$\mathcal{L}$ INVARIANTS OF p-ADICALLY UNIFORMIZED VARIETIES

AMNON BESSER AND EHUD DE SHALIT

to Glenn Stevens

1. Introduction

Let K be a finite extension of $\mathbb{Q}_p$ and E an elliptic curve with split multiplicative reduction over K. Tate has shown that, as a rigid analytic space over K,

$$E(\mathbb{C}_p) \simeq \mathbb{C}_p^\times / \langle q_E \rangle, \tag{1.1}$$

where the period $q_E \in K^\times$ is uniquely determined by the requirement $|q_E| < 1$. Fix a uniformizer π of K. The $\mathcal{L}$-invariant of E associated with the choice of π,

$$\mathcal{L}_{\pi,E} = \frac{\log_\pi(q_E)}{ord_K(q_E)} \tag{1.2}$$

is an isogeny invariant. The valuation and the p-adic logarithm are normalized by

$$ord_K(\pi) = 1, \;\; \log_\pi(\pi) = 0. \tag{1.3}$$

When $K = \mathbb{Q}_p$ it is customary, but not necessary, to take $\pi = p$.

When E is a (modular) elliptic curve defined over $\mathbb{Q}$, and p is a prime of split multiplicative reduction, then Mazur, Tate and Teitelbaum [M-T-T] discovered that the p-adic L-function $L_p(E, s)$ of E acquires a trivial zero at $s = 1$. They conjectured that the ratio of $L_p'(E, 1)$ to the algebraic part of $L(E, 1)$ should be given by $\mathcal{L}_{p,E}$. The conjecture was proved in 1993 in a remarkable paper of Greenberg and Stevens [G-S], where they have also related $\mathcal{L}_{p,E}$ to the variation of Atkin's U_p operator in a p-adic family of eigenforms depending on the weight, and specializing to E (or rather, to the modular form related to E) in weight 2.

Subsequently, $\mathcal{L}$-invariants at primes of "totally degenerate reduction" have been attached to cuspidal eigenforms (first of weight 2 and then of any weight $k \geq 2$), and similar conjectures were made. At least three distinctly different approaches to $\mathcal{L}$ invariants have emerged. The first, due to Fontaine and Mazur [Ma] is *cohomological.* It derives the $\mathcal{L}$-invariants from a comparison of de Rham and log-crystalline cohomologies (filtered (ϕ, N)-modules). The second, due to Coleman and Teitelbaum, is *rigid analytic.* It uses transcendental techniques (p-adic integration) to make $\mathcal{L}$-invariants, either directly on the modular curve, or, after applying the Jacquet-Langlands correspondence, on a Shimura curve with totally degenerate reduction at p. The third [G-S] goes, as explained above, via p-adic variation of the U_p operator. As a result of the work of several authors, it is now known that all these definitions agree with each other, and fit into the p-adic L-function machine. We refer to the survey paper by Dasgupta and Teitelbaum [D-T] for a full account and further developments.

The authors of this note are unaware of any $\mathcal{L}$-invariants attached so far to *higher dimensional* varieties[1]. A natural class of varieties to look at are the p-adically uniformized varieties, namely those obtained (rigid analytically) as quotients of Drinfeld's p-adic symmetric domain $\mathfrak{X}$ of dimension $d \geq 1$ by the action of a discrete cocompact subgroup Γ of $PGL_{d+1}(K)$. On one hand, the cohomology of these varieties is understood well-enough to allow us to make a working definition of (d, not one) $\mathcal{L}$-invariants associated with its middle-degree cohomology. In addition, thanks to their realization as a quotient of $\mathfrak{X}$, higher dimensional p-adic integration on $\Gamma\backslash\mathfrak{X}$ is amenable to computations. On the other hand, some of these varieties turn out to be Shimura varieties of unitary type, where recent progress on p-adic L functions gives hope that a precise conjecture of the Mazur-Tate-Teitelbaum type can be phrased, and perhaps even proved, one day.

The purpose of this note is three-fold.

1. To use two key results on the cohomology of $X = \Gamma\backslash\mathfrak{X}$ (orthogonality of the Hodge and Weight filtrations, and the Monodromy-Weight conjecture) in order to attach cohomological (Fontaine-Mazur type) $\mathcal{L}$ invariants to X.
2. To use p-adic integration, as developed by the first author, to make a transcendental (Coleman-Teitelbaum type) construction of $\mathcal{L}$ invariants, conjectured to agree with the cohomological $\mathcal{L}$-invariants.
3. To explore one particular 2-dimensional example where we hope to be able to relate these $\mathcal{L}$-invariants to p-adic L functions.

In the first two goals we succeed completely. Our use of p-adic integration depends on some unpublished work, which extends the integration theory of [Be1] and [Be2] from the good reduction case to the case of semi-stable reduction. In Section 3.1 we sketch part of that work which is being used here. The last part of the paper is highly speculative, as the p-adic L functions are still missing, even in the case of good reduction. Nevertheless, we believe that it is good to have a well-defined test-case in mind, to guide one in future research, so we have included it despite its speculative nature.

2. The monodromy modules attached to a p-adically uniformized variety, and the cohomological $\mathcal{L}$-invariants

2.1. p-adically uniformized varieties and their de Rham cohomology.

2.1.1. *p-adically uniformized varieties.* Throughout the paper we fix a finite extension K of $\mathbb{Q}_p$. We let π denote a uniformizer of K, and q the cardinality of its residue field. Let $d \geq 1$. Drinfeld's p-adic symmetric domain of dimension d is the rigid analytic open subdomain of $\mathbb{P}^d_K$ which is the complement of the union of all the K-rational hyperplanes,

$$\mathfrak{X} = \mathbb{P}^d_K \backslash \bigcup_a H_a. \tag{2.1}$$

Here the hyperplane $H_a \subset \mathbb{P}^d_K$ is given by the equation $a(x) = 0$, where a is a K-rational point of the projective space dual to $\mathbb{P}^d_K$. We refer to [Sch-St] for a discussion of the structure of $\mathfrak{X}$ as a rigid analytic space, and for the construction of an increasing sequence of affinoids $\mathfrak{X}_n \subset\subset \mathfrak{X}_{n+1}$ whose union is $\mathfrak{X}$ (the symbol $\subset\subset$ means *compactly embedded*).

[1] See however the recent work of Chida, Mok and Park [C-M-P]

The group $G = PGL_{d+1}(K)$ permutes the H_a, hence acts on $\mathfrak{X}$ as a group of rigid analytic motions. If $\Gamma \subset G$ is a discrete cocompact and torsion-free subgroup then $\Gamma\backslash\mathfrak{X}$ is a smooth and proper rigid analytic variety over K. A theorem of Mustafin [Mu, Theorems 3.1 and 4.1] then says that there exists a unique smooth projective algebraic variety X_Γ defined over K, whose associated rigid analytic space X_Γ^{an} is isomorphic to $\Gamma\backslash\mathfrak{X}$. For our purpose, a variety X over K will be called p-adically uniformized if it is isomorphic to such an X_Γ.

2.1.2. *The Bruhat-Tits building and the reduction map.* Let $\mathcal{T}$ be the Bruhat-Tits building of G. This is a locally finite simplicial complex whose vertices are labelled by homothety classes $[L]$ of lattices $L \subset K^{d+1}$. The k-cells of $\mathcal{T}$ are the $k+1$-sets $\sigma = \{[L_0], \ldots, [L_k]\}$, for lattices L_i satisfying

$$L_0 \supset L_1 \supset \cdots \supset L_k \supset \pi L_0, \tag{2.2}$$

all the inclusions being strict. Note that the choice of the vertex $v_0 = [L_0]$ determines an ordering of the vertices, hence an orientation of σ. We call the pair (σ, v_0) a pointed k-cell, and denote the set of pointed k-cells by $\widehat{\mathcal{T}}_k$.

We let $|\mathcal{T}|$ denote the geometric realization of $\mathcal{T}$.

There is a well-known reduction map $r : \mathfrak{X}(\mathbb{C}_p) \to |\mathcal{T}|$. We refer to [dS1, Section 6] for a description of this map and of the pre-image $\mathfrak{X}_\sigma$ of an open simplex $|\sigma|$. If σ is a vertex then $\mathfrak{X}_\sigma$ is an affinoid. On the other extreme, if σ is a d-cell then $\mathfrak{X}_\sigma$ is isomorphic to the multi-annulus of all points $x = (x_0 : \cdots : x_d)^t$ for which

$$|x_0| > |x_1| > \cdots > |x_d| > |\pi x_0|. \tag{2.3}$$

2.1.3. *The de Rham cohomology of* $\mathfrak{X}$. Since $\mathfrak{X}$ is a Stein space [Sch-St, Section 1, Proposition 4], its kth rigid de Rham cohomology is simply the space of global closed k-forms modulo the exact ones:

$$H^k_{dR}(\mathfrak{X}) = h^k(\Omega^{\cdot}(\mathfrak{X})). \tag{2.4}$$

Here $\Omega^{\cdot}$ is the complex of sheaves of rigid analytic differential forms in the rigid analytic topology, and for any complex $A^{\cdot}$ of abelian groups, $h^k(A^{\cdot})$ denotes its kth homology.

The cohomology $H^k_{dR}(\mathfrak{X})$ has been computed, as a representation of G, in [Sch-St, Section 4, Corollary 17]. An alternative approach was developed in [dS1], where the notion of the residue $res_\sigma(\omega)$ of a closed k-form ω along a pointed k-cell $\sigma \in \widehat{\mathcal{T}}_k$ has been introduced. We have also defined there a space $C^k_{har}(\mathcal{T})$ of harmonic k-cochains on $\mathcal{T}$, and proved that the map which associates to a closed k-form ω its residues along $\widehat{\mathcal{T}}_k$,

$$\omega \mapsto c_\omega, \quad c_\omega(\sigma) = res_\sigma(\omega) \tag{2.5}$$

induces an isomorphism

$$H^k_{dR}(\mathfrak{X}) \simeq C^k_{har}(\mathcal{T}). \tag{2.6}$$

In other words, a closed k-form is exact if and only if all its residues along $\widehat{\mathcal{T}}_k$ vanish, the cochains c_ω are harmonic, and every harmonic cochain is of this form.

The spaces $H^k_{dR}(\mathfrak{X})$ and $C^k_{har}(\mathcal{T})$ are G-Fréchet modules and the isomorphism between them respects, of course, the G action and the topology.

2.1.4. *The bounded cohomology.* The results of this subsection will be used in Section 3.2.1 in the study of the transcendental $\mathcal{L}$-invariant. They also play an important role in the proof of Theorem 2.3. We will be interested in the subspace $H^k_{dR}(\mathfrak{X})^b$ of cohomology classes whose associated harmonic cochain is bounded (as a function from $\mathcal{T}$ to K). We write $C^k_{har}(\mathcal{T})^b$ for the space of bounded harmonic cochains. The spaces $H^k_{dR}(\mathfrak{X})^b \simeq C^k_{har}(\mathcal{T})^b$ are p-adic Banach spaces in the sup norm, but they are dense in $H^k_{dR}(\mathfrak{X}) \simeq C^k_{har}(\mathcal{T})$ in the Fréchet topology.

The interest in the bounded cohomology stems from the fact that every bounded class is represented by a unique differential k-form of a particular kind, which is obtained from the logarithmic k-forms upon integration against a p-adic measure. These results are due to Iovita and Spiess [I-S, Theorem 4.5], see also [A-dS1]. We briefly recall them.

Let $\mathcal{A} = \mathbb{P}^d(K)^*$ be the set of K-rational hyperplanes in $\mathbb{P}^d_K$, identified with the K-rational points of the dual projective space, and endowed with the p-adic topology. The space of $k+1$-tuples $\mathcal{A}^{k+1}$ is compact. The subset $\mathcal{B}_k$ of *linearly independent* $k+1$ tuples is open. For any K-Banach space X we denote by

$$C_c(\mathcal{B}_k, X) \tag{2.7}$$

the space of X-valued continuous functions on $\mathcal{B}_k$ with compact support, and by $C_0(\mathcal{B}_k, X)$ its completion in the sup norm. We denote by $M(\mathcal{B}_k)$ the space of *bounded K-valued measures* on $\mathcal{B}_k$, the Banach dual of $C_0(\mathcal{B}_k, K)$. If $\mu \in M(\mathcal{B}_k)$ and X is a K-Banach space then

$$\int_{\mathcal{B}_k} f(S) d\mu(S) \tag{2.8}$$

(Riemann integral) makes sense for every $f \in C_0(\mathcal{B}_k, X)$, and is an element of X.

Let $(a_0, a_1, \dots, a_k)$ be a $k+1$-tuple of K-rational linear equations on $\mathbb{A}^{d+1}_K$. Since a_i/a_0 is a nowhere vanishing regular function on $\mathfrak{X}$ we can take its logarithmic derivative $d\log(a_i/a_0)$ and put

$$\omega_{a_0,\dots,a_k} = d\log(a_1/a_0) \wedge \cdots \wedge d\log(a_k/a_0) \in \Omega^k(\mathfrak{X}). \tag{2.9}$$

This is a closed k-form, which vanishes identically if the $k+1$ tuple is linearly dependent, and depends only on the images of the a_i in $\mathcal{A}$.

Fix an affinoid $\mathfrak{X}_n \subset \mathfrak{X}$ as in [Sch-St, Section 1]. The space $\Omega^k(\mathfrak{X}_n)$ is a K-Banach space. (One may use coordinates to identify it with a direct sum of copies of $\mathcal{O}(\mathfrak{X}_n)$, which is complete in the sup norm. Although this identification is not canonical, the ensuing Banach topology is.) The map

$$S = (a_0, a_1, \dots, a_k) \mapsto \omega_S|_{\mathfrak{X}_n} \tag{2.10}$$

is in $C_0(\mathcal{B}_k, \Omega^k(\mathfrak{X}_n))$ (because it extends continuously to $\mathcal{A}^{k+1}$ but vanishes on the linearly dependent tuples) hence for every $\mu \in M(\mathcal{B}_k)$ we can form

$$\omega_\mu|_{\mathfrak{X}_n} = \int_{\mathcal{B}_k} (\omega_S|_{\mathfrak{X}_n}) d\mu(S). \tag{2.11}$$

We then define ω_μ on all of $\mathfrak{X}$ by going to the limit over n. Let $\Omega^k(\mathfrak{X})^b$ be the space of all the ω_μ so obtained.

The map $\mu \mapsto \omega_\mu$ is not an isomorphism, as there are "degenerate" measures μ for which $\omega_\mu = 0$. Let $\mathcal{B}^{(k)}_{k+1}$ denote the subspace of $\mathcal{A}^{k+2}$ consisting of all the

$k+2$-tuples, any $k+1$ of which are linearly independent. Let

$$\partial_i : \mathcal{B}_{k+1}^{(k)} \to \mathcal{B}_k \tag{2.12}$$

be the map "forget the ith coordinate". Iovita and Spiess (*loc. cit.*) proved the following.

Lemma 2.1. *Let $\mu \in M(\mathcal{B}_k)$. Then $\omega_\mu = 0$ if and only if there exists a $\nu \in M(\mathcal{B}_{k+1}^{(k)})$ such that*

$$\mu = \sum_{i=0}^{k+1} (-1)^i \partial_{i*}\nu. \tag{2.13}$$

The outcome of the lemma is that we may identify the space $\Omega^k(\mathfrak{X})^b$ with the quotient of $M(\mathcal{B}_k)$ by the subspace of degenerate measures. Recall that any $\omega \in \Omega^k(\mathfrak{X})^b$ is closed. Theorem 4.5 of [I-S] is then equivalent to the following.

Theorem 2.2. *Under the map (2.5), the space $\Omega^k(\mathfrak{X})^b$ maps isomorphically and G-equivariantly onto $C^k_{har}(\mathcal{T})^b \simeq H^k_{dR}(\mathfrak{X})^b$. In other words, the following diagram is commutative ($\Omega^k(\mathfrak{X})^c$ denotes the closed k-forms):*

$$\begin{array}{ccccc} \Omega^k(\mathfrak{X})^b & \overset{\simeq}{\to} & C^k_{har}(\mathcal{T})^b & \simeq & H^k_{dR}(\mathfrak{X})^b \\ \cap & & \cap & & \cap \\ \Omega^k(\mathfrak{X})^c & \overset{\omega\mapsto c_\omega}{\to} & C^k_{har}(\mathcal{T}) & \simeq & H^k_{dR}(\mathfrak{X}) \end{array}. \tag{2.14}$$

2.1.5. *The de Rham cohomology of* $\Gamma\backslash\mathfrak{X}$. Let Γ be a discrete, cocompact and torsion-free subgroup of G. By general principles (GAGA [Kö]), passage from the algebraic category to the rigid analytic category induces an isomorphism

$$H^k_{dR}(X_\Gamma/K) \simeq H^k_{dR}(\Gamma\backslash\mathfrak{X}), \tag{2.15}$$

where by the right hand side we mean de Rham cohomology of rigid analytic forms. The covering $\mathfrak{X} \to \Gamma\backslash\mathfrak{X}$ being étale in the rigid analytic topology, it induces a covering spectral sequence [Sch-St, Section 5, Proposition 2]

$$E_2^{r,s} = H^r(\Gamma, H^s_{dR}(\mathfrak{X})) \Rightarrow H^{r+s}_{dR}(\Gamma\backslash\mathfrak{X}). \tag{2.16}$$

This covering spectral sequence has been analyzed in [Sch-St]. It degenerates at E_2 and induces the covering filtration $F^{\cdot}_\Gamma$, which is a descending filtration satisfying

$$H^k_{dR}(\Gamma\backslash\mathfrak{X}) = F^0_\Gamma H^k_{dR}(\Gamma\backslash\mathfrak{X}) \supset \cdots \supset F^k_\Gamma H^k_{dR}(\Gamma\backslash\mathfrak{X}) \supset F^{k+1}_\Gamma H^k_{dR}(\Gamma\backslash\mathfrak{X}) = 0, \tag{2.17}$$

whose graded pieces $gr^r_\Gamma H^k_{dR}(\Gamma\backslash\mathfrak{X}) = F^r_\Gamma H^k_{dR}(\Gamma\backslash\mathfrak{X})/F^{r+1}_\Gamma H^k_{dR}(\Gamma\backslash\mathfrak{X})$ are given by

$$gr^r_\Gamma H^k_{dR}(\Gamma\backslash\mathfrak{X}) \simeq H^r(\Gamma, H^{k-r}_{dR}(\mathfrak{X})). \tag{2.18}$$

Schneider and Stuhler determined also the dimension of the group cohomology

$$H^r(\Gamma, H^s_{dR}(\mathfrak{X})) \simeq H^r(\Gamma, C^s_{har}(\mathcal{T})). \tag{2.19}$$

Let $\mu(\Gamma) = \dim H^d(\Gamma, K)$. Then

$$\dim H^r(\Gamma, C^s_{har}(\mathcal{T})) = \mu(\Gamma)\delta_{r,d-s} + \delta_{r,s} \tag{2.20}$$

where $\delta_{i,j}$ is Kronecker's delta. It follows that $H^k_{dR}(\Gamma\backslash\mathfrak{X})$ is 1-dimensional (resp. 0) for even (resp. odd) $k \neq d$. The cohomology in middle degree d is $(d+1)\mu(\Gamma)$-dimensional if d is odd and $(d+1)\mu(\Gamma)+1$-dimensional if d is even.

Besides the covering spectral sequence there is also the Hodge-to-de Rham spectral sequence

$$E_1^{r,s} = H^s(X_\Gamma, \Omega^r) \Rightarrow H_{dR}^{r+s}(X_\Gamma/K) \tag{2.21}$$

(Ω^r, as usual, is the sheaf of regular r-forms), which degenerates at E_1 and induces the (descending) Hodge filtration

(2.22)
$$H_{dR}^k(X_\Gamma/K) = F_{dR}^0 H_{dR}^k(X_\Gamma/K) \supset \cdots \supset F_{dR}^k H_{dR}^k(X_\Gamma/K) \supset F_{dR}^{k+1} H_{dR}^k(X_\Gamma/K) = 0$$

whose graded pieces $gr_{dR}^r H_{dR}^k(X_\Gamma/K) = F_{dR}^r H_{dR}^k(X_\Gamma/K)/F_{dR}^{r+1} H_{dR}^k(X_\Gamma/K)$ are given by

$$gr_{dR}^r H_{dR}^k(X_\Gamma/K) \simeq H^{k-r}(X_\Gamma, \Omega^r). \tag{2.23}$$

2.1.6. *Orthogonality of the Hodge filtration and the covering filtration.* The following theorem has been conjectured by Schneider and proven by Iovita and Spiess [I-S, Theorem 5.4]. Other proofs were given later in [GK2] and [A-dS1].

Theorem 2.3. *Let $0 \le r \le d+1$. Then*

$$H_{dR}^d(X_\Gamma/K) = F_\Gamma^r H_{dR}^d(X_\Gamma/K) \oplus F_{dR}^{d+1-r} H_{dR}^d(X_\Gamma/K). \tag{2.24}$$

Corollary 2.4. *For $0 \le r \le d$ let*

$$H^{r,d-r} = F_\Gamma^r H_{dR}^d(X_\Gamma/K) \cap F_{dR}^{d-r} H_{dR}^d(X_\Gamma/K). \tag{2.25}$$

Then

(2.26)
$$H^r(\Gamma, C_{har}^{d-r}(\mathcal{T})) \simeq gr_\Gamma^r H_{dR}^d(X_\Gamma/K) \simeq H^{r,d-r} \simeq gr_{dR}^{d-r} H_{dR}^d(X_\Gamma/K) \simeq H^r(X_\Gamma, \Omega^{d-r})$$

and

$$H_{dR}^d(X_\Gamma/K) = \oplus_{r=0}^d H^{r,d-r}. \tag{2.27}$$

In all known proofs of this theorem the bounded cohomology and the logarithmic forms play a major role. In [A-dS1, Lemma 3.4] the following result, which is of independent interest, is proved along the way.

Lemma 2.5. *The inclusion $C_{har}^s(\mathcal{T})^b \subset C_{har}^s(\mathcal{T})$ induces an isomorphism*

$$H^r(\Gamma, C_{har}^s(\mathcal{T})^b) \simeq H^r(\Gamma, C_{har}^s(\mathcal{T})). \tag{2.28}$$

Note that when $r = 0$ this is obvious (a Γ-invariant cochain is bounded), so the lemma can be regarded as a generalization of this fact to cohomology in higher degrees.

2.2. **Log-crystalline cohomology and filtered (ϕ, N)-modules.**

2.2.1. *The semistable model and the log-crystalline cohomology of its special fiber.* In the previous section X_Γ was considered, whether algebraically or analytically, only over K. It is well-known, however, that $\mathfrak{X}$ has an underlying structure of a formal scheme over $Spf(\mathcal{O}_K)$, whose special fiber is strictly semi-stable (but not of finite type) over $Spec(\mathbb{F}_q)$. In fact, this structure as a formal scheme is encoded in the reduction map from the rigid analytic space to the geometric realization of the Bruhat-Tits building. Passing to $\Gamma\backslash\mathfrak{X}$, one obtains an integral structure $\mathfrak{X}_\Gamma$ of X_Γ over $\mathcal{O}_K$, whose special fiber Y_Γ is strictly semi-stable (and now of finite type), and whose degeneration complex is $\Gamma\backslash\mathcal{T}$.

In this situation Hyodo and Kato [H-K] attached to the special fiber Y_Γ, endowed with the induced log-structure (we denote this structure by $Y_\Gamma^\times$), and to every degree $0 \le k \le 2d$, a log-crystalline cohomology,

$$D^k = H^k_{crys}(Y_\Gamma^\times / K_0). \tag{2.29}$$

This is a finite dimensional vector space over the maximal unramified subfield K_0 of K (the field of fractions of $W(\mathbb{F}_q)$), which is endowed with a bijective semi-linear *Frobenius* operator $\Phi : D^k \to D^k$ and a linear nilpotent endomorphism $N : D^k \to D^k$ (the *monodromy*) satisfying

$$N\Phi = p\Phi N. \tag{2.30}$$

Let $q = p^f$. Then $\phi = \Phi^f$ acts linearly on D^k and satisfies $N\phi = q\phi N$. As a result of the work of Deligne, Katz-Messing (in the good reduction case) and Mokrane [Mo] (in the semi-stable case) it is known that ϕ acts on D^k with eigenvalues which are q-Weil numbers of weights j for $\max(0, 2k-2d) \le j \le \min(2k, 2d)$. (A q-Weil number of weight j is an algebraic integer all of whose complex embeddings are of absolute value $q^{j/2}$). We let D^k_j be the subspace of D^k which, after an extension of scalars to a finite algebraic extension of K_0, becomes the sum of the generalized eigenspaces of eigenvalues of weight j. It is clear that D^k_j is well-defined over K_0, although the generalized eigenspaces themselves may only be defined over a finite extension of K_0. It is also clear that we have a *weight decomposition*

$$D^k = \oplus_j D^k_j.$$

We define the *weight filtration* as

$$F^r_W D^k = \sum_{j \le r} D^k_j. \tag{2.31}$$

Since N maps D^k_j to D^k_{j-2}, the weight filtration is preserved by both Φ and N. The graded pieces of the weight filtration are $gr^r_W D^k = F^r_W D^k / F^{r-1}_W D^k$, and they are of course isomorphic to D^k_r. It follows that N induces a homomorphism which we denote by ν

$$\nu = gr_W N : gr^r_W D^k \to gr^{r-2}_W D^k. \tag{2.32}$$

2.2.2. *The Hyodo-Kato comparison isomorphism.* Hyodo and Kato [H-K] have also established, in our situation, a comparison isomorphism between the log-crystalline cohomology of the special fiber and the de-Rham cohomology of the generic fiber. This comparison isomorphism depends on the choice of a uniformizer π, and is an isomorphism

$$\rho_\pi : D^k \otimes_{K_0} K \simeq H^k_{dR}(X_\Gamma / K). \tag{2.33}$$

The dependence on π is given by the formula

$$\rho_{\pi'} = \rho_\pi \circ \exp(\log(\pi'/\pi) N). \tag{2.34}$$

This implies that although $\rho_\pi(D^k_j \otimes_{K_0} K)$ is not independent of π, the weight filtration

$$F^r_W H^k_{dR}(X_\Gamma / K) = \rho_\pi(F^r_W D^k \otimes_{K_0} K) \tag{2.35}$$

is. In fact we have the following.

Proposition 2.6. *If $k = 2r \neq d$ then $D^k = D^k_k$ is pure of weight k. If $k = d$ then only* even *weights $0 \leq j \leq 2k$ appear in the weight decomposition. The weight filtration and covering filtration on $H^d_{dR}(X_\Gamma/K)$ coincide, up to a change in the indexing (see [GK2, Theorem 7.3]):*

$$F^r_\Gamma H^d_{dR}(X_\Gamma/K) = F^{2d-2r}_W H^d_{dR}(X_\Gamma/K). \tag{2.36}$$

We transport the monodromy operator N to $H^k_{dR}(X_\Gamma/K)$ using the isomorphism ρ_π. Because of (2.34) this is independent of π, maps $F^{2r}_W H^k_{dR}(X_\Gamma/K)$ to $F^{2r-2}_W H^k_{dR}(X_\Gamma/K)$, and induces $\nu = gr_W N$ on the graded pieces as for D^k.

2.2.3. *The monodromy-weight conjecture.* The (p-adic) monodromy-weight conjecture for X_Γ (see [Mo]) is known. It has been proved in [dS2] (building on [A-dS2]) and independently (in l-adic cohomology) by Ito [I]. Roughly stated, it says that the monodromy operator N is as non-trivial as it could be, given the constraints imposed on it by the weights. More precisely, we have the following ([dS2, Theorem 4.1] and [A-dS2, Theorem 4.3]).

Theorem 2.7. *The graded monodromy operator $\nu = gr_W N : gr^{2r}_W H^d_{dR}(X_\Gamma/K) \to gr^{2r-2}_W H^d_{dR}(X_\Gamma/K)$ is an isomorphism, except if $d = 2m$ is even and $r = m+1$ (in which case it is injective with one-dimensional cokernel) or $r = m$ (in which case it is surjective with one dimensional kernel).*

2.2.4. *Primitive cohomology.* When d is even, there is an exception to ν being an isomorphism, when ν maps into or out of the middle graded piece $gr^d_W H^d_{dR}(X_\Gamma/K)$. To treat this exceptional case uniformly it is convenient to introduce the *primitive* cohomology as follows.

The projective embedding of X_Γ determines a Chern class $c_1 \in H^2_{dR}(X_\Gamma/K)$. If $d = 2m$, then $c_1^m \in H^d_{dR}(X_\Gamma/K)$ is non-zero. Since d is even, cup-product induces a symmetric non-degenerate pairing on $H^d_{dR}(X_\Gamma/K)$, and since $c_1^m \cdot c_1^m = c_1^d \neq 0$, the orthogonal complement of c_1^m is also a direct sum complement.

Definition 2.1. *If $d = 2m$ we let $H^d_{prim}(X_\Gamma/K)$ be the subspace of $H^d_{dR}(X_\Gamma/K)$ which is orthogonal, under cup-product, to c^1_m, so that*

$$H^d_{dR}(X_\Gamma/K) = Kc_1^m \oplus H^d_{prim}(X_\Gamma/K). \tag{2.37}$$

If d is odd, we let $H^d_{prim}(X_\Gamma/K) = H^d_{dR}(X_\Gamma/K)$.

In an analogous way we define D^d_{prim} in log-crystalline cohomology. The Hyodo-Kato comparison isomorphism respects Chern classes and the product structure on cohomology, and therefore maps primitive parts to primitive parts. Theorem 2.7 can now be strengthened.

Theorem 2.8. *The map $\nu = gr_W N : gr^{2r}_W H^d_{prim}(X_\Gamma/K) \to gr^{2r-2}_W H^d_{prim}(X_\Gamma/K)$ is an isomorphism for all r (*including *$r = m$ or $m+1$ if $d = 2m$).*

2.3. The monodromy modules associated to $H^d_{dR}(X_\Gamma/K)$ and the cohomological $\mathcal{L}$-invariant.

2.3.1. *The d monodromy modules.* Let $H = H^d_{prim}(X_\Gamma/K)$ and $D = D^d_{prim}$, and let $F^\cdot_\Gamma$ and $F^\cdot_{dR}$ be the induced "covering" and "Hodge" filtrations on H. We let $gr^\cdot_\Gamma$ and $gr^\cdot_{dR}$ be the graded pieces in these filtrations (restricted to H).

For $0 \le i \le d-1$ we consider the subquotient of length 2 in the covering filtration, which also happens to be a subquotient of length 2 in the Hodge filtration because of Theorem 2.3:

$$W_i = F_\Gamma^i / F_\Gamma^{i+2} = H/(F_{dR}^{d+1-i} \oplus F_\Gamma^{i+2}) = F_{dR}^{d-i-1}/F_{dR}^{d-i+1}. \tag{2.38}$$

Since i is held fixed, we write $W = W_i$ to facilitate the notation. Via the comparison isomorphism $\rho_\pi : D \otimes_{K_0} K \simeq H$ we transport the weight *decomposition* from D to H. It depends on π. The transported weight *filtration* coincides (up to indexing) with the covering filtration on H, and is independent of π. Therefore, the subquotient W inherits a corresponding weight decomposition as

$$W = W^0 \oplus W_\pi^2 \tag{2.39}$$

where W^0 is of weight $2d-2i-2$ and W_π^2 of weight $2d-2i$. Only W_π^2 depends on π. We also have the "cross"

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & gr_\Gamma^{i+1} = & & & & \\ & & & & H_{prim}^{i+1}(\Gamma, C_{har}^{d-i-1}) & & & & \\ & & & & \downarrow & \overset{\cong}{\searrow} & & & \\ 0 & \to & gr_{dR}^{d-i} & \to & W & \to & gr_{dR}^{d-i-1} & \to & 0 \\ & & & \underset{\cong}{\searrow} & \downarrow & & & & \\ & & & & gr_\Gamma^i = & & & & \\ & & & & H_{prim}^i(\Gamma, C_{har}^{d-i}) & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array} \tag{2.40}$$

where the diagonal arrows are isomorphisms by transversality. Here $H_{prim}^r(\Gamma, C_{har}^{d-r}) = H^r(\Gamma, C_{har}^{d-r})$ unless $2r = d$, in which case it is the intersection of the latter with H.

The monodromy operator induces a map $\nu : gr_\Gamma^i \to gr_\Gamma^{i+1}$, and is an isomorphism, even in the exceptional cases $i = d/2$ or $i = d/2 - 1$, since we are considering primitive cohomology. This is a direct consequence of Theorem 2.8 and the coincidence of the weight filtration with the covering filtration.

2.3.2. *The Fontaine-Mazur $\mathcal{L}$ invariants.* Consider the weight decomposition $W = W^0 \oplus W_\pi^2$ which has been transported to W from the analogous weight decomposition of D via ρ_π. The coincidence of the weight filtration with the covering filtration (up to indexing) implies that $W^0 = gr_\Gamma^{i+1}$ and W_π^2 projects isomorphically onto gr_Γ^i. Define

$$\lambda_\pi^{FM} : gr_\Gamma^i \to gr_\Gamma^{i+1} \tag{2.41}$$

by first lifting to gr_{dR}^{d-i} (inverting the lower diagonal arrow in (2.40)), then projecting along the weight decomposition onto W^0. Define

$$\mathcal{L}_{\pi,i+1}^{FM} = \nu^{-1} \circ \lambda_\pi^{FM} \in End(H_{prim}^i(\Gamma, C_{har}^{d-i})). \tag{2.42}$$

These $\mathcal{L}$-invariants are defined for $1 \le i+1 \le d$. As a result of the formula giving the dependence of ρ_π on the uniformizer, we have

$$\mathcal{L}_{\pi',k}^{FM} - \mathcal{L}_{\pi,k}^{FM} = -\log(\pi'/\pi) I. \tag{2.43}$$

The $\mathcal{L}$-invariants are therefore invertible for generic π.

2.3.3. *An extension of $H^j_{dR}(\mathfrak{X})$ by $H^{j-1}_{dR}(\mathfrak{X})$ and its relation to $\nu = gr_W N$.* The description of $H^j_{dR}(\mathfrak{X})$ as a space of harmonic cochains $C^j_{har}(\mathcal{T})$ allows one to make canonical extensions

$$0 \to C^{j-1}_{har}(\mathcal{T}) \to \tilde{C}^{j-1}_{har}(\mathcal{T}) \to C^j_{har}(\mathcal{T}) \to 0 \tag{2.44}$$

of G-modules ($1 \le j \le d$) and prove that $\nu = gr_W N : H^i(\Gamma, C^{d-i}_{har}(\mathcal{T})) \to H^{i+1}(\Gamma, C^{d-i-1}_{har}(\mathcal{T}))$ is in fact the connecting homomorphism arising from this short exact sequence. This is described in [A-dS2, Section 3] and [dS2, Theorem 4.1] and is the basis for our proof of the monodromy-weight conjecture. We refer to these papers for the definition of $\tilde{C}^{j-1}_{har}(\mathcal{T})$ and the proof of its relation to ν.

2.3.4. *Looking for another extension.* The discussion in this subsection is meant for motivation only. None of it will be needed later on, and the reader may skip it.

As we have just observed, the graded monodromy operator ν on the (graded pieces of the) cohomology of X_Γ comes from a *universal extension* (2.44). It is equal to the ith connecting homomorphism in the long exact sequence in Γ-cohomology derived from this extension.

The following question arises: Are there similar extensions, depending on π, of $C^j_{har}(\mathcal{T})$ by $C^{j-1}_{har}(\mathcal{T})$ ($1 \le j \le d$), which will yield as connecting homomorphisms the maps λ^{FM}_π? If so, both ν and λ^{FM}_π will be governed by "universal extensions" independent of the particular Γ, and the $\mathcal{L}$-invariant will measure the discrepancy between the two. Denoting these hypothetical extensions by $[?]_\pi$, their extension classes in Ext^1 should satisfy

$$[?]_{\pi'} - [?]_\pi = -\log(\pi'/\pi)[\tilde{C}^{j-1}_{har}]. \tag{2.45}$$

This relation will imply and explain (2.43). Of course, we should make it clear in which abelian category we wish to look for these extensions. Notice that $C^j_{har}(\mathcal{T})$ is the (full, as opposed to smooth) algebraic dual of the smooth representation

$$St_j = \frac{C^\infty(G/P_j, K)}{\sum_{P_j \subset Q} C^\infty(G/Q, K)}, \tag{2.46}$$

where P_j is the standard parabolic subgroup of G whose Levi consists of one block of size $d+1-j$, followed by blocks of size 1 along the diagonal (P_d is the Borel subgroup) [Sch-St, Section 4, Corollary 17]. In the denominator, Q runs over all the standard parabolics properly containing P_j. If $j = d$, this is the Steinberg representation. Now the representations St_j are smooth and admissible, and are examples of G-modules of compact type (direct limits of G-Banach modules under compact embeddings). In fact, they are direct limits of finite dimensional K-Banach spaces. Their full algebraic duals, endowed with the strong topology, are K-Fréchet spaces, and are reflexive: we can recover St_j as their topological duals [Sch, 16.5 and 16.10]. At first sight, this suggests to look for extensions of St_{j-1} by St_j in the category of smooth representations, and then dualize, to get $[?]_\pi$.

This, however, does not work. Sasha Orlik has shown that $Ext^1(St_{j-1}, St_j)$ (extensions in the category of smooth representations) is 1-dimensional ([Or, Theorem 1]). This means that $[?]_\pi$ would simply be a multiple of $[\tilde{C}^{j-1}_{har}]$ and the $\mathcal{L}$-invariants would come out to be scalar, which is clearly not the case. We must modify our C^j_{har} somehow, and look for extensions in a *different* category, to find $[?]_\pi$.

We take our hint from the work of Breuil [Br], who showed that $\mathcal{L}$-invariants are related to Banach space completions of locally algebraic representations. We replace $C^j_{har}(\mathcal{T})$ by the smaller Banach space $C^j_{har}(\mathcal{T})^b$ of bounded cohomology, and recall that this space is represented by bounded differential forms on $\mathfrak{X}$. We now look for an extension

$$0 \to C^{j-1}_{har}(\mathcal{T}) \xrightarrow{\alpha} [?]_\pi \xrightarrow{\beta} C^j_{har}(\mathcal{T})^b \to 0 \tag{2.47}$$

of topological G-modules, that will produce, upon passage to Γ-cohomology, connecting homomorphisms

$$\lambda_\pi : H^i(\Gamma, C^j_{har}(\mathcal{T})^b) \to H^{i+1}(\Gamma, C^{j-1}_{har}(\mathcal{T})). \tag{2.48}$$

We stress that $[?]_\pi$ will *not* be the pull-back of an extension of $C^j_{har}(\mathcal{T})$ by $C^{j-1}_{har}(\mathcal{T})$. Concrete extensions, allowing computations with cocycles, will be constructed below. Two encouraging signs are the following:

(a) By Lemma 2.5, although the extension $[?]_\pi$ may not extend to an extension of $C^j_{har}(\mathcal{T})$ by $C^{j-1}_{har}(\mathcal{T})$, for the purpose of constructing λ_π, it is just as good, since $H^i(\Gamma, C^j_{har}(\mathcal{T})^b) = H^i(\Gamma, C^j_{har}(\mathcal{T}))$, so the domain and range of λ_π are the same as those of λ^{FM}_π.

(b) When $d = 1$, the only interesting index is $j = 1$. In this case $C^1_{har}(\mathcal{T})^b \simeq \Omega^1(\mathfrak{X})^b$, the map $c \mapsto \omega_c$, inverse to $\omega \mapsto c_\omega$, being given by Teitelbaum's Poisson kernel [Tei]. The space $[?]_\pi$ is given in this case by the space of all Coleman primitives of forms from $\Omega^1(\mathfrak{X})^b$ (based on $\log_\pi$ as a branch of the logarithm). It is a known fact that this space, an extension of $\Omega^1(\mathfrak{X})^b$ by K (the constants), satisfies the right dependence on π. See [dS3], where the relation between "Coleman integration" and "Schneider integration" is studied in greater generality. We emphasize that had we stayed with $C^1_{har}(\mathcal{T}) \simeq H^1_{dR}(\mathfrak{X})$ we would have not been able to follow this path, since Coleman integrals are associated to differential forms, not to cohomology classes, and only the bounded cohomology classes have canonical forms representing them. The passage from $C^1_{har}(\mathcal{T})$ to $C^1_{har}(\mathcal{T})^b$ in the one-dimensional case is crucial.

As we shall see in the next section, p-adic integration theory, as developed by the first author, and in particular the space $H^j_{fp,\pi}(\mathfrak{X})$, supplies an extension of the type $[?]_\pi$, but of the module $\Omega^j(\mathfrak{X})^c$ of closed forms, rather than of the cohomology $H^j_{dR}(\mathfrak{X})$. The bounded cohomology $H^j_{dR}(\mathfrak{X})^b \simeq C^j_{har}(\mathcal{T})^b$ is the "common ground" where closed forms and cohomology meet.

3. Higher dimensional p-adic integration and the transcendental $\mathcal{L}$-invariant

3.1. p-adic integration for semi-stable weak formal $\mathcal{O}_K$-schemes.

3.1.1. *Semi-stable weak formal $\mathcal{O}_K$-schemes.* The p-adic integration theory of higher dimensional forms developed in [Be1] and [Be2] relies heavily on the assumption that X is a smooth (and sometimes proper) scheme over $Spec(\mathcal{O}_K)$. For various applications, and definitely for the one we have in mind, it is desirable to extend p-adic integration to $\mathcal{O}_K$-schemes (and even formal $\mathcal{O}_K$-schemes) with semi-stable reduction. This is the subject of [B-Z] and [B-L-Z]. Here we extract the minimum necessary to treat Drinfeld's p-adic symmetric domain. Some of the results still represent work in progress. We work with weak formal schemes (rather than formal schemes) and the associated dagger spaces (rather than rigid spaces) because the de Rham complex is not well-behaved in the formal/rigid setup.

Fix a uniformizer π of K. In this section we let $\mathfrak{X}$ be a weak formal $\mathcal{O}_K$-scheme in the sense of [Mer]. We assume that $\mathfrak{X}$ is locally of finite type, namely it is locally of the form $Spwf(B)$, where B is a quotient of $\mathcal{O}_K[X_1,\ldots,X_n]^\dagger$. A prime example, and in fact our motivating example, is the weak formal scheme underlying Drinfeld's rigid analytic space which was introduced in Section 2.1.1 under the same name. See [GK2, Section 6].

To every $\mathfrak{X}$ as above one can functorially associate its "generic fiber" $\mathfrak{X}_K$, which is a dagger space over K, in the sense of [GK1]. See [GK2, Section 1.1], where this space is denoted $\mathfrak{X}_\mathbb{Q}$.

There is a faithful functor from weak formal schemes to formal schemes, $\mathfrak{X}\mapsto\widehat{\mathfrak{X}}$. The "generic fiber in the sense of Raynaud" $\widehat{\mathfrak{X}}_K$ associated to $\widehat{\mathfrak{X}}$ [Ber (0.2)] is the rigid analytic space associated to the dagger space $\mathfrak{X}_K$ by Grosse-Kloenne. To sum up, the following diagram "commutes":

$$\begin{array}{ccc} \mathfrak{X} & \rightsquigarrow & \mathfrak{X}_K \\ \downarrow & & \downarrow \\ \widehat{\mathfrak{X}} & \rightsquigarrow & \widehat{\mathfrak{X}}_K \end{array}. \tag{3.1}$$

In our example, under $\widehat{\mathfrak{X}}_K$, we recover Drinfeld's space as a rigid analytic space.

We assume from now on that $\mathfrak{X}_K$ *is smooth* and $\mathfrak{X}$ is *strictly semistable.* This means that locally Zariski, $\mathfrak{X}$ is étale over

$$Spwf(\mathcal{O}_K[X_1,\ldots,X_n]^\dagger/(X_1\cdots X_r-\pi)) \tag{3.2}$$

for some $r\leq n$. (*Semistable* means the same thing where "locally Zariski" is replaced by "locally étale".) The Drinfeld space example is of this sort.

3.1.2. *The de Rham complex* $\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K)$. The de Rham complex is constructed as in [Be1, Section 5], except that since we do not need to worry about logarithmic singularities "at infinity", the construction is in fact far simpler. Thus $\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K)$ is a complex in the derived category $D^+(Mod_K)$ of K-vector spaces, whose kth homology is (by definition) $H^k_{dR}(\mathfrak{X}_K)$. Putting the stupid filtration $\tau^{\geq k}\Omega^\cdot_{\mathfrak{X}_K/K}$ on the complex of sheaves of overconvergent differential forms we define

$$Fil^k\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K)=\mathbb{R}\Gamma(\mathfrak{X}_K,\tau^{\geq k}\Omega^\cdot_{\mathfrak{X}_K/K}). \tag{3.3}$$

One has then $\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K):=Fil^0\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K)$, and there are maps

$$Fil^k\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K)\to\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K). \tag{3.4}$$

We caution the reader that the induced map on homology between these two complexes is not injective in general. For example, the kth homology of the complex $Fil^k\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K)$ is easily seen to be $\Omega^k(\mathfrak{X}_K)^c$, the global closed k-forms. Indeed, if

$$\mathfrak{X}_K=\bigcup_{U\in\mathcal{U}}U \tag{3.5}$$

is an admissible open cover of $\mathfrak{X}_K$ by dagger affinoid subdomains, then the complex (3.3) is quasi isomorphic to the *total complex* of the de Rham-Čech double complex

$$\check{C}^j(\mathcal{U},\Omega^i) \tag{3.6}$$

(in degrees $i\geq k$, $j\geq 0$ and 0 elsewhere), and its k-homology is just $\Omega^k(\mathfrak{X}_K)^c$. If $\mathfrak{X}_K$ is (the dagger space associated to) a smooth projective variety over K, then $\Omega^k(\mathfrak{X}_K)^c$ is the bottom piece of the Hodge filtration in $H^k_{dR}(\mathfrak{X}_K)$, but in general the map from $\Omega^k(\mathfrak{X}_K)^c$ to $H^k_{dR}(\mathfrak{X}_K)$ need not be injective.

If $\mathfrak{X}_K$ is Stein, so Cartan's theorem B holds and coherent cohomology in positive degrees vanishes, $\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K)$ is quasi isomorphic to the complex $\Omega^{\cdot}(\mathfrak{X}_K)$, hence in the derived category the two complexes are isomorphic. The Drinfeld symmetric domain is Stein [Sch-St, Section 1, Proposition 4].

3.1.3. *The log-rigid complex and the comparison isomorphism.* Let $\kappa = \mathcal{O}_K/\pi\mathcal{O}_K$, and let K_0 be the field of fractions of $W(\kappa)$. The special fiber Y of $\mathfrak{X}$ is a scheme, locally of finite type over κ. Being a divisor with normal crossings, Y defines on $\mathfrak{X}$ a log-structure making $\mathfrak{X}$ log-smooth over the base log-scheme

$$\mathfrak{S}^{\pi} = (Spf(\mathcal{O}_K), \mathbb{N}, 1 \mapsto \pi). \tag{3.7}$$

This last log-scheme (which, despite the notation, is independent of π), as well as

$$\mathfrak{S}^{0} = (Spf(\mathcal{O}_{K_0}), \mathbb{N}, 1 \mapsto 0), \tag{3.8}$$

are thickenings of the log-point

$$S^{0} = (Spec(\kappa), \mathbb{N}, 1 \mapsto 0). \tag{3.9}$$

Writing $Y^{\times}$ for the fiber product

$$S^0 \times_{\mathfrak{S}^{\pi}} \mathfrak{X} \tag{3.10}$$

in the category of log-schemes, $Y^{\times}$ becomes a fine log-scheme, log-smooth over S^0. This log-structure depends on $\mathfrak{X} \bmod \pi^2$, but not on the special fiber alone. By abuse of notation we continue to denote $Y^{\times}$ simply by Y.

For $a \in \{0, \pi\}$, Grosse-Kloenne constructs in [GK2, Lemma 1.4] complexes in the derived category of vector spaces (over K_0 if $a = 0$, over K if $a = \pi$)

$$\mathbb{R}\Gamma_{rig}(Y/\mathfrak{S}^{a}). \tag{3.11}$$

While $\mathbb{R}\Gamma_{rig}(Y) := \mathbb{R}\Gamma_{rig}(Y/\mathfrak{S}^0)$ is a complex of vector spaces over K_0 which carries a semi-linear action of the absolute Frobenius Φ, it is $\mathbb{R}\Gamma_{rig}(Y/\mathfrak{S}^{\pi})$ which can be *canonically identified* with $\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K)$. If $\#(\kappa) = p^f$ we let $\phi = \Phi^f$. It then acts linearly on $\mathbb{R}\Gamma_{rig}(Y)$.

The comparison isomorphism is an isomorphism *in the derived category*, depending on π,

$$\rho_{\pi} :\ \mathbb{R}\Gamma_{rig}(Y) \otimes_{K_0} K \simeq \mathbb{R}\Gamma_{rig}(Y/\mathfrak{S}^{\pi}) = \mathbb{R}\Gamma_{dR}(\mathfrak{X}_K). \tag{3.12}$$

See [GK2, Theorem 3.4, Corollary 3.7].

The log-rigid cohomology groups $H^m_{rig}(Y/K_0)$ are the homology groups of the complex $\mathbb{R}\Gamma_{rig}(Y)$.

3.1.4. *Finite polynomial cohomology.* Fix an index i and let $P \in K_0[X]$ be any polynomial. (We shall later on assume that $P(\phi)$ annihilates $H^i_{rig}(Y/K_0)$.) We denote by $P(\phi)$ also the composition of morphisms in $D^+(Mod_K)$

(3.13)

$$Fil^i\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K) \to \mathbb{R}\Gamma_{dR}(\mathfrak{X}_K) \overset{\rho_{\pi}^{-1}}{\simeq} \mathbb{R}\Gamma_{rig}(Y) \otimes_{K_0} K \overset{P(\phi)\otimes 1}{\to} \mathbb{R}\Gamma_{rig}(Y) \otimes_{K_0} K$$

and define

$$H^i_{fp,\pi,P}(\mathfrak{X}) = h^i\left(MF\{Fil^i\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K) \overset{P(\phi)}{\to} \mathbb{R}\Gamma_{rig}(Y) \otimes_{K_0} K\}\right). \tag{3.14}$$

Here, for any map of complexes $A^\cdot \xrightarrow{f} B^\cdot$, the mapping fiber (cone) $MF\{A^\cdot \xrightarrow{f} B^\cdot\}$ is the complex

$$C^i = A^i \oplus B^{i-1},\ \ d^i_C(a,b) = (d^i_A a, f(a) - d^{i-1}_B(b)), \tag{3.15}$$

and $h^i(C^\cdot)$ is the ith homology of the complex $C^\cdot$. There is an obvious exact sequence

$$0 \to B^\cdot[1] \to C^\cdot \to A^\cdot \to 0. \tag{3.16}$$

Strictly speaking, since ρ_π is only defined in the derived category, the map $P(\phi)$ is not a homomorphism of complexes, but breaks up as a chain of homomorphisms and inverses of quasi-isomorphisms. The linear algebra needed to overcome this difficulty and show that the cone construction can nevertheless be carried out, is explained (in the good reduction case) in [Be1, Section 4]. We remark that, compared to [Be1] and [Be2], where the author defines more general groups $H^i_{fp}(X,n)$, our groups should have been denoted by $H^i_{fp,\pi,P}(\mathfrak{X},i)$.

Assume that $P(\phi)$ annihilates $H^i_{rig}(Y/K_0)$. For every $\omega \in \Omega^i(\mathfrak{X}_K)^c$, representing as we have seen an element of $Fil^i\mathbb{R}\Gamma_{dR}(\mathfrak{X}_K)$, $P(\phi)(\omega)$ is, in such a case, exact. In concrete terms, an element of $H^i_{fp,\pi,P}(\mathfrak{X})$ is then nothing but a pair

$$(\omega,\gamma) \in \Omega^i(\mathfrak{X}_K) \times \mathbb{R}\Gamma^{i-1}_{rig}(Y)/d\mathbb{R}\Gamma^{i-2}_{rig}(Y) \tag{3.17}$$

such that $d\omega = 0$ and $d\gamma = P(\phi)(\omega)$. Note that the dependence on π comes only via ρ_π which is "hidden" in $P(\phi)$.

Theorem 3.1. *Assume that $P(\phi)$ annihilates $H^i_{rig}(Y/K_0)$ and write $H^i_{fp,\pi}(\mathfrak{X})$ for $H^i_{fp,\pi,P}(\mathfrak{X})$. Then this cohomology is functorial in $\mathfrak{X}$ and satisfies the following properties.*

(i) There is a short exact sequence

$$0 \to H^{i-1}_{dR}(\mathfrak{X}_K) \xrightarrow{\alpha} H^i_{fp,\pi}(\mathfrak{X}) \xrightarrow{\beta} \Omega^i(\mathfrak{X}_K)^c \to 0. \tag{3.18}$$

(ii) Assume furthermore that $P(\phi)$ is bijective on $H^{i-1}_{rig}(Y/K_0)$ (weak version of "purity"). The finite polynomial cohomology is then independent of P, up to a canonical isomorphism.

(iii) There is a cup product, which is compatible with the short exact sequence in the following two ways:

(a) $\beta(x \cup y) = \beta(x) \wedge \beta(y)$.

(b) $\alpha(x) \cup y = \alpha(x \cup [\beta(y)])$ where $[\beta(y)]$ is the cohomology class of the closed form $\beta(y)$.

Proof. The short exact sequence is immediate from the definition, together with the comparison isomorphism for H^{i-1}_{rig}. For (ii) note that if we replace P by PQ where Q is another polynomial for which $Q(\phi)$ is bijective on H^{i-1}_{rig}, the map

$$(\omega,\gamma) \mapsto (\omega, Q(\phi)\gamma) \tag{3.19}$$

induces an isomorphism between $H^i_{fp,\pi,P}(\mathfrak{X})$ and $H^i_{fp,\pi,PQ}(\mathfrak{X})$. For (iii) follow the steps in [Be1]. ∎

3.1.5. *Restriction to the smooth part.* Let Y^{sm} be the smooth locus of Y (over $Spec(\kappa)$), and $\mathfrak{X}^{sm}$ the restriction of $\mathfrak{X}$ to Y^{sm}. Let $sp : \mathfrak{X}_K \to Y$ be the specialization map ([Ber, 0.2.2] and [GK2, 1.1]) and $\mathfrak{X}'_K = sp^{-1}(Y^{sm})$. We assume that Y^{sm} is the disjoint union of connected smooth affine schemes Y_v, indexed by $v \in \mathcal{T}_0$. We let $\mathfrak{X}_v$ be the restriction of $\mathfrak{X}$ to Y_v, a weak formal scheme. Its associated dagger space $X_v = sp^{-1}(Y_v) \subset \mathfrak{X}'_K$ is an affinoid dagger space with good reduction and the comparison isomorphism

$$\mathbb{R}\Gamma_{rig}(Y_v) \otimes_{K_0} K \simeq \mathbb{R}\Gamma_{dR}(X_v/K) \tag{3.20}$$

does not depend on π. Since Y_v is affine and smooth, the rigid cohomology of Y_v is just the Monsky-Washnitzer cohomology and $\mathbb{R}\Gamma_{rig}(Y_v) \otimes_{K_0} K$ is quasi isomorphic to $\Omega^{\cdot}(X_v)$. Let

$$\widehat{H}^i_{fp}(\mathfrak{X}) = h^i \left(MF\{Fil^i \mathbb{R}\Gamma_{dR}(\mathfrak{X}_K) \stackrel{P(\phi)}{\to} \mathbb{R}\Gamma_{rig}(Y^{sm}) \otimes_{K_0} K\} \right). \tag{3.21}$$

By $P(\phi)$ we mean the map denoted by this name previously, followed by the functorial restriction to Y^{sm}. Alternatively, we can first restrict to $\mathfrak{X}'_K$, then use the map $P(\phi)$ with $\mathfrak{X}'_K$ and Y^{sm} replacing $\mathfrak{X}_K$ and Y. This cohomology does not depend on π, as is clear from the second description just given, and has the advantage that it can be written down explicitly (see below).

It follows that there is a functorial map ι sitting in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H^{i-1}_{dR}(\mathfrak{X}_K) & \stackrel{\alpha}{\to} & H^i_{fp,\pi}(\mathfrak{X}) & \stackrel{\beta}{\to} & \Omega^i(\mathfrak{X}_K)^c & \to & 0 \\ & & \downarrow & & \downarrow \iota & & || & & \\ 0 & \to & H^{i-1}_{dR}(\mathfrak{X}'_K) & \stackrel{\alpha}{\to} & \widehat{H}^i_{fp}(\mathfrak{X}) & \stackrel{\beta}{\to} & \Omega^i(\mathfrak{X}_K)^c & \to & 0 \end{array}. \tag{3.22}$$

Lemma 3.2. *Suppose that $\mathfrak{X}$ is the Drinfeld symmetric domain. Then ι is injective.*

Proof. It is enough to prove that $H^{i-1}_{dR}(\mathfrak{X}_K) \to H^{i-1}_{dR}(\mathfrak{X}'_K)$ is injective. Let $\omega \in \Omega^{i-1}(\mathfrak{X}_K)$ be a closed form. If all its restrictions to the dagger affinoids X_v are exact, all its residues along pointed $i-1$ cells of the Bruhat-Tits building vanish, and by the main theorem of [dS1], ω is exact. ∎

We shall be mainly interested in the image of ι.

It is well-known [Col, Corollary 1.1a] that, as the Y_v are smooth and affine, the Frobenius automorphism ϕ of each Y_v has a lifting to a rigid analytic (overconvergent) automorphism ϕ_v of X_v. These ϕ_v are not unique, but are all "homotopic". Fix a choice of ϕ_v for each $v \in \mathcal{T}_0$. Then an element of $\widehat{H}^i_{fp}(\mathfrak{X})$ may be concretely realized by a collection $(\omega; \gamma_v)_{v \in \mathcal{T}_0}$, where $\omega \in \Omega^i(\mathfrak{X}_K)^c$ as before, and $\gamma_v \in \Omega^{i-1}(X_v)/d\Omega^{i-2}(X_v)$ satisfy

$$d\gamma_v = P(\phi_v^*)(\omega|_{X_v}). \tag{3.23}$$

For $i = 1$ the γ_v are just overconvergent functions f_v satisfying $df_v = P(\phi_v^*)(\omega|_{X_v})$ and the data $(\omega; f_v)$ is what is needed to determine the Coleman integral of ω on X_v. See [Col, Theorem 2.1].

3.1.6. *p-adic integration on $\mathbb{G}_m$.* Before we proceed we have to settle the special case where $\mathfrak{X}$ is the standard weak formal model of $\mathfrak{X}_K = \mathbb{G}_{m,K}$. Thus $\mathfrak{X}^{sm}$ is the "Néron model of $\mathbb{G}_m$". Here Y is isomorphic to an infinite chain of $\mathbb{P}^1$'s $Y(v)$ indexed by $v \in \mathbb{Z}$, where the 0 of $Y(v)$ is glued transversally to the ∞ of $Y(v+1)$.

Note that unless we fix the parameter π, only $Y(0)$ can be canonically identified with $\mathbb{P}^1$, the isomorphisms of the other irreducible components with $\mathbb{P}^1$ being dependent on π. The smooth locus Y^{sm} is the disjoint union of $Y_v = Y(v) - \{0, \infty\}$. We let $\mathfrak{X}_v$ be the restriction of $\mathfrak{X}$ to Y_v. Note that $K^\times$ acts on $\mathfrak{X}$. On the special fiber, $a \in K^\times$ map $Y(v)$ to $Y(v + ord_\pi(a))$.

Consider the form $\omega = dz/z$ on $\mathfrak{X}_K$. Its Vologodsky integral [Vol],[B-Z] is the branch $\log_\pi$ of the logarithm satisfying $\log_\pi(\pi) = 0$, and is thus invariant under multiplication by π. It is natural to expect the same of the class in $H^1_{fp,\pi}(\mathfrak{X})$ lifting ω.

Proposition 3.3. *Every lift $\tilde{\omega}$ of $\omega = dz/z$ to $H^1_{fp,\pi}(\mathfrak{X})$ is invariant under multiplication by π.*

Proof. Rather than unravel the definitions, we found it easier, even in this simple case, to use a trick, and deduce the claim from the description of the Frobenius-monodromy modules of semi-stable curves given by Coleman and Iovita in [C-I]. See an exposition of their results also in [dS3]. Since any two $\tilde{\omega}$ differ by a constant, it is enough to prove the invariance for one lift, and it suffices to prove invariance by π^k for some $k \geq 1$.

For a suitable $q_E = \pi^k$ the Tate elliptic curve E_K with period q_E has strictly semi-stable reduction. Let E be a semi-stable proper model of E_K over $Spec(\mathcal{O}_K)$, and E_κ its special fiber, equipped with the induced log structure. If we let $\widehat{E}$ denote the weak formal completion of E along the special fiber, then $\widehat{E}$ is the quotient of $\mathfrak{X} = \widehat{\mathbb{G}}_m$ by the action of $\langle q_E \rangle$, and likewise E_K is the quotient of $\mathfrak{X}_K$. The differential form $\omega \in \Omega^1(\mathfrak{X}_K)$ descends to

$$\omega_E \in \Omega^1(E_K) \subset H^1_{dR}(E_K/K). \tag{3.24}$$

In other words, ω is the pull-back of ω_E under the quotient map.

Let $P(X) = 1 - q^{-1}X$ (where $q = \#(\kappa)$, not to be confused with q_E). It is a consequence of [C-I] that for our choice of π, the image of ω_E in $H^1_{rig}(E_\kappa/K_0) \otimes_{K_0} K$ under ρ_π^{-1}, is killed by $P(\phi^*)$ (see [B-Z]). We emphasize that this is the step where the relation between π and q_E takes its effect. Without it, all that we can say is that the image of ω_E in $H^1_{rig}(E_\kappa/K_0) \otimes_{K_0} K$ is killed by $(1 - q^{-1}\phi^*)(1 - \phi^*)$. One needs a quadratic polynomial to kill the rigid cohomology of E_κ, while the linear polynomial P suffices to kill the rigid cohomology of our $\mathfrak{X}$.

We use the polynomial P in the definition of finite polynomial cohomology. For the computation of $H^1_{fp,\pi}(\mathfrak{X})$ this is legitimate, since $P(\phi^*)$ kills $H^1_{rig}(Y/K_0)$ and is bijective on the constants. For E we *may not* identify $H^1_{fp,\pi}(E)$ with $H^1_{fp,\pi,P}(E)$. Nevertheless, by the discussion in the previous paragraph, ω_E does lift to an $\tilde{\omega}_E \in H^1_{fp,\pi,P}(E)$. As a lift of its pull-back ω to $H^1_{fp,\pi,P}(\mathfrak{X})$ we can therefore take the pull-back of $\tilde{\omega}_E$, which is evidently invariant under $q_E = \pi^k$. ∎

For each value of π there is a unique lift

$$\widetilde{\log}_\pi \in H^1_{fp,\pi}(\mathfrak{X}) \tag{3.25}$$

which, in addition to being invariant under π, is in the (-1)-eigenspace of $z \mapsto z^{-1}$.

Corollary 3.4. *We have*

$$\iota(\widetilde{\log}_\pi) - \iota(\widetilde{\log}_{\pi'}) = \alpha(\psi) \tag{3.26}$$

where $\psi \in H^0_{dR}(\mathfrak{X}'_K) = \prod_{n \in \mathbb{Z}} K$ is $n \mapsto -n \log(\pi/\pi')$.

Proof. By the proposition and functoriality, $\iota(\widetilde{\log}_\pi)$ is invariant under multiplication by π, and similarly $\iota(\widetilde{\log}_{\pi'})$ is invariant under multiplication by π'. They both map under β to dz/z so their difference must be $\alpha(\psi)$ for some ψ. This ψ must be in the (-1) eigenspace for $z \mapsto z^{-1}$, so is normailzed to vanish at $n = 0$.

The restriction of $\widetilde{\log}_\pi$ to $\mathfrak{X}_0$ is the Coleman integral of dz/z, which is just $\log(z) + C$, and as it is in the (-1)-eigenspace for $z \mapsto z^{-1}$, $C = 0$. It follows that if u is a unit of K,

$$u^*\widetilde{\log}_\pi|_{\mathfrak{X}_0} = \widetilde{\log}_\pi|_{\mathfrak{X}_0} + \log(u). \tag{3.27}$$

By the invariance under multiplication by π of $\widetilde{\log}_\pi$ (and the fact that u and π commute...), this relation continues to hold on all of $\mathfrak{X}^{sm} = \coprod_{n\in\mathbb{Z}} \mathfrak{X}_n$:

$$u^*\iota(\widetilde{\log}_\pi) = \iota(\widetilde{\log}_\pi) + \log(u). \tag{3.28}$$

A similar equation holds for $\widetilde{\log}_{\pi'}$. Letting $u = \pi/\pi'$ we find out

$$\begin{aligned} \pi^*\iota(\widetilde{\log}_\pi - \widetilde{\log}_{\pi'}) &= \iota(\widetilde{\log}_\pi) - u^*\iota(\widetilde{\log}_{\pi'}) \\ &= \iota(\widetilde{\log}_\pi - \widetilde{\log}_{\pi'}) - \log(u). \end{aligned} \tag{3.29}$$

But the ψ of the corollary is the only function which vanishes at $n = 0$ and satisfies $\pi^*\psi = \psi - \log(u)$. ∎

3.2. p-adic integration for Drinfeld's p-adic symmetric domain.

3.2.1. *An extension of* $\Omega^s(\mathfrak{X}_K)^b$ *by* $H^{s-1}_{dR}(\mathfrak{X}_K)$ *and the transcendental* $\mathcal{L}$*-invariant.* Let $\mathfrak{X}$ stand now for Drinfeld's p-adic symmetric domain over K (as a weak formal scheme). For $1 \le s \le d$ let $\Omega^s(\mathfrak{X}_K)^b$ denote the Banach space of bounded s-forms, a subspace of the space $\Omega^s(\mathfrak{X}_K)^c$ of closed forms. Let

$$0 \to H^{s-1}_{dR}(\mathfrak{X}_K) \xrightarrow{\alpha} \mathcal{E}^s_\pi \xrightarrow{\beta} \Omega^s(\mathfrak{X}_K)^b \to 0 \tag{3.30}$$

be the extension of topological G-modules which is obtained from the extension $H^s_{fp,\pi}(\mathfrak{X})$ by pull-back. Recall that $\Omega^s(\mathfrak{X}_K)^b$ maps isomorphically to $H^s_{dR}(\mathfrak{X}_K)^b \simeq C^s_{har}(\mathcal{T})^b$ and that $H^r(\Gamma, C^s_{har}(\mathcal{T})^b) = H^r(\Gamma, C^s_{har}(\mathcal{T}))$. Let

$$\lambda^{Col}_\pi : H^r(\Gamma, C^s_{har}(\mathcal{T})) \to H^{r+1}(\Gamma, C^{s-1}_{har}(\mathcal{T})) \tag{3.31}$$

be the connecting homomorphism obtained from $(\mathcal{E}^s_\pi)$. Let

$$\mathcal{L}^{Col}_{\pi,r+1} = \nu^{-1} \circ \lambda^{Col}_\pi, \tag{3.32}$$

where ν is as in (2.32). Compare Section 2.3.2.

Conjecture 3.5. *We have* $\lambda^{FM}_\pi = \lambda^{Col}_\pi$ *and similarly for the* $\mathcal{L}$*-invariants.*

3.2.2. *The dependence on* π*.* As a weak evidence towards the conjecture, we prove the following theorem.

Theorem 3.6. *Let* π *an* π' *be two uniformizers. Then for every* $1 \le k \le d$

$$\mathcal{L}^{Col}_{\pi',k} - \mathcal{L}^{Col}_{\pi,k} = -\log(\pi'/\pi)I. \tag{3.33}$$

The theorem will be proved in several steps. We shall compare the diagrams (3.22) for π and π', always operating in $\widehat{H}^s_{fp}(\mathfrak{X})$.

Let $\widetilde{\mathcal{A}} = K^{d+1} - \{0\}$ (row vectors, regarded as linear forms on column vectors) and $\mathcal{A} = \mathbb{P}^d(K)^* = \widetilde{\mathcal{A}}/K^\times$. Given a lattice $L \subset K^{d+1}$ and $a \in \widetilde{\mathcal{A}}$ we let

$$ord_L(a) = n \quad \text{if} \quad a \in \pi^n L - \pi^{n+1} L. \tag{3.34}$$

Given an $(s+1)$ tuple $S = (a_0, \ldots, a_s)$ where $a_l \in \widetilde{\mathcal{A}}$, we let $\bar{S}$ be its image in $\mathcal{A}^{s+1}$. Together with the logarithmic forms ω_S (2.9) which depend only on $\bar{S}$, we have the element

$$\tilde{\omega}_{S,\pi} = \cup_{l=1}^s \widetilde{\log}_\pi(a_l/a_0) \in H^s_{fp,\pi}(\mathfrak{X}) \tag{3.35}$$

mapping under β to ω_S. Here $\widetilde{\log}_\pi(a_l/a_0)$ is a short-hand for the pull-back under $a_l/a_0 : \mathfrak{X} \to \mathbb{G}_m$ of $\widetilde{\log}_\pi \in H^1_{fp,\pi}(\mathbb{G}_m)$. By $\mathbb{G}_m$ we understand the standard weak formal model of $\mathbb{G}_{m,K}$ discussed in length above.

Let Y be the special fiber of $\mathfrak{X}$, with its induced log structure. Let $\mathcal{T}$ be the Bruhat-Tits building of G, and recall that the connected components Y_v of the smooth part Y^{sm} of Y are labeled by its vertices $v \in \mathcal{T}_0$. Moreover, each v "is" a homothety class of lattices $L \subset K^{d+1}$. For $a, a' \in \widetilde{\mathcal{A}}$, the difference $ord_L(a) - ord_L(a')$ depends only on the images of a and a' in $\mathcal{A}$ and the homothety class of L. If $v = [L]$ we sometimes write this difference, by abuse of notation, as $ord_v(a) - ord_v(a')$.

Lemma 3.7. *We have*

$$\widetilde{\log}_\pi(a_l/a_0)|_{\mathfrak{X}_v} - \widetilde{\log}_{\pi'}(a_l/a_0)|_{\mathfrak{X}_v} = -\log(\pi/\pi')\alpha(ord_v(a_l) - ord_v(a_0)). \tag{3.36}$$

Proof. This follows from Corollary 3.4, since the restriction of a_l/a_0 to $\mathfrak{X}_v$ factors through $\mathbb{G}_{m,n}$ with $n = ord_v(a_l) - ord_v(a_0)$. ∎

Recall from [dS2, (3.25)] that for S as above and $v \in \mathcal{T}_0$, the form

$$\eta_{S,v} = -\sum_{l=0}^{s} (-1)^l ord_L(a_l)\omega_{S_l} \in \Omega^{s-1}(X_v) \tag{3.37}$$

(where S_l means S with a_l removed) is independent of the lattice L representing $v = [L]$.

Lemma 3.8. *We have*

$$\eta_{S,v} = \sum_{l=1}^{s} (-1)^{l-1}(ord_v(a_l) - ord_v(a_0)) \cdot \bigwedge_{j\neq 0,l} d\log(a_j/a_0). \tag{3.38}$$

Proof. We only have to note that

$$\sum_{l=0}^{s} (-1)^l \omega_{S_l} = 0. \tag{3.39}$$

∎

We define $\eta_S \in \Omega^{s-1}(\mathfrak{X}'_K)$ to be the form whose restriction to X_v is $\eta_{S,v}$. The following Proposition is the key to proving the Theorem.

Proposition 3.9. *We have*

$$\iota(\tilde{\omega}_{S,\pi}) - \iota(\tilde{\omega}_{S,\pi'}) = -\log(\pi/\pi') \cdot \alpha(\eta_S). \tag{3.40}$$

Proof. Fix v and compute on $\mathfrak{X}_v$. By Lemma 3.7

$$\begin{aligned}
\tilde{\omega}_{S,\pi}|_{\mathfrak{X}_v} &= \bigcup_{l=1}^{s} \widetilde{\log}_\pi(a_l/a_0)|_{\mathfrak{X}_v} \\
&= \bigcup_{l=1}^{s} (\widetilde{\log}_{\pi'}(a_l/a_0)|_{\mathfrak{X}_v} - \log(\pi/\pi')\cdot\alpha(ord_v(a_l) - ord_v(a_0)) \\
&= \tilde{\omega}_{S,\pi'}|_{\mathfrak{X}_v} - \log(\pi/\pi')\cdot \\
&\quad \{\sum_{l=1}^{s}(-1)^{l-1}\alpha(ord_v(a_l) - ord_v(a_0)) \cup \bigcup_{l\neq j=1}^{s} (\widetilde{\log}_{\pi'}(a_j/a_0)|_{\mathfrak{X}_v}\} \\
&= \tilde{\omega}_{S,\pi'}|_{\mathfrak{X}_v} - \log(\pi/\pi')\cdot \\
&\quad \alpha\left(\sum_{l=1}^{s}(-1)^{l-1}(ord_v(a_l) - ord_v(a_0)) \bigwedge_{l\neq j=1}^{s} d\log(a_j/a_0)\right). \qquad (3.41)
\end{aligned}$$

In the last two steps we have taken into account the fundamental properties of cup products in finite polynomial cohomology. See Theorem 3.1 *(iii)*. Note that part *(b)* there implies $\alpha(x)\cup\alpha(y) = 0$. The proposition follows from the expression obtained above for $\eta_{S,v}$ in the previous lemma. ∎

Let μ be a measure with compact support on $\widetilde{\mathcal{A}}^{s+1}$. We can then integrate all the expressions depending on S by the method explained in Section 2.1.4. For example,

$$\omega_\mu = \int \omega_S \cdot d\mu(S). \qquad (3.42)$$

Corollary 3.10. *With the obvious notation, we have*

$$\iota(\tilde{\omega}_{\mu,\pi}) - \iota(\tilde{\omega}_{\mu,\pi'}) = -\log(\pi/\pi')\cdot\alpha(\eta_\mu). \qquad (3.43)$$

We now conclude the proof of Theorem 3.6.

Proof. Let $\omega(\gamma_0,\dots,\gamma_r)$ be a Γ-cocycle representing a class $[\omega]$ in $H^r(\Gamma,\Omega^s(\mathfrak{X}_K)^b)$. We have to show that

$$\lambda_\pi^{Col}([\omega]) - \lambda_{\pi'}^{Col}([\omega]) = -\log(\pi/\pi')\cdot\nu([\omega]) \qquad (3.44)$$

where ν is the connecting homomorphism associated with (2.44). Let

$$c_\omega(\gamma_0,\dots,\gamma_r) \in C^s_{har}(\mathcal{T})^b \qquad (3.45)$$

be the Γ-cocycle with values in $C^s_{har}(\mathcal{T})^b$ attached to ω, see (2.5). Thus, for an oriented edge σ,

$$c_\omega(\gamma_0,\dots,\gamma_r)(\sigma) = res_\sigma\omega(\gamma_0,\dots,\gamma_r). \qquad (3.46)$$

For each $r+1$-tuple of elements of Γ there exists a measure with compact support $\mu(\gamma_0,\dots,\gamma_r)$ on $\widetilde{\mathcal{A}}^{s+1}$ such that

$$\begin{aligned}
\omega(\gamma_0,\dots,\gamma_r) &= \omega_{\mu(\gamma_0,\dots,\gamma_r)} \\
c_\omega(\gamma_0,\dots,\gamma_r) &= c_{\mu(\gamma_0,\dots,\gamma_r)}.
\end{aligned} \qquad (3.47)$$

Here $c_{\mu(\gamma_0,\dots,\gamma_r)}$ is the harmonic cochain defined in [dS2, (3.9)].

We have the corresponding Γ-cochain $\tilde{\omega}_{\mu,\pi}$ obtained from integrating $\tilde{\omega}_{S,\pi}$ against μ. The Γ-cocycle representing $\lambda_\pi^{Col}([\omega])$ is $\delta\tilde{\omega}_{\mu,\pi}$, where δ is the coboundary operator in group cohomology. It follows from Corollary 3.10 that a Γ-cocycle representing $\iota\lambda_{\pi}^{Col}([\omega]) - \iota\lambda_{\pi'}^{Col}([\omega])$ is $-\log(\pi/\pi')\cdot\delta\eta_\mu$.

We want to compare it with $\nu([c_\mu])$, the class in $H^{r+1}(\Gamma, C_{har}^{s-1}(\mathcal{T}))$ obtained from the extension (2.44) by the connecting homomorphism ν. Let $\tilde{c}_\mu$ be the Γ-cochain (with values in $\tilde{C}_{har}^{s-1}(\mathcal{T})$) lifting c_μ as in [dS2, (3.39)]. Then $\nu([c_\mu])$ is represented by the Γ-cocycle $\delta\tilde{c}_\mu$, whose values lie in $C_{har}^{s-1}(\mathcal{T})$. Note that while $c_{\mu(\gamma_0,\ldots,\gamma_r)}$ depends only on the push-down of $\mu(\gamma_0,\ldots,\gamma_r)$ to a measure on $\mathcal{A}^{s+1}$, $\tilde{c}_{\mu(\gamma_0,\ldots,\gamma_r)}$ depends on $\mu(\gamma_0,\ldots,\gamma_r)$ itself.

We look at its image $\iota(\delta\tilde{c}_\mu)$. Here ι is, as usual, the restriction to $\mathfrak{X}'_K$, i.e. the map on Γ-cohomology induced from

$$\iota : C_{har}^{s-1}(\mathcal{T}) \simeq H_{dR}^{s-1}(\mathfrak{X}_K) \to H_{dR}^{s-1}(\mathfrak{X}'_K) = \prod_{v\in\mathcal{T}_0} H_{dR}^{s-1}(X_v). \tag{3.48}$$

To conclude the proof, we must show that $\iota(\delta\tilde{c}_\mu)$ is equal (or at least cohomologous) to the Γ-cocycle $\delta\eta_\mu$. This can be checked for one v at a time. But an element of $H_{dR}^{s-1}(X_v)$ is determined uniquely by its residues along the oriented $s-1$ cells of $\mathcal{T}$ with leading vertex v. See [dS1], the discussion following (7.3). The collection $\widehat{\mathcal{T}}_{s-1}(v)$ (in the notation of [dS1]) contains all the $s-1$ cells *contiguous* to v, but the harmonicity condition easily implies that it is enough to look at cells *containing* v. Since a change in the ordering of the vertices results in a sign change, we may restrict attention to pointed cells whose leading vertex is v. Now, according to [dS2] (4.13), if $\tau \in \widehat{\mathcal{T}}_{s-1}$ and v is its leading vertex then

$$\delta\tilde{c}_\mu(\tau) = res_\tau(\delta\eta_{\mu,v}) \tag{3.49}$$

so (in $H_{dR}^{s-1}(X_v)$) $\iota(\delta\tilde{c}_\mu)|_{X_v} = \delta\eta_\mu|_{X_v}$. Since this holds for every $v\in\mathcal{T}_0$, $\iota(\delta\tilde{c}_\mu) = \delta\eta_\mu$ and the proof is complete. ∎

4. Further questions on certain 2-dimensional unitary Shimura varieties

4.1. **The Shimura varieties.** The purpose of this last section is to present a 2-dimensional case, where we believe that our $\mathcal{L}$ invariants might be eventually related to derivatives of p-adic L-functions. We shall describe the set-up in the first few paragraphs, recalling all the necessary background. At the end we shall speculate about a possible "exceptional zero conjecture". Unfortunately, the p-adic L function in question has not been yet constructed.

We depart from the notational conventions used up till now. In particular, G will denote a general unitary group, and K a compact, or compact-modulo center, subgroup. The local field denoted so far by the letter K will be simply $\mathbb{Q}_p$.

Let E be a quadratic imaginary field. Let D be a division algebra, central over E, of degree 3, i.e. $\dim_E D = 9$. Let α be an involution of the second kind on D (an involution inducing complex conjugation on E).

Let $G = GU(D,\alpha)$, a reductive algebraic group over $\mathbb{Q}$. For any $\mathbb{Q}$-algebra A

$$G(A) = \left\{(g,\nu)\in (D\otimes A)^\times \times A^\times \mid \alpha(g)\cdot g = \nu\right\}. \tag{4.1}$$

Then $G(\mathbb{R})$ is a general unitary group and we assume that

$$G(\mathbb{R}) \simeq GU(2,1). \tag{4.2}$$

One says that the signature of α at ∞ is $(2,1)$. If X_∞ is the unit ball in $\mathbb{C}^2$ then $G(\mathbb{R})$ acts transitively on X_∞, and the stabilizer K_∞ of 0 is a group which is maximal compact modulo the center, isomorphic to $G(U(2)\times U(1))$.

Let $K_f \subset G(\mathbb{A}_f)$ be a sufficiently small open compact subgroup. We assume that

$$K_f = \prod K_l \tag{4.3}$$

(the product running over all the rational primes l) where $K_l \subset G(\mathbb{Q}_l)$ is hyperspecial for all $l \notin S$ (S a finite set of primes containing the primes which are ramified in D). We write $K = K_\infty K_f$. Then the double coset space

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K = G(\mathbb{Q})\backslash (X_\infty \times G(\mathbb{A}_f)/K_f) \tag{4.4}$$

is a compact smooth (in general disconnected) complex surface. It is given by a finite union of quotients $\Gamma_j\backslash X_\infty$ where the Γ_j are arithmetic torsion-free cocompact-modulo-center subgroups of $G(\mathbb{R})$. In fact, by Shimura (see [De] and [Mi]), there exists a canonical smooth projective surface X_K (the *canonical model)* defined over E (the *reflex field*) whose complex points $X_K(\mathbb{C})$ are identified with the above complex surface.

Had we started with $M_3(E)$ instead of D, equipped with an involution of the second kind of signature $(2,1)$ at ∞, we would have ended with a similar Shimura variety, called the Picard modular surface. The only difference is that now the double coset space would be an open manifold admitting cusps, the Γ_j would not be cocompact-modulo-center, and X_K would not be projective, but rather quasi-projective.

Both in the case of D and in the case of $M_3(E)$, the resulting Shimura surface is of PEL type: it classifies abelian varieties with extra structure. In the case of $M_3(E)$, the Picard modular surface has been studied extensively. In particular, it is known to have a canonical smooth compactification $\bar{X}_K$, defined over E, where the "cuspidal divisor" $\bar{X}_K\backslash X_K$ becomes, after a finite abelian extension of the base field E, a union of elliptic curves with complex multiplication by E. In the case of D, X_K is a Shimura variety of the type considered in [H-T] p.7-8 (using the notation of Harris and Taylor take $F^+ = \mathbb{Q}$, $n = 3$, $B = D$, $\alpha(g) = \beta^{-1}g^*\beta$).

4.2. **Scalar weight modular forms.** Let X_K be the Shimura surface associated to (D,α,K) as above (defined over the field E). Let Γ be one of the Γ_j figuring in the complex uniformization of $X_K(\mathbb{C})$. From the classical perspective, (vector valued) modular forms for Γ are (vector valued) holomorphic functions on X_∞, transforming under Γ according to a matrix-valued factor of automorphy, depending on the weight. As $K_\infty \simeq G(U(2)\times U(1))$, weights are irreducible representations of $U(2)\times U(1)$ (at least as long as we forget the center). For simplicity we take only representations which are trivial on $U(2)$, and $e^{i\theta} \mapsto e^{ik\theta}$ on $U(1)$. The resulting modular forms are then scalar of weight k, and form a finite dimensional complex vector space $S_k(\Gamma)$. As Γ is cocompact modulo center, and there are no cusps, every modular form is a "cusp form". We write $S_k(K,\mathbb{C})$ for the direct sum of the $S_k(\Gamma_j)$. We say that modular forms in $S_k(K,\mathbb{C})$ are of *level* K, *weight* k and are *defined over* $\mathbb{C}$.

From an adelic perspective, modular forms in $S_k(K,\mathbb{C})$ are sections of automorphic vector bundles over X_K, and the scalar weight case corresponds to automorphic line bundles. This algebraic interpretation allows one to define what we mean by a modular form which is defined over E, or over any field L containing E, even in the absence of q-expansions. We denote these modular forms by $S_k(K,L)$. Since X_K is of PEL type, the automorphic vector bundles can be defined by linear algebra operations, starting with the tangent space of the universal abelian variety over X_K. The Kodaira-Spencer isomorphism yields then an isomorphism

$$S_3(K,L) \simeq H^0(X_{K/L}, \Omega^2_{X_K}). \tag{4.5}$$

Such an isomorphism can be deduced over $\mathbb{C}$ via an easy computation of factors of automorphy. We emphasize the fact that it can be obtained algebraically via the Kodaira-Spencer isomorphism, because both sides have an underlying E-structure, and the algebraic proof shows that the isomorphism respects it, so holds for any field L containing E. It is of course a perfect analogue of the isomorphism between cusp-forms of weight 2 and holomorphic differential forms on the classical modular curves.

4.3. p-adic uniformization.

4.3.1. *The uniformization theorem.* Let p be a prime that splits in E, $p = \mathfrak{p}\overline{\mathfrak{p}}$, but ramifies in D. Then, perhaps after changing the names of $\mathfrak{p}$ and $\overline{\mathfrak{p}}$, the Hasse invariants of D satisfy

$$inv_{\mathfrak{p}} D = 1/3, \ \ inv_{\overline{\mathfrak{p}}} D = 2/3. \tag{4.6}$$

This is because the involution α induces an isomorphism between the local division algebra $D_{\overline{\mathfrak{p}}}$ and $D_{\mathfrak{p}}^{opp}$, the opposite of $D_{\mathfrak{p}}$. Assume also that $K_p \subset G(\mathbb{Q}_p) = D_{\mathfrak{p}}^\times \times \mathbb{Q}_p^\times$ is maximal compact.

Theorem 4.1. *(Varshavsky [V, Theorem 2.2.3], Rapoport-Zink [R-Z, Theorem 6.50]). As a rigid analytic space over $\mathbb{Q}_p^{nr}$, X_K^{an} is isomorphic to a union of surfaces of the shape*

$$\Gamma_j^{int}\backslash\mathfrak{X} \tag{4.7}$$

where $\mathfrak{X}$ is Drinfeld's p-adic symmetric domain of dimension 2 over $\mathbb{Q}_p$ (viewed as a rigid analytic space over $\mathbb{Q}_p^{nr}$) and Γ_j^{int} are arithmetic discrete cocompact torsion-free subgroups of $GL_3(\mathbb{Q}_p)$ coming from

$$G^{int} = GU(D^{int}, \alpha^{int}). \tag{4.8}$$

Here D^{int}, the "interchanged" central simple algebra of rank 3 over E, is split at p, α^{int} is an involution of the second kind of signature $(3,0)$ at ∞, and otherwise (D^{int}, α^{int}) have the same local invariants as (D,α) (hence if D was ramified only *at p, $D^{int} = M_3(E)$). The adelic level (away from ∞ and p) determining the Γ_j^{int} is the level of K_f.*

For the relation between D and D^{int} see [Var, Proposition 2.1.7], but note that what we have denoted by D^{int} is denoted there D and vice versa.

4.3.2. *The $\mathcal{L}$-invariant.* Consider the two $\mathcal{L}$ invariants $\mathcal{L}_{p,i}^{FM}$ that we have associated in this case to each of surfaces $\Gamma_j^{int}\backslash\mathfrak{X}$. By Poincaré duality, one should determine the other, so let us focus on $\mathcal{L}_{p,1}^{FM}$. This is an endomorphism of $H^0(\Gamma_j^{int}\backslash\mathfrak{X}, \Omega^2)$ where Ω^2 is the sheaf of 2-forms. Taken in all the connected components together, we get an endomorphism $\mathcal{L}_{p,1}^{FM}$ of

$$H^0(X_K(\mathbb{C}_p), \Omega^2_{X_K}). \tag{4.9}$$

But as we have seen, this group is nothing but $S_3(K, \mathbb{C}_p)$, the space of $\mathbb{C}_p$-valued modular forms of weight 3 and level K.

Let $\mathbb{T}$ be the Hecke algebra generated by the Hecke operators $T_{\mathfrak{l}}$ on $S_3(K, \mathbb{C}_p)$ at primes $\mathfrak{l}$ of E not lying above S. A good reference for Hecke operators on Picard modular forms is [Fi, p.10]. The algebra $\mathbb{T}$ is commutative. Suppose $f \in S_3(K, \mathbb{C}_p)$ is a Hecke eigenform whose Hecke eigenvalues appear with multiplicity 1. It can be proved that $\mathcal{L}_{p,1}^{FM}$ commutes with the action of $\mathbb{T}$, hence ω_f, the 2-form on X_K corresponding to f, should be an eigenform of $\mathcal{L}_{p,1}^{FM}$ as well. Let $\mathcal{L}_p(f)$ be its eigenvalue.

4.4. **Speculations on an "exceptional zero conjecture".** Let $f \in S_3(K, \mathbb{C})$ be as above. Then f has a standard L function $L(f, s)$ attached to it, which is a degree 6 Euler product converging in some right half plane.

At least on the Picard modular surface, when $D = M_3(E)$ and $G = GU(D, \alpha)$ is quasi-split, these L-functions have been shown by Shimura, Shintani, and Gelbart-Piatetski Shapiro [G-PS] to have an integral representation, analytic continuation and functional equation. They have also been shown to coincide (up to finitely many bad Euler factors) with the Jacquet L-function of $BC(\pi_f)$ where π_f is the irreducible cuspidal automorphic representation of G attached to f and $BC(\pi_f)$ is its base-change to $GL_3(E)$.

Back in the case when D is a division ring, we may invoke Clozel's base change between $G = GU(D, \alpha)$ and $D^\times$ [H-T, Theorem VI.2.1], and the Jacquet-Langlands correspondence between $D^\times$ and $GL_3(E)$ [H-T, Theorem V.1.1] to get the same results for the L function of our f. With an appropriate normalization, the functional equation should relate $L(f, s)$ and $L(f, 3 - s)$. The points $s = 1, 2$ should then be interchanged by the functional equation, and (just as we chose $\mathcal{L}_{p,1}^{FM}$ over $\mathcal{L}_{p,2}^{FM}$) we focus on the value at $s = 1$.

Work in progress of Harris, Li, Skinner and Eischen [E-H-L-S] might produce one day a p-adic avatar $L_p(f, s)$ of this L-function. The local component at p of the automorphic representation attached to f is "special". Making an assumption that it is also "split", and being guided by the analogy with the classical case, does this mean that the prospective $L_p(f, s)$ acquires a "trivial zero"? If so, could $\mathcal{L}_p(f)$ be related to the ratio of $L_p'(f, 1)$ to the algebraic part of $L(f, 1)$? We do not dare to make a precise conjecture.

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Amnon Besser, Department of Mathematics, Ben-Gurion University, Be'er-Sheva, Israel

Ehud de Shalit, Einstein Institute of Mathematics, Hebrew University, Jerusalem, Israel