}$, and π_l (for a finite prime l) is a smooth irreducible representation of G_l. Assume that for $l \notin S$ it is *unramified* (also called *spherical*) i.e. $\pi_l^{K_l}$ is non-zero. Then the commutativity of the spherical Hecke algebra $\mathbb{T}_l$ (this time we take it with $\mathbb{Z}$-coefficients) implies that $\pi_l^{K_l}$ is one-dimensional, so $\mathbb{T}_l$ acts on this line via a homomorphism $\psi_l : \mathbb{T}_l \to \mathbb{C}$ called the *Hecke character* of π_l. Now, the Satake isomorphism is an isomorphism of $\mathbb{C} \otimes \mathbb{T}_l$ with

$$\mathbb{C}[\mathbf{X}_*(\mathbf{T})]^W$$

where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$, and W its Weyl group. If $\hat{T}$ is a maximal torus in $\hat{G}$ then $\mathbf{X}_*(\mathbf{T}) = \mathbf{X}^*(\hat{T})$ and a homomorphism

$$\mathbb{C}[\mathbf{X}^*(\hat{T})]^W \to \mathbb{C}$$

is the same as a point of $\hat{T}/W$, i.e. a semisimple conjugacy class in $\hat{G}$ (every semi-simple element of $\hat{G}$ can be conjugated to lie in $\hat{T}$ and two elements of $\hat{T}$ are conjugate in $\hat{G}$ if and only if they are conjugate in $N_{\hat{G}}(\hat{T})$, i.e. if and only if they lie in the same orbit of W). Thus via the Satake isomorphism, giving ψ_l is the same as giving a semisimple conjugacy class $s(\pi_l)$ in $\hat{G}$. This conjugacy class is called the *Langlands parameter* of π at the unramified prime l. It determines π_l uniquely.

Let $\mathbf{H}$ be another split reductive group defined over $\mathbb{Q}$, and let $\eta : \hat{G} \to \hat{H}$ be a homomorphism. We then have the following easy local lemma.

Lemma. *Let $\mathbf{G}$ and $\mathbf{H}$ be as above (in particular, we assume that they are split). Let $\mathbb{T}_l^G$ and $\mathbb{T}_l^H$ be the spherical Hecke algebras w.r.t. hyperspecial maximal compact subgroups at l (with complex coefficients). Let $\eta : \hat{G} \to \hat{H}$ be a homomorphism of the Langlands dual groups (as algebraic groups over $\mathbb{C}$). Then there exists a unique homomorphism*

$$\eta^* : \mathbb{T}_l^H \to \mathbb{T}_l^G,$$

such that for any Hecke character $\psi_l^G : \mathbb{T}_l^G \to \mathbb{C}$, if we let $s(\psi_l^G) \in \hat{G}$ be the corresponding parameter, and $\psi_l^H = \psi_l^G \circ \eta^*$, then $\eta(s(\psi_l^G)) = s(\psi_l^H)$.

Proof. The homomorphism η carries semisimple conjugacy classes in $\hat{G}$ to semisimple conjugacy classes in $\hat{H}$, hence induces a map $\hat{T_G}/W_G \to \hat{T_H}/W_H$, which is a morphism of affine algebraic varieties. By the above discussion and the definition of the dual group, this map comes from an algebra homomorphism $\mathbb{C}[\mathbf{X}_*(\mathbf{T}_H)]^{W_H} \to \mathbb{C}[\mathbf{X}_*(\mathbf{T}_G)]^{W_G}$. Invoking the Satake isomorphisms gives η^*. $\square$

We remark that when the groups are not split, the lemma remains true if we replace the Langlands dual by the notion of the L-group, which is a semi-direct product of the dual group with the finite Galois group of a splitting field. However, for our purposes, split groups suffice. The following is *Langlands' functoriality conjecture.*

Conjecture. *For every automorphic representation π of $\mathbf{G}(\mathbb{A})$ there exists an automorphic representation Π of $\mathbf{H}(\mathbb{A})$ satisfying the following. Let l be an unramified prime for both groups (meaning that integral models have been fixed and the corresponding groups of $\mathbb{Z}_l$ points are maximal compact hyperspecial). Then if π_l is unramified, so is Π_l, and $\eta(s(\pi_l)) = s(\Pi_l)$.*

We emphasize that in general Π need not be unique, except when $\mathbf{H} = GL_h$, in which case "strong multiplicity one" implies that knowing Π_l for all but finitely many l determines Π.

Let us return to our setting. The Langlands dual of Sp_{2n} is $SO(2n+1)(\mathbb{C})$. Consider the standard embedding η of $SO(2n+1)(\mathbb{C})$ in $GL_{2n+1}(\mathbb{C})$. According to the conjecture, to any automorphic representation π of $Sp_{2n}(\mathbb{A})$ there should correspond an automorphic representation Π of $GL_{2n+1}(\mathbb{A})$ such that, if π_l is unramified, Π_l is also unramified, and the Langlands parameter of π_l (a semi-simple conjugacy class in $SO(2n+1)(\mathbb{C})$) gets mapped under η to the Langlands parameter of Π_l (a semisimple conjugacy class in $GL_{2n+1}(\mathbb{C})$).

The *local unramified transfer* $\pi_l \mapsto \Pi_l$ is determined by the embedding η, and is dual, by the Lemma, to a homomorphism of the spherical Hecke algebras in the opposite direction. In our case, η determines a homomorphism

$$\eta^* : \mathbb{Z}_p[\sqrt{l}][Y_1^{\pm 1}, \dots, Y_{2n+1}^{\pm 1}]^{S_{2n+1}} \to \mathbb{Z}_p[\sqrt{l}][X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n} = \mathbb{T}_l[\sqrt{l}].$$

A little group theory (involving the definition of the Langlands dual) shows that this is the homomorphism taking $\{Y_1, \dots, Y_{2n+1}\}$ to $\{X_1, X_1^{-1}, \dots, X_n, X_n^{-1}, 1\}$ (since the homomorphism ought to be defined only on symmetric Laurent polynomials, any permutation of these sets will yield the same map). Note that if $\phi \in \mathbb{Z}_p[\sqrt{l}][Y_1^{\pm 1}, \dots, Y_{2n+1}^{\pm 1}]^{S_{2n+1}}$ then the image of ϕ and the image of $\dot{\phi}$ in $\mathbb{T}_l[\sqrt{l}]$ coincide. This means that $\Pi_l \simeq \Pi_l^\vee$ (is self-dual). This can be also seen from the fact that any representation into an orthogonal group is self-dual.

Using this, we can define the Hecke operator $T_{l,i}$ $(1 \le i \le 2n+1)$ to be the image of the ith elementary symmetric polynomial in the Y_j in $\mathbb{T}_l[\sqrt{l}]$. In fact, it belongs to $\mathbb{T}_l$. We can now define the Hecke polynomial to be

$$\tilde{P}_l(X) = \sum_{i=0}^{2n+1} (-1)^i T_{l,i} X^i \in \mathbb{T}_l[X].$$

2.1.4. *The global endoscopic transfer.* The Langlands functoriality conjecture is in fact known in our case, for certain π's, as a result of deep (ongoing) work of Arthur on the twisted trace formula. For reasons unclear to me, it is called the *theory of endoscopic transfer.*

Let $k > n$. Then there exists a unique discrete series representation π_k of G_∞ with minimal K_∞-type $\det^k$ and infinitesimal character[2]

$$(k-1, k-2, \ldots, k-n) \in \mathbf{X}^*(\mathbf{T}).$$

Let X_K be the Siegel modular variety of level $K_f = K_S K^S$ (notation as usual, $K = K_\infty K_f$) which is small enough (e.g. principal level subgroup of level ≥ 3). It is a smooth quasi-projective variety defined over $\mathbb{Q}$. Let X_K^* be the minimal (Baily-Borel) compactification of X_K, which is a normal projective variety. Let A be the universal (principally polarized) abelian variety over X_K and $\omega = \det(e^*\Omega^1_{A/X_K})$ (where $e: X_K \to A$ is the zero section) the Hodge bundle. Then ω is an ample line bundle and in fact

$$X_K^* = Proj(\oplus_{k=0}^\infty H^0(X_K, \omega^k)).$$

Thus $H^0(X_K, \omega^k) = H^0(X_K^*, \omega^k)$ ($n > 2$, by the Koecher principle) is the space of holomorphic modular forms of weight k on X_K. If we let $\mathcal{I}$ be the ideal sheaf of $X_K^* \setminus X_K$ then $H^0(X_K^*, \omega^k \otimes \mathcal{I})$ is the space of *cusp forms* of weight k.

The Hecke algebra $\mathbb{T} = \otimes'_{l \notin S} \mathbb{T}_l$ acts on $H^0(X_K^*, \omega^k \otimes \mathcal{I})$ by the same recipe used to define its action on singular cohomology groups of the locally symmetric spaces for GL_n, see section 1.2.3.

Recall that $k > n$. If f is a Hecke newform in $H^0(X^*_{K/\mathbb{C}}, \omega^k \otimes \mathcal{I})$ and we denote by

$$f : \mathbb{T} = \otimes'_{l \notin S} \mathbb{T}_l \to \mathbb{C}$$

also the corresponding system of Hecke eigenvalues, then there exists a cuspidal automorphic representation $\pi = \pi_f$ of $\mathbf{G}(\mathbb{A})$ associated to f, unramified outside S, whose associated Hecke character at every $l \notin S$ is f_l. The archimedean component π_∞ of π is the above-mentioned π_k.

Theorem. *(Arthur) Let π be a cuspidal automorphic representation with $\pi_\infty = \pi_k$. Assume that π is unramified outside S. Then there exist integers satisfying*

$$n_1 r_1 + \cdots + n_m r_m = 2n + 1$$

and self-dual *cuspidal automorphic representations Π_i of $GL_{n_i}(\mathbb{A})$ such that for all $l \notin S$ all the $\Pi_{i,l}$ are unramified, and the Langlands parameters satisfy*

$$\eta(s(\pi_l)) = \bigoplus_{i=1}^m (s(\Pi_{i,l}) l^{(r_i-1)/2} \oplus s(\Pi_{i,l}) l^{(r_i-3)/2} \oplus \cdots \oplus s(\Pi_{i,l}) l^{(1-r_i)/2}).$$

Furthermore, for each i, a certain twist of $\Pi_{i,\infty}$ by an integral power of $|\cdot|$ is regular L-algebraic.

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[2]Recall that if π_∞ is an irreducible admissible representation of a connected reductive Lie group G_∞ and K_∞ is a maximal (connected) compact subgroup then the subspace of K_∞-finite vectors forms a $(\mathfrak{g}, K_\infty)$-module and the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ acts on it via a character χ, called the *infinitesimal character* of the representation. The Harish-Chandra isomorphism allows us to index the central characters χ_λ by weights $\lambda \in \mathfrak{h}^*$ where $\mathfrak{h}$ is a Cartan subalgebra. The characters χ_λ and χ_μ are equal if and only if $\lambda + \delta$ and $\mu + \delta$ are in the same orbit of the Weyl group, where δ is half the sum of the positive roots. One often abuses language and refers to λ itself (rather than χ_λ) as the infinitesimal character. Recall also that π_∞ is called *regular L-algebraic* if its infinitesimal character is the infinitesimal character of an algebraic (finite dimensional) representation of $\mathfrak{g}$.

2.1.5. *The Galois representation attached to Siegel modular forms.* In [Cloz], Clozel proved the following theorem. Fix an isomorphism of $\overline{\mathbb{Q}}_p$ with $\mathbb{C}$.

Theorem. *Let Π be a self-dual cuspidal automorphic representation of $GL_n(\mathbb{A})$ such that $\Pi|\cdot|^{k/2}$ is regular L-algebraic for some integer k. Then there exists a unique continuous self-dual semisimple Galois representation*

$$\sigma_\Pi : G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{Q}}_p)$$

with the following property. For every finite prime l such that Π_l is unramified, σ_Π is unramified at l and

$$\sigma_\Pi(Fr_l)^{ss} = s(\Pi_l).$$

Combining this with the last theorem on the existence of endoscopic transfer, and our definition of the local Hecke polynomials $\tilde{P}_l$, one gets the following.

Theorem 4. *[Tors, V.1.7] Let $k > n$ and let $f : \mathbb{T} \to \overline{\mathbb{Q}}_p$ be a system of Hecke eigenvalues occuring in $H^0(X^*_{K/\overline{\mathbb{Q}}_p}, \omega^k \otimes \mathcal{I})$ (i.e. factoring through the image of $\mathbb{T}$ in the endomorphism ring of this space). Then there exists a unique continuous semisimple representation*

$$\sigma_f : G_S \to GL_{2n+1}(\overline{\mathbb{Q}}_p)$$

which is self-dual, and such that for any $l \notin S$

$$\det(1 - X \cdot \sigma_f(Fr_l)) = f(\tilde{P}_l(X)).$$

Proof. Let π be the cuspidal automorphic representation attached to f. As we have noticed, it is unramified outside S and $\pi_\infty = \pi_k$. Assume for simplicity that in Arthur's theorem there is only one (self-dual, cuspidal, regular L-algebraic) Π so that $s(\pi_l)$ maps under η to $s(\Pi_l)$. Let σ_f be the Galois representation attached by Clozel to Π. Since the Langlands parameters of π and Π at l match, by the very definition of $\tilde{P}_l(X)$, its specialization under f gives the characteristic polynomial of Frobenius. □

Remark. If $n = 2$ then $Sp_2 = SL_2$ and f corresponds to a Hecke eigenform in the classical sense. By the Eichler-Shimura congruence relation, there is a 2-dimensional representation ρ_f attached to f. The representation σ_f is nevertheless 3-dimensional and self-dual. One most surely has the relation

$$\sigma_f = Hom^0(\rho_f, \rho_f)$$

(the trace-zero endomorphisms of ρ_f).

2.2. **The cohomology of the boundary.** In this section we relate the locally symmetric spaces of Sp_{2n} and GL_n and their singular cohomologies. Since we have to use both groups, we shall use the notation X_K^G and X_K^M for these spaces. We shall also agree that if we write just X_K, we mean X_K^G. Naturally, the open compact level subgroups K should be compatible. We spell out these compatabilities. Recall that $\mathbf{M}$ was the Levi quotient of a maximal parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Thus $K_f = K_S K^S$ is open compact in $\mathbf{G}(\mathbb{A}_f)$ as usual, $K_\infty = U(n)$, $K = K_\infty K_f$ and

$$X_K = \mathbf{G}(\mathbb{Q}) \setminus \mathbf{G}(\mathbb{A})/K.$$

Let $K^P = K \cap \mathbf{P}(\mathbb{A})$. This is a compact open subgroup of $\mathbf{P}$ (warning: P_l is not reductive and its unipotent radical does not have a maximal compact subgroup!). Let K^M be the image of K^P under the projection of $\mathbf{P}(\mathbb{A})$ to $\mathbf{M}(\mathbb{A})$. It is again open and compact, and for $l \notin S$ equal to the maximal compact $M_l = GL_n(\mathbb{Z}_l)$. Recall that $\mathbf{M}$ also sits as a subgroup in $\mathbf{P}$. In the following, we shall make the assumption that K_f^M coincides with $K_f^P \cap \mathbf{M}(\mathbb{A}_f)$. This can be arranged by shrinking K if necessary.

2.2.1. *Borel-Serre compactification.* A *manifold with corners* is an m-dimensional topological manifold with boundary $(\overline{X}, \partial X)$ endowed with a differentiable structure and a stratification of the boundary ∂X by relatively open r-dimensional submanifolds ∂X_r $(0 \leq r \leq m-1)$, such that if $x \in \partial X_r$ then x has a neighborhood which is diffeomorphic to $B_r \times [0,1)^{m-r}$, where B_r is the r-ball and where x is mapped to $(0,0,\dots,0)$.

In this subsection we let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbb{Q}$, $K \subset \mathbf{G}(\mathbb{R}) = G$ a maximal compact subgroup, and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic group. Let $D = G/KA_G$ be the corresponding symmetric space and $X = \Gamma \backslash D$ the locally symmetric space. Here A_G is the connected component of the identity of the real points of the maximal $\mathbb{Q}$-split torus in the center of $\mathbf{G}$. If $\mathbf{G}$ is semisimple it is trivial. If $\mathbf{G} = GL_n$ then $A_G = \mathbb{R}_+^\times$. We assume that Γ is small enough, so that X is a real manifold.

The Borel-Serre compactification X^{BS} is a certain topological compactification of X which has the structure of a manifold with corners. For example, if X is a modular curve, then X^{BS} is obtained by gluing to X a circle at each cusp. Note that $X \hookrightarrow X^{BS}$ is a homotopy equivalence. Here is a brief sketch of the general construction (following [Gor]).

We first describe a space D^{BS} endowed with a certain "Satake topology" which is obtained from D by adding to it a boundary component e_P for every rational parabolic subgroup $\mathbf{P}$. The action of $\mathbf{G}(\mathbb{Q})$ on D will extend to it. We then let

$$X^{BS} = \Gamma \setminus D^{BS}$$

and endow it with the quotient topology.

Let $\mathbf{P}$ be a rational parabolic subgroup of $\mathbf{G}$ and $\mathbf{L}_P$ its Levi quotient. Then $\mathbf{L}_P$ is a reductive group over $\mathbb{Q}$. Let $\mathbf{A}_P$ be the maximal $\mathbb{Q}$-split torus in the center of $\mathbf{L}_P$. For example, if $\mathbf{L}_P$ is $GL_{n_1} \times \cdots \times GL_{n_k}$ then $\mathbf{A}_P = \mathbf{G}_m^k$. Let A_P be the connected component of the identity of $\mathbf{A}_P(\mathbb{R})$. By the Iwasawa decomposition $G = PK$ the group P acts transitively on D. Let $K_P = K \cap P$. Then

$$D = P/K_P.$$

Define a right action of A_P on D by $gK_P \cdot a = gaK_P$. This is well-defined. The orbits of A_P in this action are *totally geodesic submanifolds* of D (manifolds containing, together with any pair of points, also the geodesic between them). If P is a maximal parabolic, they are ordinary geodesics. We let

$$e_P = D/A_P = P/K_P A_P.$$

Note that e_P is "almost" a space like D, except that since P is not reductive, it contains a unipotent part. Define D^{BS} as a set to be the union of D and all the e_P (one for each rational parabolic subgroup $\mathbf{P}$).

Given P, we have $A_P \simeq (0,\infty)^r$ for $r > 0$ and $\dim e_P = d - r$ where $d = \dim D$. If $P \subset Q$ then $A_Q \subset A_P$ and e_Q projects to e_P. Fix P_0, a minimal rational parabolic (a rational Borel if $\mathbf{G}$ is quasi-split). We call Q a standard (rational) parabolic if it contains P_0. [We already see the structure of a "corner" with P_0 corresponding to the origin, and the maximal standard parabolics corresponding to the walls]. Let $\overline{A}_P = (0,\infty]^r$, let A_P act on it on the left by the convention that $t\cdot\infty=\infty$ and

$$D(P) = D \times_{A_P} \overline{A}_P,$$

the quotient of the product $D \times \overline{A}_P$ by the equivalence relation $(xt,a)\equiv(x,ta)$. Endow it with the quotient topology. Denote the equivalence class of (x,a) by $[x,a]$. Then $D(P)$ contains D as the set $[x,(1,1,\dots,1)]$ and also e_P as the set $[x,(\infty,\dots,\infty)]$. In fact, we easily see that

$$D(P) = \amalg_{P\subset Q} e_Q$$

where we include $Q=G$ in the union, in which case we put $e_G = D$. It comes with a natural differentiable structure of a corner. Note that while the set of Q's containing a given P is finite, the set of P contained in a given Q is countable. Thus the same space e_Q will belong to infinitely many $D(P)$ (just as $D = e_G$ is contained in *every* $D(P)$).

Theorem. *(Borel-Serre) There exists a unique topology on D^{BS} such that: (a) each $D(P)$ is open and the induced topology on it is the given one (b) the action of $\mathbf{G}(\mathbb{Q})$ extends continuously from D to D^{BS}.*

Here is a list of properties of $X^{BS} = \Gamma\setminus D^{BS}$ that will be useful to us.

- It is a space of finite type (i.e. a finite CW complex). This is clear from the construction once we realize that there are only finitely many Γ-conjugacy classes of parabolics.
- The Hecke correspondences extend from X to X^{BS}. This follows from the fact that the action of $\mathbf{G}(\mathbb{Q})$ extends from D to D^{BS}.
- The inclusion $X\hookrightarrow X^{BS}$ is a homotopy equivalence. In fact, the "geodesic flow" used to construct X^{BS} also supplies the homotopy.

Let $\partial X^{BS} = X^{BS}\setminus X$ be the boundary.

2.2.2. *Some long exact sequences in cohomology.* Let us return to the situation where $\mathbf{G} = Sp_{2n}$, $K = K_\infty K_f$, $K_f = K_S K^S$ is small enough, and $X_K = X_K^G$ as before. Then applying the Borel-Serre compactification to each connected component of X_K we get a manifold with corners X_K^{BS} and the inclusion of X_K in it is a homotopy equivalence. Let the coefficients be $\overline{\mathbb{F}}_p$ and drop them from the notation. The long exact sequence for cohomology with compact supports reads

$$\cdots\to H^i_c(X_K)\to H^i(X_K^{BS})\to H^i(\partial X_K^{BS})\to H^{i+1}_c(X_K)\to\cdots$$

and $H^i(X_K^{BS}) = H^i(X_K)$ since X_K and X_K^{BS} are homotopically equivalent. *Warning*: cohomology with compact supports is not a homotopy invariant.

Let $\mathbf{P}$ be the maximal parabolic with Levi factor $\mathbf{M} = GL_n$ that we have considered before. Let $K^P = K\cap\mathbf{P}(\mathbb{A})$. Then

$$X_K^P = \mathbf{P}(\mathbb{Q})\setminus\mathbf{P}(\mathbb{A})/A_P K^P$$

($A_P\simeq\mathbb{R}_+^\times$ since P is maximal) is a finite union of spaces $\Gamma\setminus P_\infty/K_\infty^P A_P$ which we denoted by e_P, and which appear as open submanifolds of ∂X_K^{BS}. Note that they are open in the boundary because P is maximal, so in fact $D(P) = D\cup e_P$ (in other words, they are the open walls of the corners). The open embedding

$$X_K^P \hookrightarrow \partial X_K^{BS}$$

induce natural maps

$$H^i_c(X^P_K) \to H^i(\partial X^{BS}_K) \to H^i(X^P_K)$$

(the first since compactly supported cohomology is covariant for open embeddings, and the second by contravariant functoriality of cohomology). The composition of the two maps is the map of compactly supported cohomology to ordinary cohomology.

2.2.3. *Relating the locally symmetric spaces for P and M.* Recall that K^M was the image of K^P in $\mathbf{M}(\mathbb{A})$ and $A_M = A_P$ by definition. Thus there is a projection pr^P_M from X^P_K to

$$X^M_K = \mathbf{M}(\mathbb{Q}) \setminus \mathbf{M}(\mathbb{A})/A_M K^M,$$

the locally symmetric space of GL_n which appeared before, except that at the archimedean place we have now $K^M_\infty = O(n)$ and not $SO(n)$.

Lemma. *[Tors, V.2.2] The map pr^P_M is open and proper. Its fibers are $(S^1)^{n(n+1)/2}$.*

As a result we get two maps

$$(pr^P_M)^* : H^i_c(X^M_K) \to H^i_c(X^P_K), \quad (\iota^P_M)^* : H^i(X^P_K) \to H^i(X^M_K),$$

the first by contravariance of compactly supported cohomology with respect to proper maps, and the second by pull-back along the embedding

$$\iota^P_M : X^M_K \hookrightarrow X^P_K.$$

This embedding uses the fact that $\mathbf{M}$ sits also as a subgroup of $\mathbf{P}$ and our assumption that K^M_f coincides with $K^P_f \cap \mathbf{M}(\mathbb{A}_f)$. Note that at ∞ we have $K^M_\infty = K^P_\infty$. The map ι^P_M is a section of pr^P_M, hence if we denote by

$$j^P : H^i_c(X^P_K) \to H^i(X^P_K)$$

the usual map of compactly supported cohomology into ordinary cohomology, and similarly j^M, then the composition

$$(\iota^P_M)^* \circ j^P \circ (pr^P_M)^* = j^M.$$

Let $H^i_!(X^M_K)$ be the image of this last map. It is called the *interior* cohomology. For example, when $n = 2$ and $i = 1$ both H^1_c and H^1 of the open modular curve have dimension $2g + r - 1$ where g is the genus of the complete curve and r the number of cusps, while $H^1_!$ (also called there *parabolic* cohomology) is $2g$-dimensional.

Combining the maps obtained here with the maps obtained in the previous subsection, we get maps

$$H^i_c(X^M_K) \to H^i(\partial X^{BS}_K) \to H^i(X^M_K).$$

These two maps define a homomorphism of modules

$$End(H^i(\partial X^{BS}_K)) \to Hom(H^i_c(X^M_K), H^i(X^M_K))$$

in the obvious way.

2.2.4. *Compatability with Hecke.* In this subsection we check that the last homomorphism is compatible with the action of the Hecke algebras. This is necessary later on for shifting Hecke eigensystems from the (singular) cohomology of X_K^M to the (singular) cohomology of X_K, where we can hope to attach to them Galois representations.

Consider the Hecke algebras $\mathbb{T}_l$, $\mathbb{T}_l^P$, $\mathbb{T}_l^M$, of compactly supported, $\mathbb{Z}_p$-valued functions on G_l, resp. P_l, resp. M_l which are bi-invariant under K_l, resp. K_l^P, resp. K_l^M. Restriction defines a map

$$\mathbb{T}_l \to \mathbb{T}_l^P$$

and integration along the unipotent fibers (suitably multiplied by the unimodular character of P_l) a map

$$\mathbb{T}_l^P \to \mathbb{T}_l^M.$$

The composition of the two is a "partial" Satake transform. We use the word "partial" because, had P been a Borel, and M the standard torus, we would have obtained the Satake transform S discussed before. But P is a maximal parabolic. We extend this map to the global Hecke algebras and denote it by

$$S_M^G : \mathbb{T} = \mathbb{T}^G \to \mathbb{T}^M.$$

The action of $\mathbb{T}$ on the cohomlogiy of X_K extends to the cohomology of X_K^{BS} and to the cohomology of ∂X_K^{BS}. We now have maps between the Hecke algebras of $\mathbf{G}$ and $\mathbf{M}$ and also maps between cohomology groups related to these two groups on which these algebras act. The following lemma says that they are compatible.

Lemma. *[Tors, V.2.3] The following diagram commutes*

$$\begin{array}{ccc} \mathbb{T} & \to & End(H^i(\partial X_K^{BS})) \\ S_M^G \downarrow & & \downarrow \\ \mathbb{T}^M & \to & Hom(H_c^i(X_K^M), H^i(X_K^M)) \end{array}.$$

We also have the following corollary.

Corollary. *[Tors, V.2.4] Let $\overline{\mathbb{T}}$ and $\overline{\mathbb{T}}^M$ be the images of the corresponding Hecke algebras in $End(H^i(\partial X_K^{BS}))$ and in $End(H_!^i(X_K^M))$. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathbb{T} & \to & \overline{\mathbb{T}} \\ S_M^G \downarrow & & \downarrow \\ \mathbb{T}^M & \to & \overline{\mathbb{T}}^M \end{array}$$

of $\mathbb{Z}_p$-algebras.

Proof. To deduce the corollary from the lemma one only needs to know that a Hecke operator $T \in \mathbb{T}^M$ induces the 0 map $H_c^i(X_K^M) \to H^i(X_K^M)$ if and only of it acts trivially on the interior cohomology, which should be obvious because Hecke commutes with the map between compactly supported cohomology and ordinary cohomology. □

2.2.5. *Calculating the partial Satake transform.* [To be completed. There is an annoying issue of normalization, see my remark on normalizations of the Satake isomorphisms and the Hecke polynomials earlier.] Recall that we have identified the local Hecke algebra for $\mathbf{G} = Sp_{2n}$(tensored with $\mathbb{Z}_p[\sqrt{l}]$) as

$$\mathbb{T}_l[\sqrt{l}] \simeq \mathbb{Z}_p[\sqrt{l}][X_1^{\pm 1}, \ldots, X_n^{\pm 1}]^{S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n}$$

while the local Hecke algebra for $\mathbf{M} = GL_n$ (tensored with $\mathbb{Z}_p[\sqrt{l}]$) was identified as

$$\mathbb{T}_l^M[\sqrt{l}] \simeq \mathbb{Z}_p[\sqrt{l}][X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{S_n}.$$

We have also defined a polynomial $\tilde{P}_l$ of degree $2n+1$ with coefficients in $\mathbb{T}_l$ and a polynomial P_l of degree n with coefficients in $\mathbb{T}_l^M[\sqrt{l}]$. These are the Hecke polynomials at $l \notin S$ for $\mathbf{G}$ and $\mathbf{M}$ respectively. The net result is this.

Lemma. *After possibly renormalizing the partial Satake transform (or, equivalently, multiplying the Hecke operators which appear as the coefficients of the Hecke polynomials by appropriate powers of l) S_M^G carries $\tilde{P}_l(X)$ to*

$$(1-X)P_l(X)\dot{P}_l(X)$$

where $\dot{P}$ is obtained from P by the substitution $X_i \mapsto X_i^{-1}$.

2.3. **End of the proof of Theorem 1.**

2.3.1. *Shifting Hecke eigensystems from X_K^M to X_K.* Let $\psi^M : \mathbb{T}^M \to \overline{\mathbb{F}}_p$ be a system of Hecke eigenvalues occuring in $H_!^i(X_K^M, \overline{\mathbb{F}}_p)$ (i.e. factoring through $\overline{\mathbb{T}}^M$). We ignore the slight difference between H^i, H_c^i and $H_!^i$ (with which one has to deal eventually). Recall that our goal is to attach to it a certain n-dimensional Galois representation of G_S over $\overline{\mathbb{F}}_p$. Using the last corollary we see that

$$\psi = \psi^M \circ S_M^G : \mathbb{T} \to \overline{\mathbb{F}}_p$$

factors through $\overline{\mathbb{T}}$, i.e. occurs in $H^i(\partial X_K^{BS}, \overline{\mathbb{F}}_p)$.

We now go back to the exact sequence

$$H^i(X_K, \overline{\mathbb{F}}_p) \to H^i(\partial X_K^{BS}, \overline{\mathbb{F}}_p) \to H_c^{i+1}(X_K, \overline{\mathbb{F}}_p)$$

and find out that $H^i(X_K, \overline{\mathbb{F}}_p)[\psi] \neq 0$ or $H_c^{i+1}(X_K, \overline{\mathbb{F}}_p)[\psi] \neq 0$.

Now suppose we had the following analogue of Theorem 4, but with the singular cohomology with compact supports $H_c^i(X_K, \overline{\mathbb{F}}_p)$ replacing the coherent cohomology (space of weight k cusp forms) $H^0(X^*_{K/\overline{\mathbb{Q}}_p}, \omega^k \otimes \mathcal{I})$.

Theorem 5. *[Tors, V.1.11] Let $\psi : \mathbb{T} \to \overline{\mathbb{F}}_p$ be a system of Hecke eigenvalues occuring in $H_c^i(X_K, \overline{\mathbb{F}}_p)$ (i.e. factoring through the image of $\mathbb{T}$ in the endomorphism ring of this space). Then there exists a unique continuous semisimple representation*

$$\sigma_\psi : G_S \to GL_{2n+1}(\overline{\mathbb{F}}_p)$$

which is self-dual, and such that for any $l \notin S$

$$\det(1 - X \cdot \sigma_\psi(Fr_l)) = \psi(\tilde{P}_l(X)).$$

Using this theorem we would be able to attach to any system of Hecke eigenvalues occuring in $H_c^i(X_K, \overline{\mathbb{F}}_p)$ a Galois representation with the desired properties, and by Poincaré duality, also to eigenvalues occuring in $H^i(X_K, \overline{\mathbb{F}}_p)$. Thus to ψ^M we will have attached a $2n+1$-dimensional (self-dual) representation whose Frobenius determinant at l is equal to the Hecke polynomials $\tilde{P}_l$ of $G_l = Sp_{2n}(\mathbb{Q}_l)$ for every $l \notin S$.

This is not the end of the story, because our Galois representation is $2n+1$-dimensional (and self-dual) while we are looking for an n-dimensional representation (which need not be self-dual). In the next subsection we briefly explain how to use the fact that ψ comes from ψ^M, a system of Hecke eigenvalues for GL_n, to finally construct an n-dimensional representation as in Theorem 1, whose Frobenius determinants are equal to the Hecke polynomials of GL_n. See part V.3 of [Tors], called "divide and conquer". Scholze uses a trick from [HLTT], of twisting ψ^M by infinitely many 1-dimensional Hecke characters. One also needs, of course, to relate the Hecke polynomials for Sp_{2n} and for GL_n, but this comes from the calculation of the partial Satake transform

$$S_M^G : \mathbb{T} = \mathbb{T}^G \to \mathbb{T}^M$$

which we have done above.

The bulk of Scholze's work is, however, in deducing Theorem 5 from Theorem 4. We are now in the realm of algebraic geometry (on Siegel modular varieties) but Scholze has to introduce new kinds of spaces which only live at the limit, when we add more and more level at p. This is where perfectoid spaces enter the picture.

Recall how how Galois representations are attached to eigenclasses in cohomology in the classical case of modular curves ($n = 2$). First, since we are dealing with curves, every torsion class comes by reduction modulo p from a class in $H^1_{et}(X_{K/\overline{\mathbb{Q}}}, \mathbb{Q}_p)$. But the latter is (as far as the Hecke action goes, if we forget Galois action) nothing but singular cohomology tensored with $\mathbb{Q}_p$. Assume for the moment that we have replaced X_K by the complete modular curve. The difference between the cohomologies of the two is easy to control. Then via the de-Rham isomorphism and the Hodge decomposition, the singular cohomology (tensored with $\mathbb{C}$) can be replaced by

$$H^0(X_{K/\mathbb{C}}, \Omega) \oplus \overline{H^0(X_{K/\mathbb{C}}, \Omega)}.$$

Finally, $H^0(X_{K/\mathbb{C}}, \Omega) \simeq H^0(X_{K/\mathbb{C}}, \omega^2 \otimes \mathcal{I})$ since $\Omega \simeq \omega^2 \otimes \mathcal{I}$ by the Kodaira-Spencer isomorphism (or by a direct calculation of factors of automorphy on $\mathcal{H}$, and comparison between the analytic and the algebraic categories). All this is either missing or far from tivial in higher dimensional cases, but most annoying is the fact that torsion classes in general can not be lifted to characteristic 0.

2.3.2. *Getting n-dimensional representations from $2n+1$-dimensional ones.* As promised, we have to explain how the fact that ψ comes from ψ^M allows us to construct from the $2n+1$ dimensional representation provided by Theorm 5, an n-dimensional representation as in Theorem 1.

Let Γ be a finite group, $\mathbb{F}$ an algebraically closed field, ρ a d-dimensional semi-simple representation of Γ over $\mathbb{F}$ and $P_\rho(g) = \det(1 - X\rho(g))$ the characteristic polynomial of $\rho(g)$. Suppose that there is an integer $e < d$ and for every $g \in \Gamma$ polynomials $Q(g)$ and $Q'(g)$ of degrees e and $d - e$ respectively, such that

$$P_\rho(g) = Q(g)Q'(g).$$

When is it true that $\rho = \sigma \oplus \sigma'$ with $P_\sigma = Q$ and $P_{\sigma'} = Q'$? If this is so, let χ be an arbitrary character of $\mathbb{Z}$, and consider the representation $\sigma \otimes \chi$ of the group $\Gamma \times \mathbb{Z}$ whose characteristic polynomial at (g, n) is $Q(g)(\chi(n)X)$. Similarly consider the representation $\sigma' \otimes \chi^{-1}$ whose characteristic polynomial at (g, n) is $Q'(g)(\chi^{-1}(n)X)$. Then

$$P_{\rho,\chi}(g, n) = Q(g)(\chi(n)X)Q'(g)(\chi^{-1}(n)X)$$

is the characteristic polynomial of the representation $(\sigma \otimes \chi) \oplus (\sigma' \otimes \chi^{-1})$ of $\Gamma \times \mathbb{Z}$. The following proposition says that this is also a sufficient condition.

Proposition. *[Tors V.3.8] Suppose that for every $\chi : \mathbb{Z} \to \mathbb{F}^\times$ there is a representation of $\Gamma \times \mathbb{Z}$ whose characteristic polynomial is given by $P_{\rho,\chi}(g, n)$. Then $\rho = \sigma \oplus \sigma'$ with $P_\sigma = Q$ and $P_{\sigma'} = Q'$.*

Recall that ψ^M is a system of eigenvalues of $\mathbb{T}^M$ occuring in $H^i_!(X^M_K, \overline{\mathbb{F}}_p)$. Let l_0 be an auxiliary prime not in S and $S_0 = S \cup \{l_0\}$. Let

$$\chi : \mathbb{Q}^\times \setminus \mathbb{A}^\times \to \overline{\mathbb{F}}_p^\times$$

be any character ramified only at l_0 and use the same symbol to denote the associated Galois character (a character factoring through $Gal(\mathbb{Q}(\mu_{l_0^\infty})/\mathbb{Q})$). For $l \notin S_0$ define

$$P_{l,\chi}(X) = P_l(\chi(Fr_l)^{-1}X), \quad \dot{P}_{l,\chi}(X) = \dot{P}_l(\chi(Fr_l)X)$$

(polynomials with coefficients in $\mathbb{T}^M$) and $\tilde{P}_{l,\chi}(X) = (1 - X)P_{l,\chi}(X)\dot{P}_{l,\chi}(X)$. Note that the roots of $\psi^M(P_{l,\chi})$ are the reciprocals of the roots of $\psi^M(\dot{P}_{l,\chi})$.

After shrinking K_{l_0}, one can easily check[3] that there is a system of Hecke eigenvalues[4] $\psi^M_\chi : \mathbb{T}^M \to \overline{\mathbb{F}}_p$ occuring in $H^i_!(X^M_K, \overline{\mathbb{F}}_p)$, such that for every $l \notin S_0$

$$\psi^M_\chi(P_l) = \psi^M(P_{l,\chi}), \quad \psi^M_\chi(\dot{P}_l) = \psi^M(\dot{P}_{l,\chi}).$$

It follows that if we let $\psi_\chi = \psi^M_\chi \circ S^G_M : \mathbb{T} \to \overline{\mathbb{F}}_p$ then

$$\psi_\chi(\tilde{P}_l) = \psi^M(\tilde{P}_{l,\chi}).$$

Apply Theorem 5 to this ψ_χ, which as before, occurs in the cohomology of a Siegel modular variety (with the shrunken K). We get a Galois representation

$$\sigma_{\psi,\chi} : G_{S_0} \to GL_{2n+1}(\overline{\mathbb{F}}_p)$$

which is self-dual, and such that for any $l \notin S_0$

$$\det(1 - X \cdot \sigma_{\psi,\chi}(Fr_l)) = \psi_\chi(\tilde{P}_l(X)) = (1 - X)\psi^M(P_{l,\chi})\psi^M(\dot{P}_{l,\chi}).$$

The $(1-X)$ term allows us to split off from $\sigma_{\psi,\chi}$ a copy of the trivial representation. Let Γ be the finite quotient of G_S through which σ_ψ factored. Then (with our fixed choice of l_0) for all the χ factoring through the unique $\mathbb{Z}_{l_0}$ extension of $\mathbb{Q}$, $\sigma_{\psi,\chi}$ factors through $\Gamma \times \mathbb{Z}_{l_0}$ and we are essentially in the situation of the proposition, except that $\mathbb{Z}$ is replaced by $\mathbb{Z}_{l_0}$. A little algebra allows us to deduce, as before, the existence of an n-dimensional direct summand of σ_ψ whose associated characteristic polynomial at Fr_l is $\psi^M(P_l)$, as predicted by Theorem 1.

2.4. **Scholze's main theorem and where the problem is.** Granted Theorem 4, it is clear that to deduce Theorem 5, it is enough to prove the following theorem.

Theorem 6. *[Tors I.5 and IV.3.1] Let X_K be the Siegel modular variety of some level K as before and $\psi : \mathbb{T} \to \overline{\mathbb{F}}_p$ a system of eigenvalues occuring in $H^i_c(X_K, \overline{\mathbb{F}}_p)$. Then there exists a $K' \subset K$ obtained by shrinking K_p and an integer $k > n$, such that ψ is the reduction modulo p of a system of eigenvalues Ψ which occurs in $H^0(\mathcal{X}^*_{K'/\overline{\mathbb{Z}}_p}, \omega^k \otimes \mathcal{I})$. Here $\mathcal{X}^*_{K'/\overline{\mathbb{Z}}_p}$ is an integral model of $X^*_{K'/\overline{\mathbb{Q}}_p}$.*

[3] Use the cup product

$$(pr^{K'}_K)^* H^i(X^M_K) \otimes H^0(X^M_{K'}) \to H^i(X^M_{K'})$$

[4] There is a slight abuse of notation here, as we must now take the Hecke algebra prime to S_0 and not only S

Indeed, to the system of eigenvalues Ψ occuring in $H^0(\mathcal{X}^*_{K'/\overline{\mathbb{Z}}_p}, \omega^k \otimes \mathcal{I})$ we can attach, by Theorem 4, a representation into $GL_{2n+1}(\overline{\mathbb{Z}}_p)$ whose Frobenius determinant at $l \notin S$ is $\Psi(\tilde{P}_l(X))$. The Frobenius determinant at l of the reduction modulo p of the same representation is $\psi(\tilde{P}_l(X))$, as required.

Here are three difficulties that one faces when trying to prove theorem 6:

- We are given a topological eigenclass, and we are asked to construct a geometric object: a modular form with the same eigenvalues, namely, a section of a line bundle. This requires to compare between topological and analytic (or even better, coherent algebraic) cohomologies. The prototype of this is the de Rham theorem on a complex projective variety, which is a comparison theorem between singular cohomology with coefficients in $\mathbb{C}$ and analytic or (equivalently, by GAGA) algebraic de Rham cohomology (hyper-cohomology of the complex of algebraic differential forms). p-adic comparison theorems between p-adic étale cohomology and de Rham (coherent) cohomology do exist, but they require that we enlarge the coefficients to very large rings (Fontaine's rings $B_?$) whose construction stimulated Scholze's notion of a perfectoid ring.
- A further twist is that the topological class with which we start lives in characteristic p and need not lift to characteristic 0, where the geometric object is eventually constructed. It turns out that if we add level structure at p (shrink K_p) we can lift it modulo p^m and the more level structure at p we add, the larger we can make m, but to lift it to characteristic 0 we need "to go to the limit" over K_p.
- Finally, we must relate an ith cohomology class to a global section. For this Cech cohomology is useful (and perhaps inevitable). In any reasonable cohomology theory, if $\mathcal{U} = \{U_j\}$ is a covering of a space X and $\mathcal{F}$ is a sheaf, then one has the spectral sequence
$$\check{H}^i(\mathcal{U}, \mathcal{H}^j(\mathcal{F})) \Rightarrow H^{i+j}(X, \mathcal{F}).$$
Here $\mathcal{H}^j$ is the presheaf $U \mapsto H^j(U, -)$. For this to be useful, we want the U_j to be such that for any finite set of indices $J = \{j_0, \ldots, j_k\}$, if we denote by U_J the corresponding intersection of the U_{j_i}, then U_J is $\mathcal{F}$-acyclic. That every cover has a refinement by such a cover is usually called the "Poincaré lemma". However, in our case we can not use any reasonable algebro-geometric topology (étale, or even rigid-étale) to reduce the computation of $H^i_c(X_K, \overline{\mathbb{F}}_p)$ to Cech cohomology, because the Poincaré lemma does not hold in these topologies: there need not exist a cover, for which the $\overline{\mathbb{F}}_p$-cohomology of its members vanishes. The picture that one should have in mind is the unit circle. Its universal covering space is the real line. In the classical topology we can cover it by contractible open intervals. But in the étale topology, every finite cover will be again the circle, and only in the limit will the fundamental group disappear. One might think that introducing the topological space
$$\lim_{\leftarrow} S^1$$
where the limit is, say, with respect to the maps $z \mapsto z^p$, will solve the problem. But this space (called the "solenoid") is not a manifold. A new type of space appears at the limit.

Scholze's ingenius idea is that all three problems are resolved by introducing a new class of spaces, the *perfectoid spaces,* which show up (often, but not exclusively) as limits of towers of "ordinary" rigid analytic spaces. In particular, for Shimura varieties such as X_K this is achieved by shrinking K_p.

3. Perfectoid spaces [5 weeks]

3.1. **Perfectoid fields.**

3.1.1. *Definition.* Throughout this chapter we fix a prime p. A *non-archimedean field* is a field K equipped with a non-archimedean norm $|\cdot| : K^\times \to \mathbb{R}^\times$. We denote by K^0 or by $\mathcal{O}_K$ the valuation ring and by K^{00} or by $\mathfrak{m}_K$ its maximal ideal. The residue field is K^0/K^{00} and we assume it is of characteristic p. Do not confuse the residue field with K^0/pK^0, a ring which may have many nilpotents, unless K is absolutely unramified.

Definition. A *perfectoid field* is a complete non-archimedean field K of residue characteristic p such that (i) the value group $|K^\times| \subset \mathbb{R}^\times$ is non-discrete (ii) the map $\Phi : x \mapsto x^p$ is surjective on K^0/pK^0.

Note that if the characteristic of K is p, the second condition simply means that Φ is bijective, i.e. K is perfect.

3.1.2. *Examples.*

- $\mathbb{C}_p$, the completion of a fixed algebraic closure of $\mathbb{Q}_p$.
- The completion of $\mathbb{Q}_p(\mu_{p^\infty})$
- The completion of $\mathbb{Q}_p(p^{1/p^\infty})$
- The completion of $\mathbb{F}_p((t))(t^{1/p^\infty})$
- $\mathbb{Q}_p$ is not perfectoid ((i) does not hold)
- The completion of $\mathbb{Q}_p(p^{1/l^\infty})$ is not perfectoid if $l \neq p$ ((ii) does not hold)

3.1.3. *Tilting.* Given a perfectoid field K we define another field $K^\flat$ by setting

$$\mathcal{O}_{K^\flat} = \lim_{\leftarrow \Phi} (\mathcal{O}_K/p\mathcal{O}_K)$$

where the transition maps are the Frobenius morphism Φ. This is easily seen to be an integral domain in characteristic p, and we let $K^\flat$ be its fraction field. If $char.K = p$ this is just K. Assume from now on that $char.K = 0$. If

$$x = (x_0, x_1, x_2, \ldots) \in \mathcal{O}_{K^\flat}$$

and we let $\tilde{x}_n \in \mathcal{O}_K$ be any lift of x_n then $(n \geq 0)$

$$x^{(n)} = \lim \widetilde{x}_{n+m}^{p^m}$$

is independent of the lift, $(x^{(n+1)})^p = x^{(n)}$ and this (extended to the fraction field) identifies $K^\flat$ (multiplicatively) as the set of vectors $(x^{(n)})$ with entries from K satisfying $(x^{(n+1)})^p = x^{(n)}$. The addition can then be defined

$$(x^{(n)}) + (y^{(n)}) = (z^{(n)})$$

with $z^{(n)} = \lim_m (x^{(n+m)} + y^{(n+m)})^{p^m}$. Write $x^\# = x^{(0)}$. If we define a norm on $K^\flat$ by

$$|x| = |x^\#|$$

then it is easily checked that $K^\flat$ becomes a perfectoid field in characteristic p, called the *tilt* of K, with valuation ring $\mathcal{O}_{K^\flat}$. The map $x \mapsto x^\#$ is multiplicative and (by definition) norm preserving. It is in general not surjective, e.g. in the second example above, K does not contain all the p-power roots of p. Note that there does not exist a map "$x \mapsto x^\flat$" in the opposite direction, namely from K to $K^\flat$.

For example, if K is the completion of $\mathbb{Q}_p(p^{1/p^\infty})$ and we let

$$t = (p, p^{1/p}, p^{1/p^2}, \ldots) \in \mathcal{O}_{K^\flat}$$

then $K^\flat$ becomes the completion of $\mathbb{F}_p((t))(t^{1/p^\infty})$.

Remark. It is extremely interesting to study, for a given perfectoid field L in characteristic p, the "moduli space" of all the perfectoid fields K in characteristic 0 yielding L as their tilt. This leads to the Fargues-Fontaine "curve" [FF].

In the sequel we fix any element $0 \neq \pi \in K$ with $|p| \leq |\pi| < 1$. Then, as in the above example, there exists an element $\pi^\flat \in K^\flat$ with $|\pi^\flat| = |\pi|$, and then

$$\mathcal{O}_K/\pi\mathcal{O}_K \simeq \mathcal{O}_{K^\flat}/\pi^\flat\mathcal{O}_{K^\flat}, \ \ \mathcal{O}_K/\mathfrak{m}_K \simeq \mathcal{O}_{K^\flat}/\mathfrak{m}_{K^\flat}.$$

For the first isomorphism note that $\pi^\flat\mathcal{O}_{K^\flat}$ is the kernel of the surjective homomorphism $\mathcal{O}_{K^\flat} \ni x \mapsto x_0 mod \pi \in \mathcal{O}_K/\pi\mathcal{O}_K$. The isomorphism between the residue fields follows at once.

Fontaine and Wintenberger [FW] proved the following important theorem, which allows one to reduce many questions about local fields in characteristic 0 to local fields in characteristic p.

Theorem. *Let K be a perfectoid field. Every finite extension of K is again a perfectoid field. Tilting induces a natural equivalence between the category of finite extensions of K and the category of finite extensions of $K^\flat$ (although one concerns fields in characteristic 0 and the other fields in characteristic p). In particular, there is a canonical isomorphism between the absolute Galois groups of K and of $K^\flat$.*

For more on this theme see [Del]. For two applications of these ideas, to explicit reciprocity laws, and to Néron models of tori, see [dS1] and [dS2].

3.2. **Almost mathematics.**

3.2.1. *The category of almost modules.* Faltings' notion of "almost mathematics", expounded in the book of Gabber and Romero [Ga-Ra], is a systematic approach that explains the argument behind the theorem of Fontaine and Wintenberger. Observe first that if K is a perfectoid field, then $\mathfrak{m}_K^2 = \mathfrak{m}_K$ because the valuation is non-discrete. Call an $\mathcal{O}_K$-module *almost zero* if it is annihilated by $\mathfrak{m}_K$. (More generally, the almost-zero elements in M are $M[\mathfrak{m}_K]$.) The almost-zero modules make up a full subcategory $\mathcal{C}$ of the category $\mathcal{O}_K$-mod. Call a morphism between two $\mathcal{O}_K$-modules an *almost isomorphism* if its kernel and cokernel are almost zero. Then thanks to $\mathfrak{m}_K^2 = \mathfrak{m}_K$ an extension of almost-zero modules is almost zero, and the composition of two almost isomorphisms is again an almost isomorphism. Thus the category $\mathcal{C}$ is a thick Serre subcategory (closed under sub-objects, quotients and extensions) and we may localize by it (i.e. by the multiplicative set of almost isomorphisms) and form the category

$$\mathcal{O}_K^a\text{-mod}= \mathcal{O}_K\text{-mod}/\mathcal{C}$$

which we call the category of almost $\mathcal{O}_K$-modules. Recall that its objects are the same as those of $\mathcal{O}_K$-mod, although we denote a module M by M^a when considered in this new category. The morphisms, however, are such that all almost-isomorphisms are inverted. As a result, two distinct morphisms between M and N in the category of modules, may become the same morphism between M^a and N^a in the almost category, and it does not make sense to talk about the image of an element of M under a morphism from M^a to N^a. Note that $\mathcal{O}_K/\mathfrak{m}_K$ is almost isomorphic to 0, but $\mathcal{O}_K/p\mathcal{O}_K$ is not. We follow the excellent exposition [AM] by Bhatt.

In general, computing in localized categories is a nuisance. In our case it is easy, because the functor $M \to M^a$ has right and left adjoints. In fact, for $M \in \mathcal{O}_K$-mod, let

$$M_* = Hom_{\mathcal{O}_K}(\mathfrak{m}_K, M), \quad M_! = \mathfrak{m}_K \otimes_{\mathcal{O}_K} M.$$

Then

$$Hom_{\mathcal{O}_K}(M_!, N) = Hom_{\mathcal{O}_K^a}(M^a, N^a) = Hom_{\mathcal{O}_K}(M, N_*).$$

The map from $Hom_{\mathcal{O}_K}(M_!, N)$ to $Hom_{\mathcal{O}_K^a}(M^a, N^a)$ associates to ϕ the equivalence class of $\phi \circ (x \otimes 1)$ where $x : \mathcal{O}_K \to \mathfrak{m}_K$ is the inverse of the almost isomorphism $\mathfrak{m}_K \to \mathcal{O}_K$. One proves that this map is bijective and respects the $\mathcal{O}_K$-module structure. That $Hom_{\mathcal{O}_K}(M_!, N) = Hom_{\mathcal{O}_K}(M, N_*)$ is standard.

Exercise. (a) $(\mathcal{O}_K)_* = (\mathfrak{m}_K)_* = \mathcal{O}_K$, $(\mathcal{O}_K/\mathfrak{m}_K)_* = \{0\}$. (b) $(\mathcal{O}_K)_! = (\mathfrak{m}_K)_! = \mathfrak{m}_K$, $(\mathcal{O}_K/\mathfrak{m}_K)_! = \{0\}$. (c) For any N, $Hom_{\mathcal{O}_K}(\mathcal{O}_K/\mathfrak{m}_K, N) = N[\mathfrak{m}_K]$ is the module of almost-zero elements in N, but $Hom_{\mathcal{O}_K^a}((\mathcal{O}_K/\mathfrak{m}_K)^a, N^a) = 0$.

Note that $(M_!)^a$ and M^a are canonically isomorphic (in the almost-category) and similarly $(M_*)^a$ and M^a, but M and $M_!$ or M and M_* are in general not isomorphic (in the usual category of modules). Thus going from the almost world to the usual world and back does not change anything, but not vice versa.

The module N_* is called *the module of almost-elements* in N. Note that it is equal to $Hom_{\mathcal{O}_K^a}(\mathcal{O}_K^a, N^a)$ and that it does not contain any almost-zero elements except 0. There is a canonical map of N to N_* whose kernel is $N[\mathfrak{m}_K]$ and one has $(N_*)_* = N_*$. Here is another example.

Exercise. (a) Assume that K is as in the third example above. The module $(\mathcal{O}_K/p\mathcal{O}_K)_*$ contains "elements" of the form

$$x = \sum_{n=1}^{\infty} p^{1-1/p^n} x_n$$

where $x_n \in \mathcal{O}_K/p\mathcal{O}_K$ are arbitrary, but only finite sums (i.e. sums where all but finitely many x_n vanish) are in $\mathcal{O}_K/p\mathcal{O}_K$. What we mean by x is the homomorphism $\mathfrak{m}_K \to \mathcal{O}_K/p\mathcal{O}_K$ sending a to ax. Notice that ax makes sense because ax_n vanishes for almost all n. (b) $(\mathcal{O}_K/\mathfrak{m}_K p)_* = (\mathcal{O}_K/p\mathcal{O}_K)_*$.

The category $\mathcal{O}_K^a$-mod is an abelian tensor category (tensor product is inherited from the one between usual modules) and has internal hom's[5]. These are the almost homomorphisms

$$alHom(M^a, N^a) := Hom_{\mathcal{O}_K^a}(M^a, N^a)^a.$$

[5]An object $\underline{Hom}(B, C) \in \mathcal{A}$ in an abelian tensor category $\mathcal{A}$ is called the internal hom of B to C if it represents the contravariant functor $\mathcal{A} \rightsquigarrow AbelianGroups$ sending A to $Hom_{\mathcal{A}}(A \otimes B, C)$.

We emphasize that although objects of the almost-category are sets, since morphisms are not set-maps, kernels of morphisms are defined only up to unique isomorphisms, so the kernel of $\phi : M^a \to N^a$ may not be identified with any submodule of M. However, the module of almost elements $\ker(\phi)_*$ is an ordinary submodule of M_*.

Exercise. Exactness properties: (a) $M \rightsquigarrow M_!$ is an exact functor on $\mathcal{O}_K$-mod (b) $N \rightsquigarrow N_*$ is left exact, but in general not right-exact (hint: use the previous exercise) (c) the functor $M \rightsquigarrow M^a$ is an exact functor from $\mathcal{O}_K$-mod to $\mathcal{O}_K^a$-mod (since it has both right and left adjoints).

Since tensor products are defined and satisfy the usual rules, it is possible to define $\mathcal{O}_K^a$-algebras ("almost $\mathcal{O}_K$-algebras") by the standard diagrams in the category of almost-modules. If A is an "almost $\mathcal{O}_K$-algebra" then A_* is an ordinary $\mathcal{O}_K$-algebra. Since $A = (A_*)^a$ every "almost $\mathcal{O}_K$-algebra" comes via localization from an ordinary $\mathcal{O}_K$-algebra.

3.2.2. *Almost commutative algebra.* Let R be an $\mathcal{O}_K$-algebra and M an R-module. We call M^a *(almost) flat* over R^a if the functor $X \to X \otimes_R M$ is exact in the almost category. This is equivalent to the standard Tor groups $Tor_i^R(M, -)$ being almost zero for $i > 0$. Flatness over $\mathcal{O}_K^a$ is particularly simple: M^a is flat over $\mathcal{O}_K^a$ if and only if M_* is flat over $\mathcal{O}_K$.

We call M^a *almost finitely presented* if for any $\epsilon \in \mathfrak{m}$ there exists a finitely presented R-module M_ϵ and a map $f_\epsilon : M_\epsilon \to M$ whose kernel and cokernel are killed by ϵ. It is *uniformly almost f.p.* if there is an integer n such that all the M_ϵ are generated by n elements. For example, $\mathfrak{m}_K$ is almost finitely presented, but not finitely presented in the ordinary sense.

An A-algebra B is called *unramified* (in the almost category, or *almost unramified*) if there exists an almost element $e \in (B \otimes_A B)_*$ satisfying

- $e^2 = e$
- $\mu(e) = 1$
- $ker(\mu) \cdot e = 0$

where $\mu : B \otimes_A B \to B$ is the multiplication map. (Note that in standard commutative algebra, if B is of finite type over A, then the existence of $e \in B \otimes_A B$ satisfying the three properties is an equivalent condition for B/A being unramified. Geometrically, if $X = SpecA$ and $Y = SpecB$ it means that the diagonal embedding of Y in $Y \times_X Y$ is an open immersion. We call e therefore a diagonal idempotent. See [Mi], Proposition I.3.5). It is *(almost) étale* if it is (almost) flat and (almost) unramified and *finite étale* if it is in addition almost finitely presented.

There are several competing definitions for these notions in the almost category. It turns out that the given ones work the best.

Example. Let $p > 2$ and consider the fields

$$K_n = \mathbb{Q}_p(u_n)$$

where $u_0 = p$ and $u_n^p = u_{n-1}$. Let K_∞ be their union and K its completion, a perfectoid field. Now consider $L_0 = \mathbb{Q}_p(v_0)$ where $v_0^2 = u_0 = p$ and $L_n = L_0 K_n$, $L = L_0 K$. It is easily seen that L_n and L are quadratic extensions of K_n and K respectively. Now L/K is unramified, of course, and this is reflected by the isomorphism

$$L \otimes_K L \simeq L \oplus L, \quad x \otimes y \to (xy, x\overline{y}).$$

The idempotent e corresponding to $(1,0)$ under this isomorphism is the unique element of $L \otimes_K L$ satisfying the above equations. If $v \in L$ is any element with $\bar{v} = -v$ and if $u = v^2 \in K$ we can take

$$e = \frac{v \otimes v}{2u} + \frac{1 \otimes 1}{2}.$$

Now, suppose we want to do the same with the rings of integers. Then $\mathcal{O}_{L_n} = \mathcal{O}_{K_n}[v_n]$ with $v_n^2 = u_n$. To see this it is enough to note that for appropriate integers i, j the element $v_0^i u_n^j$ is a $2p^n$ root of p. Thus choosing v_n for v shows that we can make the denominator in the first term in e very close to a unit, although, strictly speaking, it will never be a unit, and we will not be able to take $e \in (\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L)$. However, the argument shows that $e \in (\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L)_*$ so $\mathcal{O}_L$ is almost unramified over $\mathcal{O}_K$. This idea, that the ramification is swallowed in the tower, is due to Tate.

The same type of argument shows that although $\mathcal{O}_L$ is not a finite $\mathcal{O}_K$-module[6], it is almost finitely presented. In fact, it is *uniformly* almost finitely presented in the sense that all the M_ϵ can be taken of the same rank (here, 2). Flatness is easy, so altogether this shows that the extension is *finite étale* in the almost category although it is neither finite nor étale in the usual sense.

We remark that $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K} = 0$ (and is not only almost zero). Indeed, at a finite level $v_n^2 = u_n$ yields $2v_n dv_n = 0$, and this in itself only shows that the different at a finite level is $(2v_n)$ which becomes smaller and smaller. However, assuming $v_{n+1}^p = v_n$ yields $pv_{n+1}^{p-1}dv_{n+1} = dv_n$ so $dv_n = 0$ in the $n+1$ step already, hence in the limit. Since $\mathcal{O}_L = \bigcup \mathcal{O}_K[v_n]$, the module of Kähler differentials vanishes at the limit.

Here is another example.

Example. Let K be a perfectoid field of characteristic p and $0 \neq t \in \mathfrak{m}_K$. Let A be a flat $\mathcal{O}_K$ algebra, integrally closed in the generic fiber $A' = A[t^{-1}]$. Let B' be a finite étale A'-algebra and B the integral closure of A in B'. Then if A is perfect ($\Phi : A \to A$ is surjective), B^a is almost finite étale over A^a.

Proof. Let $e \in B' \otimes_{A'} B'$ be the diagonal idempotent. Then for some $N \geq 0$ $t^N e \in B \otimes_A B$. If A is perfect then A' hence B' hence B are all perfect. Since $e^{1/p} = e$ we have $t^{N/p^n} e \in B \otimes_A B$ for all n and hence e is an almost element of $B \otimes_A B$. This shows that B is almost unramified over A. Note how the fact that A was perfect is used in the proof.

Let tr be the trace map from B' to A'. Since B is integral over A, and A is integrally closed, it maps B to A. Now let $\epsilon \in \mathfrak{m}_K$ and consider $\epsilon e = \sum_{i=1}^N x_i \otimes y_i \in B \otimes_A B$. The endomorphism of B

$$b \mapsto tr \otimes 1\,((b \otimes 1)e)$$

is multiplication by ϵ. But it factors as $B \to A^N \to B$ where the two maps are

$$b \mapsto (tr(bx_i)), \quad (a_i) \mapsto \sum a_i y_i.$$

Thus for any ϵ the module ϵB is finitely generated over A, which means that B is almost finitely generated. With a little more effort one can prove that B is flat and almost finitely presented over A (see [Perf, Prop. 4.10]). Since we have already seen that it is almost unramified, B is almost finite étale over A. □

[6] Since $\mathcal{O}_L$ is integral over $\mathcal{O}_K$ this means that $\mathcal{O}_L$ is not even of finite type as an $\mathcal{O}_K$-algebra.

3.2.3. *Deforming almost finite étale algebras over nilpotent thickenings.* The most important property of finite étale maps is that they lift uniquely over nilpotent thickenings. The same holds in the almost category. As before, fix a non-zero $\pi \in \mathfrak{m}_K$ which divides p.

Theorem. *[Perf, Thm 4.17][Ga-Ra, Thm 5.3.27] Let A be an $\mathcal{O}_K$-algebra and assume that in the category of $\mathcal{O}_K^a$-modules A is flat over $\mathcal{O}_K^a$ and π-adically complete: $A \simeq \lim_{\leftarrow}(A/\pi^n A)$. Then the functor $B \mapsto B \otimes_A A/\pi$ is an equivalence of categories*

$$A_{f\acute{e}t} \simeq (A/\pi)_{f\acute{e}t}$$

between the corresponding categories of almost finite étale algebras. Any $B \in A_{f\acute{e}t}$ is again flat over $\mathcal{O}_K^a$ and π-adically complete.

The theorem of Fontaine and Wintenberger is now a consequence of the following string of equivalences between categories

$$K_{\text{fét}} \simeq (\mathcal{O}_K^a)_{\text{fét}} \simeq (\mathcal{O}_K^a/\pi)_{\text{fét}} \simeq (\mathcal{O}_{K^\flat}^a/\pi^\flat)_{\text{fét}} \simeq (\mathcal{O}_{K^\flat}^a)_{\text{fét}} \simeq K^\flat_{\text{fét}}.$$

The equivalence in the middle is tautological as $\mathcal{O}_K^a/\pi = \mathcal{O}_{K^\flat}^a/\pi^\flat$, and being almost finite étale over this ring is the same whether we view it as an $\mathcal{O}_K^a$-algebra or $\mathcal{O}_{K^\flat}^a$-algebra. The two equivalences on both sides of it follow from the theorem.

Finally, the outermost equivalences are properties of perfectoid fields and will be discussed in the next section, in the more general setting of perfectoid algebras. One has to show, as we have seen explicitly in one example, that for any finite separable field extension L/K, the corresponding extension $\mathcal{O}_L/\mathcal{O}_K$ is almost finite étale. If K is of characteristic p, this is the case $A = K$ of the second example above, as $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L. In general, the proof is similar.

3.3. **Perfectoid algebras.**

3.3.1. *Definition.*

Definition. Let K be a perfectoid field. A *perfectoid K-algebra* is a commutative Banach K-algebra R such that the subring R^0 of power-bounded elements is a bounded subring, and such the Frobenius $\Phi : x \mapsto x^p$ is surjective on R^0/pR^0. Morphisms between perfectoid rings are continuous morphisms.

The condition on R^0 is made to ensure that the topology induced by the basis at 0 consisting of all multiples of R^0 by scalars from K, coincides with the metric topology induced by the Banach norm. Recall that the norm on a Banach algebra (unlike a valued field) is only assumed to be sub-multiplicative. In particular, this condition implies that perfectoid algebras are reduced, i.e. contain no non-zero nilpotents. One can thus replace the given Banach norm by the spectral Banach norm $||.||_R$, whose unit ball is precisely R^0, obtaining an isomorphic (albeit not necessarily isometric) Banach algebra. A perfectoid K-algebra R embeds in a *perfectoid field* if and only if its spectral norm is multiplicative (and not only sub-multiplicative).

Note that we consider perfectoid K-algebras only up to isomorphism, and not up to isometry. In other words, we care about the ensuing Banach topology, but we allow to change the Banach norm to an equivalent one.

3.3.2. *Examples.*

- The basic example is $R = K\left\langle T^{1/p^\infty}\right\rangle$. This is, by definition, $K \otimes_{\mathcal{O}_K} \mathcal{O}_K\left\langle T^{1/p^\infty}\right\rangle$ where $\mathcal{O}_K\left\langle T^{1/p^\infty}\right\rangle$ is the p-adic completion of $\mathcal{O}_K[T^{1/p^\infty}] = \bigcup \mathcal{O}_K[T^{1/p^m}]$. Geometrically, we should think of this as the inverse limit of the closed unit disc under raising to power p map. We can give the same example in several variables. Here the norm is actually multiplicative and $R^0 = \mathcal{O}_K\left\langle T^{1/p^\infty}\right\rangle$.
- [Perf, Prop. 5.23] Let R be a perfectoid K-algebra and S an R-algebra which is finite étale over R. Then: (a) S is naturally equipped with a Banach topology which makes it into a perfectoid K-algebra, (b) The ring of power bounded elements S^0 is almost finite étale over R^0. This generalizes what we have seen for $R = K$ and $S = L$ a finite field extension.

3.3.3. *Tilting.* Let R be a perfectoid K-algebra. Fontaine's construction

$$R^{\flat 0} = \varprojlim_{\Phi}(R^0/pR^0)$$

defines an $\mathcal{O}_{K^\flat}$-algebra and

$$R^\flat = R^{\flat 0} \otimes_{\mathcal{O}_{K^\flat}} K^\flat$$

is identified, as usual, multiplicatively, with $\varprojlim_{\Phi} R$, i.e. with sequences

$$x = (x^{(0)}, x^{(1)}, \ldots)$$

of elements of R with $x^{(i+1)p} = x^{(i)}$. We let $x^\# = x^{(0)}$. Then $x \mapsto x^\#$ is a multiplicative homomorphism from $R^\flat$ back to R. We call $R^\flat$ the *tilt* of R. It is a perfectoid $K^\flat$-algebra with subring of power-bounded elements $R^{\flat 0}$. Note that Φ is bijective on $R^{\flat 0}$. In fact it is bijective on S^0 (hence also on S) for any perfectoid $K^\flat$-algebra S because $S^0 = S^0/pS^0$, Φ is surjective by assumption, and injective since perfectoid algebras are reduced.

Theorem. *[Perf, Thm 5.2] The functor $R \rightsquigarrow R^\flat$ is an equivalence between the category K-Perf of perfectoid K-algebras and the similar category of perfectoid $K^\flat$-algebras. Under tilting, $R^\flat$ is a perfectoid field if and only if R is a perfectoid field.*

This theorem generalizes the theorem of Fontaine and Wintenberger from fields to the relative set-up. In fact, more is true. Scholze defines what he means by a perfectoid $\mathcal{O}_K^a$-algebra (in the category of almost algebras). This is a π-adically complete, flat $\mathcal{O}_K^a$-algebra A for which

$$\Phi : A/\pi^{1/p}A \simeq A/\pi A.$$

Here by $\pi^{1/p}$ we mean any element of K with $|\pi^{1/p}| = |\pi|^{1/p}$ (π itself need not be a pth power). Scholze then shows that a perfectoid K-algebra, which is an object "over the generic fiber of $\mathcal{O}_K$", has a canonical extension to a perfectoid $\mathcal{O}_K^a$-algebra, and the latter is canonically and uniquely determined by its reduction modulo π. Moreover, any perfectoid $\mathcal{O}_K^a/\pi$-algebra (i.e. a perfectoid $\mathcal{O}_K^a$-algebra killed by π) has a unique deformation to a perfectoid $\mathcal{O}_K^a$-algebra. Thus, there are equivalences of categories

$$K\text{-Perf} \cong \mathcal{O}_K^a\text{-Perf} \cong (\mathcal{O}_K^a/\pi)\text{-Perf}$$

of perfectoid algebras in the correspodning categories of algebras (or almost algebras), and similarly between these three categories and the three tilted categories over $K^\flat$, with the element $\pi^\flat$ replacing π.

In these equivalences of categories, the first (extension from the generic fiber in the almost-world) is relatively easy, and only uses "almost mathematics": To go from $R \in K$-Perf to $A \in \mathcal{O}_K^a$-Perf take $A = R^{0a}$. To go from A to R take $R = A_*[\pi^{-1}]$ and put on it the Banach structure in which A_* is the unit ball. In showing that these two maps are inverse to each other one has to use the fact that for R^0, the ring of power bounded elements in R, $(R^{0a})_* = R^0$ (not true for a general A).

On the other hand, the second equivalence (uniquely deforming a perfectoid $\mathcal{O}_K^a/\pi$-algebra to a perfectoid $\mathcal{O}_K^a$-algebra) is more subtle. We have already encountered this equivalence for almost-finite-étale algebras. In general, it is done via the theory of the cotangent complex of Quillen, as generalized by Illusie. See [Perf, Theorem 5.10].

We make a few general remarks on deformations. If $f : A \to B$ is a map of commutative rings then there is a complex $\mathbb{L}_{B/A}$ in the derived category of B-modules (in non-positive degrees) $D^{\leq 0}(B\text{-mod})$ which controls the deformation theory of f. We give two simple-minded examples. If f is smooth then

$$\mathbb{L}_{B/A} = \Omega^1_{B/A}[0].$$

If $B = A/I$ and I is generated by a regular sequence of elements (f then corresponding to a nice closed embedding) then

$$\mathbb{L}_{B/A} = (I/I^2)[-1]$$

is the conormal bundle (in degree -1). In both these examples it is known that deformation theory is related to $\mathbb{L}_{B/A}$. Let $X = \mathit{SpecA}$, $Y = \mathit{SpecB}$ and globalize, leaving the affine case, so that f corresponds now to a map $f : Y \to X$. In the first case the obstruction to *lifting* $f : Y \twoheadrightarrow X$ to a square-zero thickening $\tilde{X}$ of X lies in $H^2(Y, \mathcal{T}_{Y/X})$ (Kodaira-Spencer), and if the obstruction vanishes the isomorphism classes of these liftings become a torsor under $H^1(Y, \mathcal{T}_{Y/X})$. In the second case, the obstruction of *extending* $f : Y \rightarrowtail X$ to a square zero thickening $\tilde{Y}$ of Y lies in $H^1(Y, \mathcal{N}_{Y/X})$ and the isomorphism classes of extensions make up a torsor under $H^0(Y, \mathcal{N}_{Y/X})$. Note that the shift by 1 in the degree of the cohomology reflects the shift by 1 in the degree in which $\Omega^1_{Y/X} = \mathcal{T}^\vee_{Y/X}$ or $I/I^2 = \mathcal{N}^\vee_{Y/X}$ are placed in $\mathbb{L}_{Y/X}$. In some cases, e.g. when in the first case $\tilde{X} = X \times_k k[\epsilon]$ ($\epsilon^2 = 0$, k a field) the obstruction vanishes, as there is a trivial deformation. See [Luc]. What we have to remember from all this is only the extreme case: when $\mathbb{L}_{Y/X} = 0$, f deforms uniquely.

For example, suppose that $A = \mathbb{F}_p$ and B is a perfect reduced A-algebra. Then $\Phi : B \simeq B$ is an isomorphism so by functoriality $d\Phi : \mathbb{L}_{B/A} \simeq \mathbb{L}_{B/A}$. But in characteristic p we always have $d\Phi = 0$, hence $\mathbb{L}_{B/A} = 0$ and B deforms uniquely over nilpotent thickenings of A. For example, there exists a unique flat $\mathbb{Z}/p^n\mathbb{Z}$-algebra $W_n(B)$ lifting B, explicitly given by the Witt vectors construction.

Gabber-Ramero and Scholze extended this to the perfectoid world. If $A \to B$ is now a morphism of $\mathcal{O}_K^a$-algebras, they show that there exists a complex $\mathbb{L}_{B/A} \in D(B\text{-mod})$ with the expected properties, and show that if B is a *perfectoid* $\mathcal{O}_K^a/\pi\mathcal{O}_K^a$-algebra then $\mathbb{L}_{B/(\mathcal{O}_K^a/\pi)} = 0$. The proof of this relies on the fact that the relative Frobenius is again an isomorphism between $B/\pi^{1/p}B$ and B. Furthermore the unique lifting of B to an $\mathcal{O}_K^a/\pi^n\mathcal{O}_K^a$-algebra is again perfectoid. Going to the limit over n gives the desired equivalence

$$\mathcal{O}_K^a\text{-Perf} \cong (\mathcal{O}_K^a/\pi)\text{-Perf}.$$

Another way to prove the theorem "directly at the level of generic fibers", avoiding almost mathematics altogether, is to write down explicitly an inverse functor using the map Θ of p-adic Hodge theory (see [CDM], p.6, or [Perf], Remark 5.19). This invokes (not surprisingly) Witt vectors again and we do not give it here. However, the extra information provided by the equivalence with the categories of perfectoid almost-algebras is needed elsewhere.

3.3.4. *"Almost purity" theorem.* Note that the equivalence of categories

$$K_{\text{fét}} \simeq (\mathcal{O}_K^a)_{\text{fét}} \simeq (\mathcal{O}_K^a/\pi)_{\text{fét}}$$

that we have encountered in the Fontaine-Wintenberger theorem is "embedded" in the equivalence between categories of pefectoid algebras, because every finite étale K-algebra is automoatically perfectoid (and similarly over $\mathcal{O}_K^a$). The same holds when we replace K by any $R \in K$-Perf (resp. $\mathcal{O}_K^a$ by R^{0a} where R^0 is the ring of power-bounded elements in R).

Proposition. *[Perf. Theorem 7.9 and Prop. 5.3, especially discussion following it] Let R be a perfectoid K-algebra and $S \in R_{fét}$. Then $S \in K$-Perf (resp. $S^{0a} \in \mathcal{O}_K^a$-Perf, resp. $S^{0a}/\pi \in (\mathcal{O}_K^a/\pi)$-Perf), S^{0a} is (almost) finite étale over R^{0a} and this establishes an equivalence of categories*

$$R_{fét} \cong R^{0a}_{fét} \cong (R^{0a}/\pi)_{fét}.$$

3.4. **Perfectoid spaces.**

3.4.1. *Perfectoid affinoid K-algebras.* Let K be a perefectoid field as before and R a perfectoid K-algebra. We think of R as the ring of "functions" on a certain space (which is yet to be defined). Let R^+ be an open and integrally closed (in R) subring of R^0. Note that R^0 itself is open and integrally closed (easy exercise). In general R^+ must contain all topologically nilpotent elements (because it is integrally closed and open), so $\mathfrak{m}_K R^0 \subset R^+ \subset R^0$, R^+ is almost isomorphic[7] to R^0, and for all practical purposes one may asssume that the two are equal. One should think of R^+ as the subring of functions which are "everywhere bounded by 1 in absolute value".

Definition. A *perfectoid affinoid K-algebra* is a pair (R, R^+) where R is a perfectoid K-algebra and R^+an open and integrally closed subring of R^0.

As with perfectoid K-algebras, the categories of perfectoid affinoid K-algebras and perfectoid affinoid $K^\flat$-algebras are equivalent under tilting.

[7]Scholze does not even assume that R^+ is an $\mathcal{O}_K$-algebra, although this actually holds in almost every application. Here we implicitly assume it.

3.4.2. *Valuations.* Let Γ be a totally ordered abelian group (example: $\mathbb{R}^d$ with the lexicographic ordering). Write Γ multiplicatively.

Definition. A *valuation* on a commutative ring R with value group Γ is a surjective map $|\cdot| : R \to \Gamma \cup \{0\}$ such that $|0| = 0$, $|1| = 1$, $|xy| = |x||y|$ and $|x + y| \leq \max\{|x|, |y|\}$. The *kernel* of the valuation is the set $\mathfrak{n}$ of elements with $|x| = 0$. It is a prime ideal and $R/\mathfrak{n}$ is a domain. The valuation extends to a valuation of the field of fractions F of $R/\mathfrak{n}$. Let

$$D = \{x \in F||x| \leq 1\}.$$

Then D is a valuation ring in F (for every x, x or x^{-1} lies in D). Let $\mathfrak{m}$ be the unique maximal ideal of D. Then $\mathfrak{p} = \mathfrak{m} \cap (R/\mathfrak{n})$ is a prime ideal and $(R/\mathfrak{n})_{\mathfrak{p}} \subset D$ and is dominated by D. If R is a topological ring the valuation is called *continuous* if for every $\gamma \in \Gamma$ the set of x with $|x| < \gamma$ is open. Two valuations are called *equivalent* if $|x| \leq |y|$ holds in one if and only if $|x|' \leq |y|'$ holds in the other.

Note that we do not insist that Γ is of rank 1 (equivalently, embeddable in $\mathbb{R}$). A typical non-rank 1 valuation is obtained when $\Gamma = \mathbb{Z}^2$ with $(n, m) < (n', m')$ if $n < n'$ or $n = n'$ and $m < m'$. For R we take $\mathbb{Z}_p[t]$. We let $v(f) = (-n, -m)$ if $n = ord_0 f$ and $m = ord_p a_n(f)$ where $f = \sum_{k=n}^{\infty} a_k(f)t^k$. In this case $\mathfrak{n} = 0$, $F = \mathbb{Q}_p(t)$, D is the collection of f for which $|f(0)| \leq 1$, $\mathfrak{m}$ the collection of f with $|f(0)| < 1$ and $\mathfrak{p}$ the maximal ideal (p, t) of R. Here $R_{\mathfrak{p}} \subset D$ but the element t/p is not in $R_{\mathfrak{p}}$. Note that $\mathfrak{m} = pD$ so D is a local ring whose maximal ideal is principal, but it is of Krull dimesnion 2, the ideal $\mathfrak{q}$ of all f vanishing at 0 being a prime ideal. Note also that D is non-noetherian, as $\mathfrak{q}$ is not finitely generated. In fact, $\mathfrak{q} = (t, t/p, t/p^2, \ldots)$ and this infinite sequence of generators does not have a finite subsequence which generates it.

3.4.3. *The adic spectrum of a perfectoid affinoid K-algebra.* Let (R, R^+) be a perfectoid affinoid K-algebra. Following Huber [Hu] we define $Spa(R, R^+)$ (the *adic spectrum* of (R, R^+)) to be the set of equivalence classes of continuous valuations on R for which $|f| \leq 1$ for every $f \in R^+$. If $x \in Spa(R, R^+)$ we usually denote $x(f)$ by $|f(x)|$. We can recover R^+ as the set of $f \in R$ for which $|f(x)| \leq 1$ at every $x \in Spa(R, R^+)$.

Remark. (a) This definition works also for R a (usual, not perfectoid) *finite type affinoid algebra*, i.e. a quotient R of the Tate algebra $K\langle T_1, \ldots, T_n\rangle$ by some (necessarily closed and finitely generated) ideal. In this case one considers in rigid analysis the *maximal spectrum* $Spm(R)$ consisting of maximal ideals $\mathfrak{m}$ of R, with the associated valuation being the unique extension of the valuation of K to a valuation of $R/\mathfrak{m}$ (a finite field extension of K). It is then true that $Spm(R) \subset Spa(R, R^0)$, but $Spa(R, R^0)$ contains "points" corresponding to non-maximal ideals in R. However, rigid analytic geometry, or its variants defined by Berkovich and Huber, always invoke noetherianity assumptions. Prefectoid rings are never noetherian. (b) The Berkovich spectrum of R, denoted $Sp(R)$, is the set of rank-1 continuous valuations on R. A rank-1 continuous valuation automoatically satisfies $|f(x)| \leq 1$ for every $f \in R^0$ (easy exercise) so $Sp(R)$ is just the rank-1 valuations in $Spa(R, R^+)$ and is independent of the choice of R^+. (c) As sets we have therefore

$$Spm(R) \subset Sp(R) \subset Spa(R, R^+)$$

The topology that is induced on $Sp(R)$ by the topology defined below on $Spa(R, R^+)$ is the same as the topology defined by Berkovich on $Sp(R)$. In the case of $Spm(R)$ this is a little more subtle, as the topology defined in rigid analysis on $Spm(R)$ is not an ordinary topology but a Grothendieck topology.

Let $f_1, \ldots, f_n, g \in R$ be such that $R = \sum_{i=1}^{n} Rf_i$. The set

$$U = U(f_1, \ldots, f_n|g) = \{x \in Spa(R, R^+) || f_i(x)| \leq |g(x)|\}$$

is called a rational subset of $Spa(R, R^+)$. It is clear that g does not vanish on it, but one can actually prove more, that if $\pi \in K$ and $|\pi|_K < 1$ then for some N, $|\pi(x)|^N \leq |g(x)|$ on U. It is easily verified that the rational sets form a basis for a topology on $Spa(R, R^+)$, and that they are quasi-compact.

Let (R, R^+) be a perfectoid affinoid K-algebra and $(R^\flat, R^{\flat+})$ the tilted perfectoid affinoid $K^\flat$-algebra. For $x \in Spa(R, R^+)$ define $x^\flat \in Spa(R^\flat, R^{\flat+})$ by setting

$$|f(x^\flat)| = |f^\#(x)|$$

(note that this is compatible with the valuations on K and $K^\flat$).

Theorem. *This assignment defines a homeomorphism between $Spa(R, R^+)$ and $Spa(R^\flat, R^{\flat+})$ mapping rational subsets to rational subsets.*

This is far from obvious. As Scholze remarks in [ICM], it is not clear a-priori that $x^\flat$ is a valuation at all, because the map $\#$ is multiplicative but not additive, so one should show that the strong triangle inequality persists. This is easy, but even if we grant this, it is not clear that $x \mapsto x^\flat$ is injective, because the image of $\#$ is not dense in general. What is crucial in proving the injectivity is a certain approximation lemma, which roughly says that for every $f \in R$ there is a $g \in R^\flat$ such that $|g^\#(x)| = |f(x)|$ except where both f and $g^\#$ are small. See [Perf], Lemma 6.5, and [ICM], Lemma 6.6.

The theorem is used in the proof that the structure sheaf $\mathcal{O}_X$ constructed below is indeed a sheaf.

3.4.4. *The structure sheaf $(\mathcal{O}_X, \mathcal{O}_X^+)$ on $X = Spa(R, R^+)$.* Let X be the topological space that has just been defined. If U is a rational set of the shape given above, equip $R[g^{-1}] = R[x]/(xg-1)$ with the norm for which the image of $R^+[f_1g^{-1}, \ldots, f_ng^{-1}]$ is the unit ball, let

$$R\left\langle f_1g^{-1}, \ldots, f_ng^{-1}\right\rangle$$

be its completion (a Banach K-algebra) and $R\left\langle f_1g^{-1}, \ldots, f_ng^{-1}\right\rangle^+$ the completion of the integral closure of $R^+[f_1g^{-1}, \ldots, f_ng^{-1}]$. Then the pair

$$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (R\left\langle f_1g^{-1}, \ldots, f_ng^{-1}\right\rangle, R\left\langle f_1g^{-1}, \ldots, f_ng^{-1}\right\rangle^+)$$

turns out to be independent of the choices of the f_i and g, i.e. depends only on the set U and not on how this set was constructed. Moreover, restriction of valuations defines a map

$$Spa((\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to Spa(R, R^+)$$

which is a homoeomorphism onto U, preserving rational subsets. The pair of rings $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ satisfies a universal property with respect to maps of $Spa(S, S^+)$ into $Spa(R, R^+)$ factoring *set-theoretically* through U : any such map is obtained from a unique map of $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ to (S, S^+) by pull-back of valuations.

Define $\mathcal{O}_X(W)$, for any open subset $W \subset X$ as the inverse limit of $\mathcal{O}_X(U)$ over the rational subsets $U \subset W$ and similarly for $\mathcal{O}_X^+(W)$.

One of the main theorems of Huber, in the case where R was a finite type affinoid K-algebra, was that this $\mathcal{O}_X$ is a sheaf of K-algebras on X and that it satisfies Cartan's theorem A and B. Huber relied strongly on noetherianity. However, in the context of perfectoid affinoid K-algebras, Scholze managed to prove the same using the perfectoid assumption as a substitute for noetherianity.

Theorem. *[Perf, Theorem 6.3],[ICM, Theorem 6.8] (i) Let (R, R^+) be a perfectoid affinoid K-algebra. Then the presheaf $\mathcal{O}_X$ just constructed is a sheaf of K-algebras and $\mathcal{O}_X^+$ is a sheaf of sub-$\mathcal{O}_K^a$-algebras. (ii) These sheaves behave well with respect to tilting, i.e. for every rational subset U the pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is a perfectoid affinoid K-algebra which tilts to $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$. (iii) For any $i > 0$ the cohomology $H^i(X, \mathcal{O}_X) = 0$, while $H^i(X, \mathcal{O}_X^+)$ is almost-zero (killed by $\mathfrak{m}_K$).*

We emphasize that the presheaf $\mathcal{O}_X^+$ satisfies the sheaf axiom only in the almost-category, i.e. the corresponding diagrams of $\mathcal{O}_K$-algebras are only almost exact. This is the source of the "almostness" in part (iii). The topological space X with the pair of sheaves $(\mathcal{O}_X, \mathcal{O}_X^+)$ is called a *perfectoid affinoid space*.

Scholze's strategy for proving the theorem is: (a) First prove part (ii), about tilting, independently of (i) and (iii). (b) Prove the rest of the theorem in characteristic p, in the special case where R is the completed perfection of a reduced affinoid K-algebra S of topological finite type (a nilpotent-free quotient of the Tate algebra $K\langle T_1, \ldots, T_n\rangle$). I.e. take such an S, let S^+ be an open and integrally closed sub-$\mathcal{O}_K$-algebra of S (e.g. S^0) and

$$R^+ = \varprojlim_n((\varinjlim_\Phi S^+)/\pi^n), \quad R = K \otimes_{\mathcal{O}_K} R^+.$$

Perfectoid affinoid K-algebras (R, R^+) of this type are called p-finite. In this case one deduces easily both the sheaf proprety and the vanishing of $\mathcal{O}_X$-cohomology in positive degrees from the analogous theorems proven by Tate (Tate's acyclicity theorem). Taking direct limit over Φ extends the vanishing of $\mathcal{O}_X$-cohomology to an almost-vanishing of $\mathcal{O}_X^+$-cohomology. (c) Prove the theorem in the general characteristic p case by writing an arbitrary perfectoid affinoid K-algebra (R, R^+) as the completed direct limit of p-finite algebras as in step (b). (d) Deduce parts (i) and (iii) in characteristic 0 from the characteristic p case using tilting and part (ii).

As usual, one denotes by $\mathcal{O}_{X,x}$ the stalk of the structure sheaf at $x \in X$. It is a local ring whose maximal ideal consists of all the (germs of) functions in the kernel of the valuation x, i.e. for which $|f(x)| = 0$. The residue field $k(x)$ inherits the valuation $|\cdot|_x$. For example (in the context of Huber adic spaces, the reader may easily transport the example to the perfectoid world) if $X = Spa(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$ is the unit disc and x is a closed (type 1) K-rational point, $k(x) = K$. But if x is the generic point (the Gauss norm on $K\langle T\rangle$) its kernel is 0, and $k(x)$ is the field of fractions of $K\langle T\rangle$.

3.4.5. *Perfectoid spaces in general.* The final step in defining the category of perfectoid spaces is standard. A perfectoid space is a space that is glued from perfectoid affinoid spaces. One works in a very general category $\mathcal{V}$ of locally ringed topological spaces $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of complete topological K-algebras, and where we further equip the pair $(X, \mathcal{O}_X)$ with continuous valuations $f \mapsto |f(x)|$ on the stalks $\mathcal{O}_{X,x}$ for each $x \in X$. Morphisms in $\mathcal{V}$ are defined in the obvious way. Huber defines the category $\mathcal{A}dic$ of *adic spaces* which is the full subcategory of $\mathcal{V}$ consisting of objects which locally are of the form $Spa(R, R^+)$, where R is, quite generally, a Tate K-algebra (a complete topological K-algebra whose topology is defined by a basis for the topology of the form aR_0, $a \in K$, R_0 a fixed open subring). Perfectoid rings (as well as affinoid K-algebras topologically of finite type) are examples of Tate K-algebras. A *perfectoid space* is an adic space that is locally isomorphic in $\mathcal{V}$ to a perfectoid affinoid space.

One way in which perfectoid spaces are better than general adic spaces is that they admit fiber products (the category of adic spaces does not admit fiber products in general).

Proposition. *[Perf, 6.18] Let $X \to Y$ and $Z \to Y$ be morphism of perfectoid spaces over K. Then the fiber product $X \times_Y Z$ exists in the category $\mathcal{A}dic$* and is again a perfectoid space.

3.4.6. *An open problem.* It is an open problem whether being a "prefectoid affinoid" is a local property. More precisely, suppose R is a Tate K-algebra and R^+ an open integrally closed subring. Suppose the adic affinoid $X = Spa(R, R^+)$ is a perfectoid space, i.e. is covered by (finitely many) perfectoid affinoids. Is R necessarily perfectoid, i.e. is X itself a perfectoid affinoid?

3.4.7. *Inverse limits.* An *(adic) affinoid K-algebra* is a pair (R, R^+) where R is a Tate K-algebra (not necessarily topologically of finite type) and R^+ is an open, integrally closed, subring. For any affinoid K-algebra one can define the adic space $Spa(R, R^+)$ (as an object in the category $\mathcal{V}$) by the same procedure as before. If (R, R^+) is topologically of finite type then we recover rigid analytic affinoids in Huber's sense. If (R, R^+) is a perfectoid affinoid K-algebra we recover the perfectoid affinoid spaces defined above.

The category of affinoid K-algebras does not admit filtered direct limits, hence the category of adic affinoid spaces does not admit inverse limits.

Let (X_i) $(i \geq 1)$ be a tower of reduced adic spaces of finite type over K (e.g. the adic spaces associated to quasi-projective varieties over K) with finite transition maps $X_{i+1} \to X_i$. Let X be a perfectoid space and

$$f_i : X \to X_i$$

a compatible systems of maps of adic spaces (recall that X is also reduced).

Definition. (i) If all are affinoids, $X_i = Spa(R_i, R_i^+)$ with R_i a Tate K-algebra of topological finite type, and $X = Spa(R, R^+)$ with R a perfectoid K-algebra, then we say that X is a naive inverse limit of the X_i if R^+ is the π-adic completion of $\varinjlim R_i^+$. (ii) In general, say that X is a naive inverse limit of the X_i and write

$$X \sim \varprojlim X_i$$

if this holds locally on X.

In this case

- $|X| \simeq \varprojlim |X_i|$ for the topological spaces
- if $x \in |X|$ maps to (x_i) then $\varinjlim k(x_i) \to k(x)$ (on residue fields) has dense image.

[One can show that there is a good category of "locally spectral adic spaces" such that X becomes the true inverse limit of the X_i in this category.]

3.4.8. *Example: the perfectoid projective space.* Scholze gives examples of perfectoid toric varieties in Section 8 of [Perf]. Projective space is a sub-example. We do here $\mathbb{P}^1_K$, the generalization to $\mathbb{P}^n_K$ being obvious. We first have to recall the adic projective line over K, $(\mathbb{P}^1_K)^{adic}$. This is obtained by gluing $Spa(K\langle X\rangle, \mathcal{O}_K\langle X\rangle)$ and $Spa(K\langle Y\rangle, \mathcal{O}_K\langle Y\rangle)$ along $Spa(K\langle Z, Z^{-1}\rangle, \mathcal{O}_K\langle Z, Z^{-1}\rangle)$ using $X = Z$, $Y = Z^{-1}$. Recall that points of the adic unit disk $Spa(K\langle X\rangle, \mathcal{O}_K\langle X\rangle)$ come in 5 types: (i) type-1 points are classical points correspodning to maximal ideals in $K\langle X\rangle$ whose residue field is a finite extension$L$ of K. Here $|f(x)|$ is just the p-adic absolute value of the evaluation of f at a point $x \in L$. (ii), (iii) Points of these types map f to $\sup_D |f|$ where D is a closed sub-disk of the unit disk. If D degenerates to a point we recover type (i), if the radius of D is positive and belongs to $|K|$ the point is of type (ii), and if the radius does not belong to $|K|$ it is of type (iii). The case of D being the whole unit disk is called the Gauss point, and corresponds to the Gauss norm. Points of type (iv) correspond to phantom disks, i.e. to infinite sequences of closed disks $D_1 \supset D_2 \supset \cdots$ with $\cap D_i = \emptyset$ (examples occuring only if K is not spherically complete). Points of type (i)-(iv) are all Berkovich points (rank 1 valuations) and the topology on them is that of a tree, rooted in the Gauss point, with bifurcations occuring at points of type (ii), “true ends” corresponding to points of type (i), and points of type (iv) corresponding to “dead ends” of the tree. Points of type (v) are rank 2 valuations and sit infinitesimally close to points of type (ii), at the root of every branch coming out of it. Note that when $K = \mathbb{C}_p$ for example, there are infinitely many branches at each bifurcation point of the tree.

The two pictures, for the two half-spaces of $\mathbb{P}^1_K$, are glued in the obvious way. One obtains the standard tree with the action of $PGL_2(K)$.

Now to define $(\mathbb{P}^1_K)^{perf}$ we need to fix a lifting of Frobenius, which is *non-canonical.* The simplest choice in the case of $\mathbb{P}^1_K$ is to take

$$\varphi(x : y) = (x^p, y^p).$$

Proposition. *There exists a unique perfectoid space* $(\mathbb{P}^1_K)^{perf}$ *such that*

$$(\mathbb{P}^1_K)^{perf} \sim \varprojlim_{\varphi} (\mathbb{P}^1_K)^{adic}.$$

Indeed, one simply glues $Spa(K\langle X^{1/p^\infty}\rangle, \mathcal{O}_K\langle X^{1/p^\infty}\rangle)$ and the corresponding other half-space $Spa(K\langle Y^{1/p^\infty}\rangle, \mathcal{O}_K\langle Y^{1/p^\infty}\rangle)$ as before ($XY = 1$). It should be stressed that in characteristic 0, the resulting space $(\mathbb{P}^1_K)^{perf}$ depends on the choice of the lifting of Frobenius φ, although its tilt, which is $(\mathbb{P}^1_{K^\flat})^{perf}$, is unique. The map

$$|(\mathbb{P}^1_{K^\flat})^{ad}| \simeq |(\mathbb{P}^1_{K^\flat})^{perf}| \simeq |(\mathbb{P}^1_K)^{perf}| \simeq \varprojlim_{\varphi} |(\mathbb{P}^1_K)^{adic}| \to |(\mathbb{P}^1_K)^{adic}|$$

is $(x : y) \mapsto (x^\# : y^\#)$. The first isomorphism stems from the fact that in characteristic p the map φ is bijective, as a map of topological spaces.

3.5. **The étale topology of a perfectoid space.**

3.5.1. *Étale morphisms.*

Definition. (i) Let $(R, R^+) \to (S, S^+)$ be a morphism of (adic) affinoid K-algebras (thus R and S are Tate K-algebras). It is called *finite étale* if S is finite étale over R (in the usual sense), has the induced topology, and S^+ is the integral closure of R^+ in S (but S^+ can well be ramified over R^+). (ii) A morphism $f : X \to Y$ of adic spaces over K is called finite étale if there is a cover of Y by open adic affinoids $V \subset Y$ such that $U = f^{-1}(V)$ is an adic affinoid, and the associated map

$$(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_Y(U), \mathcal{O}_Y^+(U))$$

is finite étale. (iii) A morphism $f : X \to Y$ of adic spaces is called étale if for any $x \in X$ there are neighborhoods $x \in U'$, $y \in V$, a finite étale morphism $f_U : U \to V$ and an open embedding $j : U' \subset U$, such that

$$f|U' = f_U \circ j.$$

As we have noted before, if (R, R^+) is a perfectoid affinoid K-algebra, then so is (S, S^+) and in this case S^{0a} is (almost) finite étale over R^{0a}. Since $R^0$and R^+ are almost isomorphic, the same holds for S^{+a} over R^{+a}. The fiber product of étale morphisms between perfectoid spaces is again étale.

The étale site of a pefectoid space X is the category X_{et} of all $Y \to X$ which are étale. Morphisms are X-morphisms $Y \to Y'$, and are necessarily étale (this is a fundamental property of étale morphisms in any category). Coverings are families of (étale) morphisms $\{f_\alpha : Y_\alpha \to Y\}$ (over X) for which $|Y| = \bigcup f_\alpha(|Y_\alpha|)$. The étale topos $X_{et}^{\sim}$ is the category of abelian sheaves on X_{et}. The following theorem is an immediate consequence of all that has been said on tilting.

Theorem. *Let X be a perfectoid space over K. Let $X^\flat$ be the tilted perfectoid space over $K^\flat$. Then tilting induces an isomorphism of sites $X_{et} \simeq X_{et}^\flat$.*

If $X \to Y$ and $Z \to Y$ are morphisms between perfectoid spaces, and if the first is étale (resp. finite étale), then so is $X \times_Y Z \to Z$.

3.5.2. *Vanishing of $H^i(X_{et}, \mathcal{O}_X)$ for perfectoid affinoids X.* The analogue of Tate's acyclicity theorem holds in the étale topology as well (with a similar proof):

Proposition. *Let $X = Spa(R, R^+)$ be a perfectoid affinoid space. Then*

$$H^i(X_{et}, \mathcal{O}_X) = 0$$

for $i > 0$ and $H^i(X_{et}, \mathcal{O}_X^+)$ is almost zero.

Here is an application of this proposition and tilting. Let Y be a perfectoid space in characteristic p. Then one has the Artin-Schreier exact sequence of sheaves on Y_{et},

$$0 \to \mathbb{F}_p \to \mathcal{O}_Y \overset{\Phi-1}{\to} \mathcal{O}_Y \to 0.$$

It follows from the proposition that if Y is a perfectoid *affinoid* and $i > 1$, then $H^i(Y_{et}, \mathbb{F}_p) = 0$. So far we have not used the fact that Y was perfectoid. Now let X be a perfectoid affinoid in characteristic 0 and $Y = X^\flat$ its tilt. Since the étale sites of X and Y are canonically isomorphic we get the following corollary.

Corollary. *Let X be a perfectoid affinoid over K. Then*

$$H^i(X_{et}, \mathbb{F}_p) = 0$$

for every $i > 1$.

Another corollary of the proposition is that for any perfectoid space X and any *locally free sheaf* $\mathcal{L}$ on X_{et}, $H^i(X_{et}, \mathcal{L})$ coincides with $H^i(X, \mathcal{L})$ (the latter computed in the analytic topology), and that both vanish if X is a perfectoid affinoid and $i > 0$. This is proved as for schemes, because the proposition reduces the computation of cohomology to Cech cohomology w.r.t. a covering by perfectoid affinoid subdomains.

Caution: there does not seem to be a good notion of coherent sheaves for perfectoid spaces. Perfectoid rings are not noetherian and localization is not flat, and this makes many of the standard foundational results difficult to prove. See the work of Kedlaya and Liu on p-adic Hodge theory I (in Astérisque) and II (in preparation).

To be able to apply étale cohomology of perfectoid spaces to the study of usual (rigid) analytic varieties, we need the following proposition.

Proposition. *[Perf, Corollary 7.18] Let $\{X_n\}$ be an inverse system of (noetherian rigid) analytic spaces, and suppose that X is a perfectoid space such that $X \sim \varprojlim X_n$ in the sense described above. Let F be any sheaf of abelian groups on X_{et} and similarly F_n a sheaf on $X_{n,et}$. Assume that the sheaves are compatible in the sense that the pre-image of F_n on $X_{m,et}$ for $m \geq n$ is F_m and likewise on X_{et} we get F. Then for any $i \geq 0$*

$$H^i(X, F) = \varinjlim H^i(X_n, F_n).$$

3.5.3. *The pro-étale topology of a rigid analytic space.* Let X be a locally noetherian adic space. Recall that the site X_{et} is a certain category equipped with a Grothendieck topology. The pro-category Pro-X_{et} is the category whose objects are inverse systems $U = (U_i)_{i \in I}$ where I is a directed set, and whose morphisms are

$$Mor(U, V) = \lim_j co\lim_i Mor_{X_{et}}(U_i, V_j).$$

Here lim is inverse limit and $co\lim$ is direct limit. We would like to think of U as a new geometric object that lives in the limit, but there is a problem: the image of an étale morphism is an open set, and these open sets may shrink indefinitely, so that in the limit we will be left with no image at all in X. We therefore define the category X_{proet} to be the full subcategory of Pro-X_{et} consisting of systems (U_i) where the $U_i \to X$ are étale but the $U_i \to U_j$ are (surjective) finite étale.

Next we turn this category into a site by singling out (for U and V as above) good morphisms in $Mor(U, V)$ that are called *pro-étale morphisms*. First, a morphism $U \to V$ is called étale (resp. finite étale) if it is of the form

$$U = U_0 \times_{V_0} V \to V$$

for a single étale (resp. finite étale) morphism $U_0 \to V_0$ in X_{et}. Then, using the fact that the category Pro-X_{et} has cofiltered inverse limits (this amounts to combining a directed system of directed systems I_i into a single directed system), we can "repeat" the condition used to define morphisms in X_{proet} and say that a morphism $U \to V$ is pro-étale if it can be written as a cofiltered inverse system $U_i \to V$ (each $U_i = (U_{ij})_{j \in J_i}$ itself an inverse system) where each $U_i \to V$ is étale and $U_i \to U_j$ is finite étale.

This defines the "admissible morphisms" of the Grothendieck topology. "Coverings" of $V = (V_j)$ are families of such morphisms $U^t \to V$ that cover $|V| = \lim |V_j|$ set theoretically.

It can be shown then that this is indeed a site, that pro-étale morphisms are open and that the category X_{proet} has finite projective limits. Finally cohomology of abelian sheaves behaves well, in the sense that

$$H^i(U, \nu^* F) = co\lim_n H^i(U_n, F)$$

if F is a sheaf in X_{et} and $\nu : X_{proet} \to X_{et}$ is the obvious projection.

It may happen that an object $U = (U_i)$ in X_{proet} is represented by a perfectoid space Z in the sense that $Z \sim \lim U_i$ as we have encountered before. In this case we shall say that "U is perfectoid".

The key result is this.

Proposition. *Every locally noetherian adic space (in particular every rigid analytic space) is covered in the pro-étale topology by perfectoids.*

Example. Consider $X = Spa(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$, the affinoid unit disc, as before. The family of coverings induced by $T = T_n^{p^n}$ is étale over the affinoid subdomain $|T| = 1$, i.e. over $U = Spa(K\langle T^{\pm 1}\rangle, \mathcal{O}_K\langle T^{\pm 1}\rangle)$, and in the limit become perfectoid. Embedding X as a small affinoid disk around 1 in U and restricting the coverings to it does the same job for X (an observation of Colmez). The idea behind this example works in all cases, and in higher dimensions as well, to prove the proposition.

3.6. **Cohomology of rigid analytic spaces.**

3.6.1. *The Main Theorem on comparison of étale cohomology and coherent cohomology.* The main use of perfectoid spaces to the cohomology of rigid analytic varieties is in proving the following theorem:

Theorem 7. *[CDM, Theorem 3.3] [pHT, Theorem 5.1] Let X be a proper smooth locally noetherian adic space over $\mathbb{C}_p$. Let $\mathcal{L}$ be an étale local system of $\mathbb{F}_p$-modules on X. Then:*

(i) $H^i(X_{et}, \mathcal{L})$ is a finite-dimensional $\mathbb{F}_p$-vector space and vanishes for $i > 2\dim X$.

(ii) There exists an almost isomorphism of $\mathcal{O}_p^a$-modules (here $\mathcal{O}_p = \mathcal{O}_{\mathbb{C}_p}$)

$$H^i(X_{et}, \mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_p/p\mathcal{O}_p \simeq H^i(X_{et}, \mathcal{L} \otimes \mathcal{O}_X^+)^a.$$

Part (i) of the theorem holds for any proper smooth locally noetherian adic space. It has been previously known (by Faltings) for the analytification of smooth projective (or maybe even proper) varieties, but there are many rigid analytic or adic spaces which are not algebraizable. Finiteness of cohomology is false for non-proper spaces, even for $H^1(X_{et}, \mathbb{F}_p)$ when X is the closed unit disk, due to the existence of many étale Artin-Schreier coverings. Part (ii) of the theorem establishes the key relation between étale cohomology and coherent coohmology. It has a version for $\mathbb{Z}_p$-local systens, but we shall need only local systems of $\mathbb{F}_p$-vector spaces.

One of the applications of this theorem is a B_{dR}-comparison theorem between p-adic étale cohomology and de Rham cohomology of proper smooth varieties over $\mathbb{Q}_p$. This has been done by Fontaine-Messing, Faltings, and other authors, but is extended here to non-algebraizable situations. It should be considered a p-adic analogue of the classical de Rham theorem. Other applications include (a) the degeneration of the Hodge-to-de Rham spectral sequence for proper smooth rigid analytic varieties, or more generally proper smooth locally noetherian adic spaces defined over a discretely valued field.[8] (b) a Hodge-Tate decomposition of p-adic étale cohomology tensored with $\mathbb{C}_p$, if X is defined over a discretely valued subfield of $\mathbb{C}_p$. See [pHT], Theorem 8.4.

[8] No analogue of the Kähler condition is neeed in the p-adic setup!

Note that the theorem implies a remarkable behavior of the cohomology groups $H^i(X_{et}, \mathcal{O}_X^+)$. After inverting p in the coefficients, these groups become the (analytic) cohomologies of the structure sheaf $H^i(X_{et}, \mathcal{O}_X) = H^i(X, \mathcal{O}_X)$, so vanish for $i > \dim X$ (and if X is algebraizable, they coincide with the algebraic coherent cohomology by GAGA). On the other hand, the cohomology groups of the sheaf $\mathcal{O}_X^+$ taken modulo p coincide with $H^i(X_{et}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_p/p\mathcal{O}_p$. In the region $\dim X < i \leq 2\dim X$ the groups $H^i(X_{et}, \mathcal{O}_X^+)$ are therefore torsion. From the exact sequence

$$0 \to \mathcal{O}_X^+ \xrightarrow{p} \mathcal{O}_X^+ \to \mathcal{O}_X^+/p\mathcal{O}_X^+ \to 0$$

we get the exact sequence

$$0 \to H^i(X_{et}, \mathcal{O}_X^+)/p \to H^i(X_{et}, \mathcal{O}_X^+/p\mathcal{O}_X^+) \to H^{i+1}(X_{et}, \mathcal{O}_X^+)[p] \to 0.$$

The essence of the proof of the theorem is to "cover" X by perfectoid spaces, and for these use (a) the result that $H^i(Z, \mathcal{O}_Z^+)$ is almost 0 if Z is a perfectoid affinoid space (b) the Artin-Schreier exact sequence in characteristic p, or a variant of it in characteristic 0. Strictly speaking, however, it is not possible to cover X by perfectoids in the étale topology, but only in the pro-étale topology. This is where the pro-étale site enters the game.

3.6.2. *Proof of the main theorem (sketch).* **Step 1**. First one proves a variant of (i), that $H^j(X_{et}, \mathcal{L} \otimes \mathcal{O}_X^+)^a$ is almost finitely generated over $\mathcal{O}_p$ and is almost 0 for $j > 2\dim X$. See [pHT], Lemma 5.8. One knows (by de-Jong and van-der-Put) that the cohomological dimension of X_{an} (i.e. X with the analytic topology) is $n = \dim X$. On the other hand if $V \subset X$ is a nice (see *loc. cit.* Lemma 5.6) smooth affinoid adic space with an étale map $V \to \mathbb{T}^n$ to the n-dimensional torus we can take $\widetilde{V} = V \times_{\mathbb{T}^n} \widetilde{\mathbb{T}}^n$ as a pro-étale affinoid perfectoid cover of V where $\widetilde{\mathbb{T}}^n$ is the standard pro-étale perfectoid cover of the torus. We then have by "almost purity" that $H^i(\tilde{V}, \mathcal{L} \otimes \mathcal{O}_V^+)^a = 0$ for $i > 0$. Since $\widetilde{V}/V$ is a Galois pro-étale cover with profinite Galois group isomorphic to $\mathbb{Z}_p^n$, and since the cohomological dimension of $\mathbb{Z}_p^n$ is n (for continuous group cohomology), we get by the Leray spectral sequence[9] that

$$H^i(V_{et}, \mathcal{L} \otimes \mathcal{O}_X^+)^a = H^i(V_{proet}, \mathcal{L} \otimes \mathcal{O}_X^+)^a = 0$$

for $i > n$. Sheafifying this means that if $\lambda : X_{et} \to X_{an}$ is the projection between the two sites, $R^i\lambda(\mathcal{L} \otimes \mathcal{O}_X^+)^a = 0$ for $i > n$. Another application of a Leray spectral sequence[10] shows that this result, together with the fact that the cohomological dimension of X_{an} is n, imply $H^j(X_{et}, \mathcal{L} \otimes \mathcal{O}_X^+)^a = 0$ for $j > 2n$. The statement about almost-finite-generation is more difficult, but uses similar ideas.

Step 2. A purely algebraic lemma on almost finitely generated $\mathcal{O}_p/p\mathcal{O}_p = \mathcal{O}_p^\flat/p^\flat\mathcal{O}_p^\flat$-modules which can be lifted to a "p-divisible group" of almost-f.g. modules over $\mathcal{O}_p^\flat/(p^\flat)^k\mathcal{O}_p^\flat$ for all k, satisfying some functorial properties, allows one to deduce that such modules are almost free [pHT Lemma 2.12]. Using tilting, Scholze proves that this is indeed the case with the almost-finitely generated module $H^i(X_{et}, \mathcal{L} \otimes \mathcal{O}_X^+)^a$. We therefore have

$$H^i(X_{et}, \mathcal{L} \otimes \mathcal{O}_X^+)^a \simeq (\mathcal{O}_p/p\mathcal{O}_p)^r$$

[9]The functor of global sections on V is the composition of the functor of Galois invariants on the functor of global sections on $\widetilde{V}$, and this leads as usual to a spectral sequence.

[10]Here one uses that global sections of a sheaf F on X_{et} can be obtained as global sections of the sheaf $\lambda_* F$ on X_{an}.

for some integer r. Apply now cohomology to the short exact sequence of sheaves on X_{et}

$$0 \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_X^+ \overset{\Phi-1}{\to} \mathcal{L} \otimes \mathcal{O}_X^+ \to 0$$

to get

$$H^i(X_{et}, \mathcal{L}) \simeq H^i(X_{et}, \mathcal{L} \otimes \mathcal{O}_X^+)^{\Phi=1} \simeq \mathbb{F}_p^r.$$

Note that $\Phi - 1$ is surjective on $\mathcal{O}_p/p\mathcal{O}_p$ (because $\mathbb{C}_p$ is algebraically closed). Tensoring back with $\mathcal{O}_p/p\mathcal{O}_p$ gives the desired result.

4. From étale cohomology classes to modular forms [4 weeks]

4.1. **Recall.**

4.1.1. *Recall of notation and what was left to prove.* Recall that $\mathbf{G}$ denoted the group $Sp_{2n/\mathbb{Z}}$ and S a finite set of rational primes which contained ∞ and p. Recall that $K = K_\infty K_f \subset \mathbf{G}(\mathbb{A})$ where $K_\infty = U(n)$ is the standard (connected) maximal compact subgroup of $G_\infty = \mathbf{G}(\mathbb{R})$. We furthermore assume that $K_f = K^S K_S$ where $K^S = \prod_{l \notin S} \mathbf{G}(\mathbb{Z}_l)$, $K_S = \prod_{l \in S} K_l$, each K_l is open compact in $\mathbf{G}(\mathbb{Z}_l)$ and K_S is "small enough" (e.g. of full tame level $N \geq 3$). In the following we shall assume that K^p is held fixed and K_p shrinks. For example, we may let it run over a the sequence of principal level-p^r subgroups, i.e. the kernels of $\mathbf{G}(\mathbb{Z}_p) \to \mathbf{G}(\mathbb{Z}/p^r\mathbb{Z})$.

With each K we have the Siegel modular variety X_K "of level K" and its Satake-Baily-Borel (normal, projective) compactification X_K^*. These are varieties defined over $\mathbb{Q}$, and in what follows we consider them always over the algebraically closed perfectoid field $C = \mathbb{C}_p$. The reader may fix an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$ and identify $\mathbb{C}_p$-valued modular forms with complex ones, but this is not necessary.

On X_K^* there exists a canonical ample line bundle ω, which on the open variety X_K is the determinant of the relative cotangent bundle of the universal abelian variety. Global sections of ω^k for $k \geq 0$ are nothing but C-valued Siegel modular forms of scalar weight k. We let $\mathcal{I}$ be the ideal sheaf of $\partial_K = X_K^* - X_K$.

We have introduced the local spherical Hecke algebras

$$\mathbb{T}_l = \mathbb{Z}_p[K_l \setminus G_l / K_l]$$

with coefficients in $\mathbb{Z}_p$, for all $l \notin S$. Then $\mathbb{T}_S = \otimes'_{l \notin S} \mathbb{T}_l$ is their restricted tensor product. It is a large commutative algebra that "acts on everything in sight".

Our goal in this seminar has been reduced to proving the following theorem, which was called Theorem 6 before.

Theorem. *[Tors I.5 and IV.3.1] Let $\psi : \mathbb{T} \to \overline{\mathbb{F}}_p$ be a system of eigenvalues occuring in $H_c^i(X_K, \overline{\mathbb{F}}_p)$ (étale cohomology with compact supports) Then there exists a $K' \subset K$ obtained by shrinking K_p, and an arbitrarily large integer k, such that ψ is the reduction modulo p of a system of eigenvalues Ψ which occurs in $H^0(X_{K'}^*, \omega^k \otimes \mathcal{I})$.*

Part of the statement is that the system of eigenvalues Ψ is in $\mathcal{O}_C$. By "reduction modulo p" we mean reduction modulo the maximal ideal of $\mathcal{O}_C$.

4.1.2. *How theorem 7 is put to use.* It is convenient to introduce the *compactly supported completed cohomology* with coefficients in $\mathbb{Z}/p^n\mathbb{Z}$ as

$$\tilde{H}^i_{c,K^p}(\mathbb{Z}/p^n\mathbb{Z}) = \lim_{\to} H^i_c(X_{K^pK_p}, \mathbb{Z}/p^n\mathbb{Z})$$

where the direct limit is taken with respect to the transition maps when K_p shrinks. The cohomology groups on the right are étale cohomology groups with compact supports of $X_{K^pK_p}$ over the algebraically closed field $C = \mathbb{C}_p$, but if we identify C with the complex numbers, can be identified with the usual singular cohomology groups with compact supports of the open Siegel variety. Here we use the fact that if $K' \subset K$ then the map $X_{K'} \to X_K$ is finite and flat, so compactly supported cohomology is contravarient for such maps. [We shall only need the case $n = 1$, but note in passing that if one takes now an inverse limit over n, one gets Emerton's p-adic completed cohomology. It is important to take the limits in the right order: first, with fixed finite coefficients, a direct limit over the level, then an inverse limit over the coefficients. It is the case that torsion classes at finite levels build up to give p-adic classes in the completed cohomology that survive after we tensor with $\mathbb{Q}_p$. The resulting cohomology is significantly richer than the direct limit of the usual groups $H^i_c(X_{K^pK_p}, \mathbb{Q}_p)$.]

The Hecke algebra $\mathbb{T}$ acts on each $H^i_c(X_{K^pK_p}, \mathbb{Z}/p^n\mathbb{Z})$ and commutes with the transition maps, so induces an action on the completed cohomology. Since the tame level will be fixed throughout we abbreviate

$$\tilde{H}^i_c(\mathbb{F}_p) = \tilde{H}^i_{c,K^p}(\mathbb{F}_p).$$

We shall denote by $\mathcal{X}_K$ and $\mathcal{X}^*_K$ the corresponding adic spaces over $Spa(C, \mathcal{O}_C)$ obtained by the analytification of the open or closed Siegel modular variety. The space

$$H^0(\mathcal{X}^*_K, \omega^k)$$

is the space of p-adic modular forms of weight k and level $K = K_pK^p$, and by GAGA is identified with $H^0(X^*_K, \omega^k)$. The same is true for cusp forms, if we consider sections of $\omega^k \otimes \mathcal{I}$. Our goal is thus to relate the Hecke eigensystems appearing in spaces of p-adic modular forms and in completed étale cohomology.

Theorem 7 is a prototype of such a comparison theorem. It cannot be applied directly, though, because "on the coherent side" it does not quite give the desired spaces of modular forms (even modulo p). However, the idea behind the proof of Theorem 7 was to use a perfectoid cover of our adic space X, work at the level of the perfectoid cover, and descend to deduce results on the cohomology of X. It turns out that in our case we have such an explicit perfectoid cover at hand !

4.2. **The perfectoid Siegel modular variety and the Hodge-Tate map.**

4.2.1. *The Siegel varieties become perfectoid in the limit over K_p.* Fix the tame level K^p as before.

Theorem 8. *There exists a prefctoid space $\mathcal{X}^* = \mathcal{X}^*_{K^p}$ over C which is similar (in the precise sense defined before) to the inverse limit of the Siegel spaces when K_p shrinks, i.e.*

$$\mathcal{X}^*_{K^p} \sim \lim_{\leftarrow} \mathcal{X}^*_{K^pK_p}.$$

This perfectoid space is clearly unique. Its construction goes hand in hand with the construction of a certain map of adic spaces

$$\pi_{HT} : \mathcal{X}^* \to \mathcal{F}l$$

where $\mathcal{F}l$ is the adic space associated with the Grassmanian of all maximal isotropic subspaces in $(C^{2n}, \langle , \rangle)$, i.e. the analytic space behind $\mathbf{G}/\mathbf{P}$ where $\mathbf{P}$ is the standard parabolic subgroup discussed earlier. This is the *Hodge Tate period* map. It exists only at the infinite (perfectoid) level: it does *not* come as a limit of morphisms of algebraic varieties defined at the finite levels.

We shall not say much more about the proof of Theorem 8, except that it is done in two steps. One first studies a certain piece $\mathcal{X}^*(0)_a$ of the ordinary locus $\mathcal{X}^*(0)$ of $\mathcal{X}^*$ (the so called anti-canonical ordinary locus) and a strict neighborhood $\mathcal{X}^*(\epsilon)_a$ of it where the Hasse invariant is not too small, or alternatively, where the abelian variety has a canonical subgroup. One shows that this piece is perfectoid. In the second step one uses the action of G_p on the tower to prove the perfectoid-ness of the whole $\mathcal{X}^*$.

4.2.2. *The Hodge Tate decomposition for abelian varieties.* Let A be an abelian variety over $\mathcal{O}_C$, i.e. an abelian variety over C with good reduction. The case of bad reduction can be treated similarly, but requires the introduction of Néron models, and we shall skip it. There are various ways to get the Hodge-Tate exact sequence for A. We follow the original construction by Tate [pdiv].

For any p-torsion module M we let

$$T_p(M) = Hom_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, M) = \lim_{\leftarrow} M[p^n], \ \ V_p(M) = Hom_{\mathbb{Z}_p}(\mathbb{Q}_p, M) = T_p(M) \otimes \mathbb{Q}$$

be its integral (resp. rational) Tate modules. We write $T_p(A)$ and $V_p(A)$ for the corresponding modules with $M = A(C)[p^\infty]$.

Denote by A^t the dual abelian scheme over $\mathcal{O}_C$. Recall that $Lie(A^t) = H^1(A, \mathcal{O}_A)$ canonically. Since $A[p^n]_{/\mathcal{O}_C}$ is the Cartier dual of the finite group scheme $A^t[p^n]_{/\mathcal{O}_C}$, a point $x_n \in A[p^n](\mathcal{O}_C)$ is a homomorphism of group schemes $A^t[p^n] \to \mu_{p^n}$ over $\mathcal{O}_C$. A point $x \in T_p(A)$ is therefore a homomorphism of p-divisible groups $A^t(p) \to \mu_{p^\infty}$. Passing to the Lie algebras over $\mathcal{O}_C$ it yields a homomorphism from $Lie(A^t)$ to $Lie(\mu_{p^\infty})$. But the latter is canonically $\mathcal{O}_C$. This gives a map

$$\alpha_A : T_p(A) \otimes_{\mathbb{Z}_p} \mathcal{O}_C \to Lie(A^t)^\vee.$$

Replacing A by A^t and dualizing we get

$$\alpha_{A^t}^\vee : Lie(A) \to T_p(A^t)^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}_C \simeq T_p(A)(-1) \otimes_{\mathbb{Z}_p} \mathcal{O}_C$$

where the last step involves the (perfect) Weil pairing $T_pA \times T_pA^t \to \mathbb{Z}_p(1)$. Taken together, and tensoring with $\mathbb{Q}$, one obtains the Hodge-Tate sequence

$$0 \to Lie(A_{/C})(1) \to V_p(A) \otimes_{\mathbb{Q}_p} C \to Lie(A^t_{/C})^\vee \to 0$$

which, as Tate proved, is exact. Moreover, when A is defined over a discretely valued subfield[11] K of C this sequence is split-exact, as a sequence of $Gal(\bar{K}/K)$-modules. This follows from the fundamental theorems of Tate on the continuous Galois cohomology of $C(j)$. Dualizing again, Tate got in this case the (canonical) Hodge-Tate decomposition

$$H^1_{et}(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \simeq H^0(A, \Omega^1_{A/K}) \otimes_K C(-1) \oplus H^1(A, \mathcal{O}) \otimes_K C.$$

Fontaine-Messing (in the good reduction case) and Faltings (in general) have generalized this to the étale cohomology of any proper smooth variety over K, in any dimension. Finally, in [pHT], Scholze extended the Hodge-Tate decomposition to any proper smooth rigid analytic space (not necessarily algebraizable).

Suppose now that $A_{/C}$ comes equipped with a polarization λ (as is the case for the A's parametrized by the Siegel modular variety X_K). Then $H^1_{et}(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$

[11] I appologize for using the same letter for the compact group K and the discretely valued field. No confusion should occur, as the field K will soon disappear and we shall return to work over C.

is equipped with a symplectic polarization pairing $\langle,\rangle_\lambda \otimes 1$ and one proves that the subspace $H^1(A, \mathcal{O}_A) = Lie(A^t) \simeq Lie(A)$ is a maximal isotropic subspace.

Finally we note that if L is any complete subfield of C over which all the p-power torsion $A[p^\infty]$ is defined, then the Hodge-Tate exact sequence (but not its splitting) is defined over L.

4.2.3. *Another way to define the Hodge-Tate exact sequence (after Fontaine).* Suppose again that A is defined over a discretely valued subfield K. One notes first that the map

$$\mathcal{O}_{\bar{K}} \otimes_{\mathbb{Z}_p} \mu_{p^\infty} \to \Omega^1_{\mathcal{O}_{\bar{K}}/\mathcal{O}_K}$$

given by $(a, \zeta) \mapsto a\frac{d\zeta}{\zeta}$ is surjective with a kernel that is killed by some power of p (depending on the absolute different of K).

Consider now $\omega \in H^0(A, \Omega^1_{A/\mathcal{O}_K})$. If $x \in A(\mathcal{O}_{\bar{K}})$ then $x^*(\omega) \in \Omega^1_{\mathcal{O}_{\bar{K}}/\mathcal{O}_K}$. The two pairings that we have described yield first an isomorphism

$$V_p(\Omega^1_{\mathcal{O}_{\bar{K}}/\mathcal{O}_K}) \simeq C(1)$$

(Tate twist of C) and secondly a pairing

$$H^0(A, \Omega^1_{A/\mathcal{O}_K}) \times V_p(A) \to V_p(\Omega^1_{\mathcal{O}_{\bar{K}}/\mathcal{O}_K}) \simeq C(1).$$

We get the Galois equivariant map

$$\phi_A : H^0(A, \Omega^1_{A/K}) \otimes_K C \to Hom(V_p(A), C(1)) = H^1_{et}(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C(1).$$

As before, it is possible to get the map from $H^1(A, \mathcal{O}) \otimes_K C$ to $H^1_{et}(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$ from ϕ_A playing with duality and the dual abelian variety.

4.2.4. *The Hodge-Tate period morphism.* Consider now a C-valued point $x \in |\mathcal{X}^*| = \lim_{\leftarrow} |\mathcal{X}^*_{K^p K_p}|$. As we are taking an inverse limit over all the level subgroups at p, such a point parametrizes a principally polarized abelian variety A over C, a tame level structure away from p, and a full p^∞-level structure, i.e. a trivialization of $V_p(A)$, or, by duality, a symplectic isomorphism

$$(H^1_{et}(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C,\ \langle,\rangle_\lambda \otimes 1) \simeq (C^{2n}, \langle,\rangle_J).$$

Here $\langle,\rangle_\lambda$ is the Weil pairing in cohomology induced by the principal polarization λ, and $\langle,\rangle_J$ is the standard symplectic form given by the matrix J. This symplectic isomorphism carries $H^1(A, \mathcal{O}) \otimes_K C$ to a certain maximal isotropic subspace of C^{2n}, hence defines a C-valued point $\pi_{HT}(x)$ of $\mathcal{F}l$. We emphasize that to get it we needed to work with full level structure. This defines the Hodge-Tate morphism at the level of C-valued points. See [Tors, Lemma III.3.4]. Scholze proves the following.

Theorem 9. *[Tors, III.3.18 and IV.1.1] (i) There is a unique G_p-equivariant map of adic spaces over $\mathbb{Q}_p$, $\pi_{HT} : \mathcal{X}^* \to \mathcal{F}l$ which realizes the above map on C-valued points.*

(ii) Let $\mathcal{X}^*(0)$ denote the ordinary locus in $\mathcal{X}^*$ (the locus parametrizing ordinary abelian varieties, including the boundary ∂). Then $\mathcal{X}^*(0)$ is the pre-image of $\mathcal{F}l(\mathbb{Q}_p)$ under π_{HT}.

(iii) Let s_j (where j runs over all the subsets of size n of $\{1, 2, \ldots, 2n\}$) be the standard coordinates on $\mathcal{F}l$ obtained from the embedding in $Gr(n, 2n)$. Let $\mathcal{F}l_j$ be the affinoid subdomain where $|s_j| \geq |s_{j'}|$ for all j'. Then the pre-image $\mathcal{V}_j$ of $\mathcal{F}l_j$ under π_{HT} is an affinoid perfectoid, say $Spa(R_j, R_j^+)$. For all sufficiently large m

it is the pre-image of an affinoid $\mathcal{V}_{j,m} = Spa(R_{j,m}, R^+_{j,m})$ *in the Siegel modular variety of full level* p^m *at* p *and* R^+_j *is the* p*-adic completion of* $\lim_{\to} R^+_{j,m}$.

(iv) The maps π_{HT} *satisfy the obvious compatability with respect to shrinking the tame level.*

(v) For any $\gamma \in \mathbf{G}(\mathbb{A}^p_f)$ *the map* $\gamma_* : \mathcal{X}^*_{K^p} \to \mathcal{X}^*_{\gamma^{-1}K^p\gamma}$ *satisfies* $\pi_{HT} \circ \gamma_* = \pi_{HT}$.

(vi) Let $W_{\mathcal{F}l} \subset \mathcal{O}^{2n}_{\mathcal{F}l}$ *be the tautological totally isotropic sub-bundle. Then over the open set* $\mathcal{X} \subset \mathcal{X}^*$ *there is a natural isomorphism*

$$LieA \simeq LieA^t \simeq \pi^*_{HT} W_{\mathcal{F}l}$$

and therefore also a natural isomorphism

$$\omega \simeq \pi^*_{HT}\omega_{\mathcal{F}l}$$

where $\omega_{\mathcal{F}l} = \det(W_{\mathcal{F}l})^\vee$ *is the natural ample line bundle on* $\mathcal{F}l$.

We make a comment on (ii). It means that the Hodge-Tate subspace $Lie(A^t_{/C}) \simeq H^1(A, \mathcal{O}_A)$ of $H^1_{et}(A, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$ comes from a $\mathbb{Q}_p$-rational subspace of $H^1_{et}(A, \mathbb{Q}_p)$ via base-change if and only if A is ordinary. If A is ordinary and denfined over a discretely valued field K then this $\mathbb{Q}_p$-rational subspace of $H^1_{et}(A, \mathbb{Q}_p)$ can be readily described. In this case the $Gal(\bar{K}/K^{nr})$-module $V_p(A)$ has a filtration

$$0 \to \mathbb{Q}_p(1)^n \to V_p(A) \to \mathbb{Q}_p^n \to 0$$

(this is essentially equivalent to the definition of ordinariness) and it can be shown that the Hodge-Tate filtration comes from the dual filtration on $H^1_{et}(A, \mathbb{Q}_p)$. Nevertheless, the existence of a morphism as in (ii) is bizarre, and can occur only with large perfectoid spaces such as $\mathcal{X}^*$. For example, when $n = 1$, the flag variety is $\mathbb{P}^1$. All of the ordinary locus gets contracted by π_{HT} to $\mathbb{P}^1(\mathbb{Q}_p)$, and for an ordinary elliptic curve equipped with a trivialization of its Tate module, the associated point in $\mathbb{P}^1(\mathbb{Q}_p)$ measures the location of the canonical p-divisible group (the part in the kernel of reduction) vis-a-vis the trivialization. On the other hand the supersingular locus in $\mathcal{X}^*$ gets mapped under π_{HT} onto the Drinfeld p-adic upper half plane. Note that the cusps are "ordinary".

Point (iii) is a by-product of the construction of $\mathcal{X}^*$. Points (iv) and (v) are natural, and are needed to study the action of the Hecke operators away from the bad primes and p. Point (vi) is clear from the modular interpretation of the map π_{HT} described above for C-points.

4.3. **Completed cohomology versus p-adic automorphic forms.**

Proposition. *There are natural isomorphisms of almost* $\mathcal{O}_C$*-modules*

$$\tilde{H}^i_c(\mathcal{X}^*_{et}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p\mathcal{O}_C \simeq H^i(\mathcal{X}^*_{an}, \mathcal{I}^+/p\mathcal{I}^+).$$

Proof. Let $K = K^pK_p$ be a fixed level. We have seen that for any local system $\mathcal{L}$ of $\mathbb{F}_p$-vector spaces in the étale topology

$$H^i(X^*_{K,et}, \mathcal{L}) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p\mathcal{O}_C \simeq H^i(X^*_{K,et}, \mathcal{L}\otimes\mathcal{O}^+_X)$$

(an almost isomorphism). This is extended in [CDM], Theorem 3.13, to constructible sheaves of $\mathbb{F}_p$-vector spaces (not necessarily locally constant). Let j be the open embedding of X_K in X^*_K. Note that $\mathcal{I}^+/p\mathcal{I}^+ = j_!(\mathbb{F}_p) \otimes O^+_X$ (the stalks of $\mathcal{I}^+/p\mathcal{I}^+$ at the boundary vanish since a function from $\mathcal{O}^+_X$ which vanishes at the boundary, is divisible by p in some open neighborhood of it) and that

$H^i(X^*_{K,et}, j_!(\mathbb{F}_p)) = H^i_c(X_{K,et}, \mathbb{F}_p)$. Take the limit over K_p, to get the proposition with $\mathcal{X}^*_{et}$ instead of $\mathcal{X}^*_{an}$ on the right hand side.

Now there is a short exact sequence

$$0 \to \mathcal{I}^+/p\mathcal{I}^+ \to \mathcal{O}^+_{\mathcal{X}}/p\mathcal{O}^+_{\mathcal{X}} \to \mathcal{O}^+_{\partial}/p\mathcal{O}^+_{\partial} \to 0.$$

Etale and analytic cohomologies of $\mathcal{O}^+_{\mathcal{X}}/p\mathcal{O}^+_{\mathcal{X}}$ agree: they both (almost) vanish on affinoid subsets of $\mathcal{X}^*$(here we finally use the fact that this is a perfectoid), and the rest is given by the Cech spectral sequence. The same holds true with $\mathcal{O}^+_{\partial}/p\mathcal{O}^+_{\partial}$, hence also with $\mathcal{I}^+/p\mathcal{I}^+$. □

Corollary. *The cohomology groups $\tilde{H}^i_c(\mathcal{X}^*_{et}, \mathbb{F}_p)$ vanish for $i > d = \dim \mathcal{X}^*$.*

Proof. By the theorem, it is enough to show that the cohomological dimension of $\mathcal{X}^*_{an}$ (the maximal dimension in which cohomology of torsion abelian sheaves may not vanish) is at most d, but this is the contents (in the analytic topology!) of the theorem of de Jong and van der Put mentioned above. □

4.4. **Hecke algebras and conclusion of the proof.** It is now possible to complete the proof of the theorm. Consider the Hodge-Tate period map as a map of $\mathbb{Q}_p$-adic spaces

$$\pi_{HT} : \mathcal{X}^* \to \mathcal{F}l \hookrightarrow \mathbb{P}^{N-1}$$

where $N = \begin{pmatrix} 2n \\ n \end{pmatrix}$, using the Plücker embedding. The line bundle $\mathcal{O}(1)$ pulls back to the tautological line bundle $\omega_{\mathcal{F}l}$ and then to ω. Let $s_1, \ldots, s_N$ be the standard sections of $\mathcal{O}(1)$ and $\mathcal{U}_i \subset \mathbb{P}^{N-1}$ the open affinoid where

$$|s_j| \le |s_i|$$

for every j. If $\emptyset \neq J \subset \{1, \ldots, N\}$ let $\mathcal{U}_J = \bigcap_{j \in J} U_j$ and $s_J = \prod_{j \in J} s_j$. If $\bar{\mathcal{U}}_J$ is the reduction modulo p^n of $\mathcal{U}_J$ and $\bar{s}_J$ the reduction of the corresponding section of $\mathcal{O}(\# J)$ then $\bar{\mathcal{U}}_J$ is the subscheme of $\mathbb{P}^{N-1}$ (over $\mathbb{Z}/p^n\mathbb{Z}$) where $\bar{s}_J$ is invertible.

Let $\mathcal{V}_J = \pi^{-1}_{HT}(\mathcal{U}_J)$, which by (iii) of Theorem 9 is an affinoid perfectoid of the form

$$\mathcal{V}_J = Spa(R_J, R^+_J).$$

Let $\psi : \mathbb{T} \to \bar{\mathbb{F}}_p$ be a system of Hecke eigenvalues which occurs in $\tilde{H}^i_c(\bar{\mathbb{F}}_p)$. Then by the Proposition it occurs in $H^i(\mathcal{X}^*, \mathcal{I}^+/p\mathcal{I}^+)$ (sheaf cohomology in the analytic topology). Since the cohomology of $\mathcal{I}^+/p\mathcal{I}^+$ over any affinoid (perfectoid) subdomain of $\mathcal{X}^*$ vanishes in positive degrees, we may compute the latter cohomology as the homology of the Cech complex whose terms are the almost modules

$$H^0(\mathcal{V}_J, \mathcal{I}^+/p\mathcal{I}^+) = \{f \in R^+_J |\ f|_\partial = 0\}^a / p\{f \in R^+_J |\ f|_\partial = 0\}^a.$$

As a consequence of (iv) and (v) of Theorem 9, the map π_{HT} commutes with the Hecke operators away from p. It follows that the $\mathcal{V}_J$ are stable by the Hecke correspondences, and $\mathbb{T}$ acts on each of the terms in the Cech complex separately. Thus if ψ occurs in $H^i(\mathcal{X}^*, \mathcal{I}^+/p\mathcal{I}^+)$, it also occurs in some $H^0(\mathcal{V}_J, \mathcal{I}^+/p\mathcal{I}^+)$.

For all K_p sufficiently small, the $\mathcal{V}_i$ are the pull-back via the projection to X^*_K of an affinoid subdomain $\mathcal{V}_{iK}$ of X^*_K and $\lim_{\to} H^0(\mathcal{V}_{iK}, \omega)$ is dense in $H^0(\mathcal{V}_i, \omega)$. We can therefore approximate the sections s_j on $\mathcal{V}_i$ by

$$s^{(i)}_j \in H^0(\mathcal{V}_{iK}, \omega)$$

so that

$$|\frac{s_j - s_j^{(i)}}{s_i^{(i)}}| \leq |p|$$

on $\mathcal{V}_i$ for all j. Using these sections we get a formal model $\mathfrak{X}_K^*$ (over $Spf(\mathbb{Z}_p)$) of the rigid analytic space $\mathcal{X}_K^*$, and an open cover $\{\mathfrak{V}_{iK}\}_{i=1}^N$ by affine formal schemes. (The relation between affinoid coverings of rigid analytic varieties and formal models is classical[12], see [Tors], Lemma II.1.1.) This formal model comes equipped with an ample line bundle which we still denote ω and a sheaf of ideals $\mathcal{J}$ that, on the "generic fiber", yield ω and $\mathcal{I}$.

These objects are independent of the choice of the approximating sections $s_j^{(i)}$ used to approximate s_j. The sections

$$s_j^{(i)} \mod p$$

glue to give a section $\bar{s}_j$ of $\omega \mod p$ on $\mathfrak{X}_K^*$, which is independent of any choice. The open formal subscheme $\mathfrak{V}_{J,K}$ is the non-vanishing locus of $\bar{s}_J = \prod \bar{s}_j$. These sections are compatible with change of tame level at K and with the action of $G(\mathbb{A}_f^p)$ as in part (v) of Theorem 9. As a result, multiplication by $\bar{s}_J$ commutes with the action of $\mathbb{T}$ and therefore preserves eigenvectors and systems of Hecke eigenvalues. Scholze calls these sections "substitutes of the Hasse invariants".

We remark that the integral structures which result from the formal models $\mathfrak{X}_K^*$ are very far from the familiar integral models of the Shimura varieties X_K^* defined in terms of the moduli problem, and are not related to them. By definition we have

$$H^0(\mathcal{V}_i, \mathcal{I}^+/p\mathcal{I}^+) = \lim_{\rightarrow} H^0(\mathcal{V}_{i,K}, \mathcal{I}^+/p\mathcal{I}^+) = \lim_{\rightarrow} H^0(\mathfrak{V}_{i,K}, \mathcal{J}/p\mathcal{J})$$

and similarly for any $J \subset \{1, 2, \ldots, N\}$. Thus it is enough to assume that ψ occurs in some $H^0(\mathfrak{V}_{J,K}, \mathcal{J}/p\mathcal{J})$.

Now

$$H^0(\mathfrak{V}_{J,K}, \mathcal{J}/p\mathcal{J}) = \lim_{\rightarrow} H^0(\mathfrak{X}_K^*, \omega^{k\#(J)} \otimes \mathcal{J}/p\mathcal{J})$$

where the limit is with respect to the map of multiplication by $\bar{s}_J$. As we have noticed above, multiplication by $\bar{s}_J$ commutes with the Hecke action. We may therefore assume that ψ occurs in $H^0(\mathfrak{X}_K^*, \omega^k \otimes \mathcal{J}/p\mathcal{J})$ and that furthermore, k is arbitrarily large. But for k large enough, by the ample-ness of ω,

$$H^1(\mathfrak{X}_K^*, \omega^k \otimes \mathcal{J}) = 0$$

hence $H^0(\mathfrak{X}_K^*, \omega^k \otimes \mathcal{J}/p\mathcal{J}) = H^0(\mathfrak{X}_K^*, \omega^k \otimes \mathcal{J})/pH^0(\mathfrak{X}_K^*, \omega^k \otimes \mathcal{J})$, i.e. for a large enough weight, every mod p modular form lifts to a characteristic 0 modular form. But by comparison of formal, analytic and algebraic cohomologies

$$H^0(\mathfrak{X}_K^*, \omega^k \otimes \mathcal{J}) \otimes_{\mathcal{O}_C} C = H^0(\mathcal{X}_K^*, \omega^k \otimes \mathcal{I}) = H^0(X_K^*, \omega^k \otimes \mathcal{I})$$

so ψ is the reduction modulo p of a Hecke eigensystem Ψ occuring in $H^0(X_K^*, \omega^k \otimes \mathcal{I})$. QED.

4.5. **More on the perfectoid Siegel space and the Hodge-Tate map.**

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[12]Basically, to get from an adic affinoid $\mathcal{V}_{iK} = Spa(R, R^+)$ to a formal scheme, take $\mathfrak{V}_{i\mathfrak{K}} = Spf(R^+)$. The gluing is done along the formal schemes $\mathfrak{V}_{ij,K}$ which are open formal subschemes of both $\mathfrak{V}_{iK}$ and $\mathfrak{V}_{jK}$. The resulting formal scheme is projective and the line bundle ω glues to become an ample line bundle on $\mathfrak{X}$.

4.5.1. *Set-up.* In the remaining time we shall focus on the construction of $\mathcal{X}^*$ and the map π_{HT} for $n = 1$, i.e. in the classical case of modular curves. I shall follow notes from lectures of Mark Kisin in MSRI in 2/14. The notation and the assumptions are a little different than what was used till now.

Fix $N \geq 3$. Fix a generator of the Tate module of μ_{p^∞} and identify $\mathbb{Z}_p(1)$ with $\mathbb{Z}_p$. For $n \geq 0$ let $X(p^n)$ be the modular curve over $\mathbb{Q}_p^{cycl} = \mathbb{Q}(\mu_{p^\infty})\hat{}$ parametrizing elliptic curves A, a full level-N structure, and a full level-p^n structure for which the Weil pairing becomes the standard symplectic pairing on $(\mathbb{Z}/p^n\mathbb{Z})^2$. (Here we use the identification of $\mathbb{Z}_p(1)$ with $\mathbb{Z}_p$.) Let $X_0(p^n)$ be the modular curve with $\Gamma_0(p^n)$ level structure, i.e. parametrizing elliptic curves, tame level structure, and a cyclic subgroup of order p^n. By $X(p^n)$ or $X_0(p^n)$ we mean the complete curve (including the cusps). Write $X = X(1)$.

Let (L, L^+) be a complete non-archimedean extension of $\mathbb{Q}_p^{cycl}$ and its ring of integers. Define

$$X(p^\infty)(L) = \lim_{\leftarrow} X(p^n)(L).$$

Thus outside the cusps, a point $x \in X(p^\infty)(L)$ "is" an elliptic curve $A_{/L}$, a full level-N structure, and a symplectic isomorphism

$$T_pA \simeq \mathbb{Z}_p^2.$$

The Hodge-Tate exact sequence attached to x a canonical line

$$Lie(A)(1) \hookrightarrow T_pA \otimes_{\mathbb{Z}_p} L \simeq L^2$$

hence a point of $\mathcal{F}(L) = \mathbb{P}^1(L)$. This is the map

$$\pi_{HT} : X(p^\infty)(L) \to \mathbb{P}^1(L).$$

We want to construct a perfectoid space $\mathcal{X}$ over $\mathbb{Q}_p^{cycl}$ which is similar to $\lim_{\leftarrow} X(p^n)^{ad}$ and a map of adic space $\pi_{HT} : \mathcal{X} \to \mathbb{P}^1$ which agree with the above inverse limit and projection on L-points. At any finite level we have

$$X(p^n)^{ad} = X(p^n)^{ord} \amalg X(p^n)^{ss}$$

where the ordinary part is a (disconnected) affinoid and the super-singular part a rigid analytic space covering the open supersingular disks in X^{ad}. The cusps belong (by definition) to $X(p^n)^{ord}$. As these decompositions are compatible with the inverse limit they define a similar decomposition of $X(p^\infty)$. Similar notation will be used for Γ_0-level structure.

The idea is to prove "perfectoid-ness" of $X(p^\infty)$ on an open part first, then use $SL_2(\mathbb{Q}_p)$-action to show that it is all perfectoid.

4.5.2. *The canonical subgroup.* Let S be an $\mathbb{F}_p$-scheme and A/S an elliptic curve. The Verschiebung isogeny $Ver : A^{(p)} \to A$ induces $V^* : \omega_{A/S} \to \omega_{A/S}^{(p)} \simeq \omega_{A/S}^p$ hence a canonical section $Ha(A/S)$ of $\omega_{A/S}^{p-1}$ called the Hasse invariant of A over S. One knows that A/S is (fiber-wise) ordinary if and only if $Ha(A/S)$ is invertible. Now assume ($\epsilon \in [0, 1)$) that $|p^\epsilon| \in |L|$ where $L = \mathbb{Q}_p^{cycl}$ and S is a scheme over L^+/pL^+. We say that $Ha(A/S)$ divides p^ϵ if there is a section η of $\omega_{A/S}^{1-p}$ such that $Ha(A/S) \cdot \eta = p^\epsilon$. This is a measure of "how supersingular" A is.

Let $X(p^n)[\epsilon]$ be the affinoid subdomain of $X(p^n)^{ad}$ parametrizing elliptic curves A for which $Ha(A_1)$ is divisible by p^ϵ. Here A_1 is $A \mod p$. When $n = 0$ this is a strict neighborhood of the ordinary locus, including an annulus in each supersingular disk, and in general it is the inverse image of this affinoid in $X(p^n)^{ad}$. Note that $X(p^n)[0] = X(p^n)^{ord}$. Over this affinoid A is ordinary and $A[p^m]$ has, for every m, a canonical subgroup scheme C_m which is finite flat of rank p^m, and whose reduction modulo p is the kernel of $Fr^m : A_1 \to A_1^{(p^m)}$. The next theorem tells us, in a quantitative way, how the canonical subgroup "overconverges". In part (i) we work over a certain general base $Spec(R)$. In part (ii) we work in neighborhoods of X^{ord} where $X = X(1)$ (no p^n-level yet) and these neighborhoods become smaller and smaller with m. In part (iii) we show that if we introduce $\Gamma_0(p^m)$-level structure, these neighborhoods are mapped to certain open and closed components of $X_0(p^m)[\epsilon]$ with a *fixed width* ϵ, and that the transition maps between them become Frobenius modulo $p^{1-\epsilon}$.

Theorem 10. *[Tors, Corollary III.2.6, Proposition III.2.8] Let $\epsilon \in [0, 1/2)$ and $m \geq 1$. Let R be a p-adically complete flat $\mathbb{Z}_p^{cycl}$-algebra, and A/R an elliptic curve for which $Ha(A_1/R_1)^{p^m}$ divides p^ϵ.Then:*

(i) There exists a canonical finite flat subgroup scheme $C_m \subset A[p^m]$ of rank p^m such that $C_m \equiv ker(Fr^m)$ modulo $p^{1-\epsilon}$. This subgroup scheme behaves functorially in R and satisfies

$$C_m(R) = \{s \in A[p^m](R) | \ s \equiv 0 \mod p^{(1-\epsilon)/p^m}\}.$$

(ii) In particular, C_1 exists in $\mathcal{A}[p]$ where $\mathcal{A}$ is the universal elliptic curve over $X[p^{-1}\epsilon]$ and $\mathcal{A} \mapsto \mathcal{A}/C_1$ induces a map

$$\tilde{F} : X[p^{-m}\epsilon] \to X[p^{1-m}\epsilon]$$

for every $m \geq 1$ (thus $\mathcal{A}/C_1$ is "more supersingular" than $\mathcal{A}$ but in a controled way). These affinoids have nice integral structures[13] and when we reduce them modulo $p^{1-\epsilon}$ the map $\tilde{F}$ reduces to the relative Frobenius morphism of the scheme $X[p^{-m}\epsilon] \mod p^{1-\epsilon}$.

(iii) The map

$$A \mapsto (A, C_m) \mapsto (A/C_m, A[p^m]/C_m)$$

induces open embeddings $X[p^{-m}\epsilon] \hookrightarrow X_0(p^m)^{ad}$ whose image is an open and closed subset

$$X_0(p^m)[\epsilon]_a$$

of $X_0(p^m)[\epsilon]$.

The affinoid $X_0(p^m)[\epsilon]_a$ is called the anti-canonical affinoid. If $\epsilon = 0$ it is the unique component of $X_0(p^m)^{ord}$ where the cyclic subgroup scheme of rank p^m reduces to an étale group scheme, i.e. is disjoint from the kernel of reduction. In general, the anticanonical affinoid can be described similarly, as the unique component of $X_0(p^m)[\epsilon]$ where the cyclic subgroup scheme of rank p^m is "as far as possible from canonical".

[13]These integral structures arise from viewing the affinoid $X[p^{-m}\epsilon]$ as the generic fiber of a certain formal scheme $\mathfrak{X}[p^{-m}\epsilon]$ which represents a certain natural functor, see [Tors, Definition III.2.12].

4.5.3. *The perfectoid-ness of the $\Gamma_0(p^m)$ tower.* Using part (iii) of the last theorem we get a commutative diagram

$$\begin{array}{ccc} X[p^{-m-1}\epsilon] & \overset{A\mapsto A/C_{m+1}}{\hookrightarrow} & X_0(p^{m+1})^{ad} \\ \downarrow \tilde{F} & & \downarrow \\ X[p^{-m}\epsilon] & \overset{A\mapsto A/C_m}{\hookrightarrow} & X_0(p^m)^{ad} \\ \downarrow & & \downarrow \\ X[p^{-1}\epsilon] & \overset{A\mapsto A/C_1}{\hookrightarrow} & X_0(p)^{ad} \end{array}$$

The images of the horizontal arrows are the tower of $X_0(p^m)[\epsilon]_a$ and one checks easily that the induced transition maps in that tower are the natural maps between level p^{m+1} to level p^m. As modulo $p^{1-\epsilon}$ these masp are just the relative Frobenius maps, we see that modulo $p^{1-\epsilon}$ the tower of rings of definition[14] on the right looks like $\lim_{\rightarrow}$ of a fixed ring w.r.t. the map of raising to power p, hence is perfect. If we take the completion of this direct limit we get an affinoid perfectoid $\mathcal{X}_0[\epsilon]_a$ such that

$$\mathcal{X}_0[\epsilon]_a \sim \lim_{\leftarrow} X_0(p^m)[\epsilon]_a.$$

This is the first step in the construction of $\mathcal{X}$.

4.5.4. *From $\mathcal{X}_0[\epsilon]_a$ to $\mathcal{X}[\epsilon]$.* Consider now full level p^∞-structure and the map

$$(A, \alpha : T_pA \simeq \mathbb{Z}_p^2) \mapsto (A, \alpha^{-1}(\mathbb{Z}_p \oplus (0)))$$

from $\mathcal{X}$ to $\mathcal{X}_0$. Then setting $\mathcal{X}[\epsilon]_a$ to be the pre-image of $\mathcal{X}_0[\epsilon]_a$ we see that this is also a perfectoid (note the ϵ is unchanged because A is unchanged).

Next, let $SL_2(\mathbb{Z}_p)$ act on the tower. It is easy to see that

$$\mathcal{X}[\epsilon] = SL_2(\mathbb{Z}_p) \cdot \mathcal{X}[\epsilon]_a.$$

We conclude that $\mathcal{X}[\epsilon]$ is perfectoid.

4.5.5. *From $\mathcal{X}[\epsilon]$ to $\mathcal{X}$.* At this point it is essential to study π_{HT} and also to take $\epsilon > 0$ (till now we could work with $\epsilon = 0$, i.e. with the ordinary parts only).

Lemma. *(i)* $\pi_{HT}^{-1}(\mathbb{P}^1(\mathbb{Q}_p)) = \mathcal{X}[0]$.

(ii) Assume $\epsilon > 0$. There exists an open rigid analytic space $U \subset (\mathbb{P}^1)^{ad}$ containing $\mathbb{P}^1(\mathbb{Q}_p)$ (actually a union of finitely many affinoid disks, how many depends on ϵ) such that

$$\pi_{HT}^{-1}(U) \subset \mathcal{X}[\epsilon].$$

The following lemma is an easy exercise:

Lemma. *Let $\mathbb{P}^1(\mathbb{Q}_p) \subset U \subset (\mathbb{P}^1)^{ad}$ be as above. Then $SL_2(\mathbb{Q}_p) \cdot U = (\mathbb{P}^1)^{ad}$.*

Combining the two lemmas and using the $SL_2(\mathbb{Q}_p)$-equivariance of π_{HT} we conclude that $SL_2(\mathbb{Q}_p) \cdot \mathcal{X}[\epsilon] = \mathcal{X}$ and so $\mathcal{X}$, like $\mathcal{X}[\epsilon]$, is perfectoid.

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[14] Recall that if $X^{ad} = Spa(R, R^+)$ then R^+ is called the ring of definition of X^{ad}.

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