KIRILLOV MODELS AND INTEGRAL STRUCTURES IN p-ADIC SMOOTH REPRESENTATIONS OF $GL_2(F)$

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Abstract. Let F be a local field of residual characteristic p, and ρ a smooth irreducible representation of $GL_2(F)$, realized over the algebraic closure of $\mathbb{Q}_p$. Studying its Kirillov model, we exhibit a necessary and sufficient criterion for the existence of an integral structure in ρ. We apply our criterion to tamely ramified principal series, and get a new proof of a theorem of M.-F. Vigneras.

1. Introduction

1.1. **Background.** Let F be a local field of residual characteristic p, and G the group $GL_2(F)$. Let C be a fixed algebraic closure of the field $\mathbb{Q}_p$ (or its completion). A smooth representation (V,ρ) of G over C is a C-vector space V equipped with a homomorphism $\rho : G \to GL(V)$, for which the stabilizer of every $v \in V$ is open in G.

At least when F has characteristic 0, so can be embeded in C, a good reason to realize smooth representations of G over C rather than over $\mathbb{C}$ is that we can put them in a much larger tensor category that encompasses the finite dimensional rational representations as well. This has been recognized early on by Schneider and Teitelbaum, and became crucial with the the emergence of the (yet largely conjectural) theory of p-adic local Langlands correspondence [B-B-C].

A key natural question that arises in this context is the existence and classification of integral structures. Although the question makes perfect sense for any reductive p-adic group, and for a much larger class of representations (see, for example, the paper by M.Emerton [E]) we shall adhere to the simplest case of smooth representations of $GL_2(F)$. Let $\mathcal{O}_C$ be the ring of integers in C.

Definition 1.1. *An* integral structure *in (V,ρ) is an $\mathcal{O}_C$-submodule V_0 of V, stable under $\rho(G)$, spanning V over C, which contains no C-line. Two integral structures are* commensurable *if each of them is contained in a scalar multiple of the other.*

Remark 1.1. *Any algebraic isomorphism $\iota : \mathbb{C} \simeq C$ carries smooth representations over $\mathbb{C}$ to smooth representations over C, but the existence of integral structures is in general sensitive to the choice of ι, as is already evident for characters of $GL_1(F)$.*

Remark 1.2. *If (V,ρ) is* irreducible *it is enough to assume that the $\mathcal{O}_C[G]$-module V_0 is neither 0 nor the full V. Indeed, CV_0, as well as the union of all the C-lines in V_0, are then subrepresentations of V, so by irreducibility the first must be V, and the second 0. In the irreducible case V admits an integral structure if and only if the $\mathcal{O}_C[G]$-span of any non-zero vector is not the whole space. The $\mathcal{O}_C[G]$-span of any two such vectors are then commensurable, but not every integral structure is necessarily commensurable with such a "minimal" one.*

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The question of existence of integral structures tends to be, in the words of M.-F. Vigneras [Vig], either "obvious" or "very hard". Assume from now on that (V,ρ) is irreducible, and let $\omega_\rho : F^\times \to C^\times$ be its *central character.* We assume that ω_ρ is *unitary,* namely $|\omega_\rho| = 1$, since otherwise V has no chance of admitting an integral structure.

If ρ is supercuspidal, integral structures are abundant. Let v' be any smooth linear functional on V, and consider, for $v \in V$, the matrix coefficient $c_{v,v'}(g) = v'(\rho(g)v)$, which, for ρ supercuspidal, is compactly supported modulo the center. Since the central character is unitary, this is a bounded function on G. The map $v \mapsto c_{v,v'}$ embeds (V,ρ) in the space of bounded smooth functions on G, the group acting by right translation. Pulling back the obvious integral structure induced by the sup norm on functions, we get an integral structure on V.

If V is a twist of the Steinberg representation by a unitary character, an integral structure can be exhibited explicitly.

There remains the case of irreducible principal series, for which integral structures need not exist, in general. A necessary condition for their existence is easily established, but its sufficiency has only been proved by Vigneras [Vig] when the representation is tamely ramified. See Theorem 1.2 below for the precise statement. Only when $F = \mathbb{Q}_p$ the question of sufficiency is completely settled, by round-about methods, as a result of Colmez' proof of the p-adic local Langlands correspondence.

1.2. **Principal series.** Let χ_1, χ_2 be smooth characters of $F^\times$, and B the Borel subgroup of upper triangular matrices in G. The principal series

$$(V,\rho) = Ind_B^G(\chi_1,\chi_2) \tag{1}$$

(smooth induction) is the space of all functions $f : G \to C$ for which (i)

$$f\left(\begin{pmatrix} t_1 & s \\ 0 & t_2 \end{pmatrix} g\right) = \chi_1(t_1)\chi_2(t_2)f(g) \tag{2}$$

and (ii) there exists an open subgroup $H \subset G$, depending on f, such that $f(gh) = f(g)$ for all $h \in H$. The group G acts on V by right translation:

$$(\rho(g)f)(g') = f(g'g). \tag{3}$$

Notice that unlike the classical case we prefer to work with non-normalized induction. This is to avoid the sign ambiguity in the choice of a square root of $\omega(t_1/t_2)$, where $\omega : F^\times \to C^\times$ is the unramified character

$$\omega(a) = q^{-v(a)}. \tag{4}$$

Here, and in the rest of the paper, q is the cardinality of the residue field of F, π is a fixed prime element of $\mathcal{O}_F$, and v, the valuation of F, is normalized by $v(\pi) = 1$. We emphasize that $\omega(a)$ is C-valued, while the p-adic absolute value is real. Properly normalized, the latter takes rational values and coincides with ω, but note that $|\omega(\pi)| > 1$.

The *contragredient* of $Ind_B^G(\chi_1,\chi_2)$ is the representation $Ind_B^G(\chi_1^{-1}\omega, \chi_2^{-1}\omega^{-1})$.

The *twist* of $Ind_B^G(\chi_1,\chi_2)$ by χ is $\chi \circ \det \otimes Ind_B^G(\chi_1,\chi_2) \simeq Ind_B^G(\chi\chi_1, \chi\chi_2)$.

The center of G acts on $Ind_B^G(\chi_1,\chi_2)$ via the character $\chi_1\chi_2$. *We assume that it is unitary.*

The Jacquet module V_N of V is two-dimensional. The torus B/N acts on it via the *exponents* χ_1 and $\chi_2\omega$.

If $\chi_1 = \chi_2$ our ρ is reducible, admitting $\chi_2 \circ \det$ as a one-dimensional subspace, the quotient being isomorphic to $\chi_2 \circ \det \otimes St$. If $\chi_1 = \chi_2\omega^2$ then ρ is again reducible, this time admitting a representation isomorphic to $\chi_2\omega \circ \det \otimes St$ as a subspace, with a 1-dimensional quotient isomorphic to $\chi_2\omega \circ \det$. We exclude these two cases from now on. In all other cases ρ is irreducible.

Among the irreducible principal series we have $Ind_B^G(\chi_1, \chi_2) \simeq Ind_B^G(\chi_2\omega, \chi_1\omega^{-1})$, and no other isomorphisms.

For technical reasons we also exclude the "middle case" in which the two exponents of the Jacquet module of ρ coincide: $\chi_1 = \chi_2\omega$. In this case the Kirillov model of ρ, denoted by $\mathcal{K}$ below, has to be modified, and with it all our arguments.

1.3. **The main result.** Studying the Kirillov model of the principal series representation, we can easily exhibit a function-theoretic criterion for the existence of an integral structure in $Ind_B^G(\chi_1, \chi_2)$. Fix once and for all a non-trivial additive character

$$\psi : F \to C^\times, \tag{5}$$

and write $\psi_b(x) = \psi(bx)$.

Let $C_c^\infty(F)$ be the space of smooth C-valued functions with compact support on F. Let

$$\mathcal{K} = \chi_1 C_c^\infty(F) + \chi_2\omega C_c^\infty(F), \tag{6}$$

viewed as a space of smooth functions on $F^\times$ (if either f_1 or f_2 does not vanish at 0, $\chi_1 f_1 + \chi_2\omega f_2$ does not extend to F). The Kirillov model of ρ is given by an explicit action of G on $\mathcal{K}$ (*cf.* Section 2). Let $F_0' = \chi_1 \cdot 1_{\mathcal{O}_F}$ and $F_0'' = \chi_2\omega \cdot 1_{\mathcal{O}_F}$, where 1_A always denotes the characteristic function of the set A. Let $F_k'(x) = F_0'(\pi^{-k}x)$ and likewise $F_k''(x) = F_0''(\pi^{-k}x)$.[1] Let

$$\Lambda = Span_{\mathcal{O}_C}\{\psi_b F_k', \psi_b F_k'' \mid k \in \mathbb{Z}, b \in F\} \subset \mathcal{K}. \tag{7}$$

Proposition 1.1. *The representation $Ind_B^G(\chi_1, \chi_2)$ admits an integral structure if and only if $\Lambda \neq \mathcal{K}$.*

The $\mathcal{O}_C$-module Λ is by construction stable under B. It is finitely generated as an $\mathcal{O}_C[B]$-module, and spans $\mathcal{K}$ over C. From these facts the proposition follows at once, since G/B is compact[2].

Our main theorem is an application to the case where the characters χ_i are at most tamely ramified. We reprove the results found by Vigneras [Vig] by completely different methods. To our regret, we have not been able, so far, to apply the criterion to wildly ramified principal series.

Theorem 1.2. *Suppose that χ_i are at most tamely ramified, and that $\chi_1\chi_2$ is unitary. Then $Ind_B^G(\chi_1, \chi_2)$ admits an integral structure if and only if*

$$1 \leq |\chi_1(\pi)| \leq |q^{-1}|. \tag{8}$$

The *necessity* of the estimate $1 \leq |\chi_1(\pi)| \leq |q^{-1}|$ is easy, and holds under no ramification restriction. The proof that the same estimate is *sufficient* to guarantee an integral structure is somewhat tricky, and is carried out in the last section, first

[1] No confusion should arise from the use of the letter F to denote both the field and a function.

[2] We are indebted to C.Breuil for pointing this out to us. Previously we have used elaborate computations involving an explicit description of $\rho(w)$ and the Bruhat decomposition to prove the same result.

in the unramified case, and then in the tamely ramified case. It is interesting to note that the proof eventually relies on the well-known properties of Gauss sums. Vigneras' approach makes use of them too.

In [Dat], Dat introduced the notion of a *locally integral structure*. Let H be an open compact subgroup of G. The Hecke algebra $\mathcal{H}(G//H)$ of double cosets of G modulo H, with $\mathbb{Z}$ coefficients, acts on the finite dimensional space V^H. We say that V admits a locally integral structure if for every H there is an $\mathcal{O}_C$-lattice in V^H stable under $\mathcal{H}(G//H)$. Dat proves ([Dat], Prop.3.2) that the above estimate on $\chi_1(\pi)$ is always a necessary and sufficient condition for the existence of a *locally* integral structure. In fact, his proof generalizes to smooth irreducible representations of any p-adic reductive group, where the estimates become estimates on the p-adic valuations of the exponents of the Jacquet module of the representation. It is not known however, even in the case of $GL_2(F)$, if admitting a locally integral structure is equivalent to admitting an integral structure, or is a strictly weaker condition.

Let V_{sm} denote an irreducible smooth representation, and V_{alg} an irreducible algebraic representation of G. The tensor product

$$V = V_{sm} \otimes_C V_{alg} \tag{9}$$

is then irreducible (an observation due to Prasad) and if its central character is unitary, there is an estimate similar to the above which is necessary for the existence of an integral model in V. It is likely that the study of a Kirillov model for V would lead to generalizations of our results to the "locally algebraic" representation V. We hope to pursue this direction in future work.

2. A criterion for the existence of an integral structure

2.1. **Notation.** F- a local filed of residual characteristic p, v the normalized valuation, q the cardinality of the residue field, π a prime element, $\mathcal{O}_F$ its ring of integers, $\mathfrak{p}_F = \pi\mathcal{O}_F$, U_F the units, $U_F^n = 1 + \mathfrak{p}_F^n$ the principal units of level n.

C - a fixed algebraic closure of $\mathbb{Q}_p$, or its completion. The absolute value on C is normalized so that $|p| = 1/p$. All characters and functions are C-valued.

ψ - a non-trivial additive character on F of level 0, i.e. $\mathcal{O}_F$ is its own annihilator under the bilinear pairing $(x, y) \mapsto \psi(xy)$.

ω - the unramified character on $F^\times$ with $\omega(\pi) = q^{-1}$.

dx - C-valued Haar distribution on F, normalized by $\int_{\mathcal{O}_F} dx = 1$. Let $d^*x = \omega(x)^{-1}dx$. Then $d(ax) = \omega(a)dx$, while d^*x is invariant under homotheties.

$C_c^\infty(F)$ - locally constant functions with compact support on F.

$C^\infty(F^\times)$ - locally constant functions on $F^\times$, $C_c^\infty(F^\times)$ the subspace of functions with compact support.

ϕ_k - the characteristic function of $\pi^k U_F$.

1_k - the characteristic function of $\pi^k\mathcal{O}_F$.

Γ - group of smooth characters: $U_F \to C^\times$. If $\xi \in \Gamma$, then we extend it to $F^\times$ by $\xi(\pi) = 1$, so we regard Γ also as the group of smooth characters on $F^\times$ which are trivial on $\langle\pi\rangle$. We let $n(\xi)$ be the conductor of ξ : the smallest n such that ξ is trivial on U_F^n.

If $a \in F^\times$, let $\tilde{a} = a\pi^{-v(a)}$. If $\chi : F^\times \to C^\times$ is a smooth character, we let $\tilde{\chi}(a) = \chi(\tilde{a})$. Then $\tilde{\chi} \in \Gamma$.

$G = GL_2(F)$, B the Borel subgroup of upper triangular matrices, N its unipotent radical.

For $\phi \in C_c^\infty(F)$, define its Fourier transform $\widehat{\phi} \in C_c^\infty(F)$ by

$$\widehat{\phi}(x) = \int_F \psi(xy)\phi(y)dy. \tag{10}$$

2.2. **Gauss sums.** If $\xi : F^\times \to C^\times$ is any smooth character and $n = n(\xi)$, we put

$$\tau(\xi) = \sum_{u \in U_F/U_F^n} \psi(\pi^{-n}u)\xi^{-1}(\pi^{-n}u). \tag{11}$$

This is independent of the choice of π (but does depend, in an obvious way, on the choice of ψ). It is well-known that

$$\tau(\xi)\tau(\omega\xi^{-1}) = \xi(-1). \tag{12}$$

Recall that ϕ_k is the characteristic function of $\pi^k U_F$, and 1_k of $\pi^k \mathcal{O}_F$. A direct computation yields the following.

Proposition 2.1. *Let $\xi \in F^\times \to C^\times$ be a smooth character.*

(i) If ξ is ramified, $n = n(\xi) \geq 1$, then

$$\widehat{\xi^{-1}\phi_k} = \tau(\xi) \cdot \xi\omega^{-1}\phi_{-k-n}. \tag{13}$$

(ii) If ξ is unramified,

$$\widehat{\xi^{-1}\phi_k} = \omega\xi^{-1}(\pi)^k \cdot \left(-\frac{1}{q}\phi_{-k-1} + \frac{q-1}{q} 1_{-k} \right). \tag{14}$$

Corollary 2.2. *Let $1 \neq \xi \in \Gamma$, $n = n(\xi)$. Then*

$$(\xi^{-1}\phi_k)(x) = \frac{\tau(\xi)}{q^n} \cdot \sum_{u \in U_F/U_F^n} \xi(u) \cdot \psi(-\pi^{-k-n}ux)\phi_k(x). \tag{15}$$

In particular, $q^n\xi^{-1}\phi_k$ is an integral linear combination of functions of the form $\psi_b\phi_k$.

2.3. **Kirillov model of the principal series.** Let $(V, \rho) = Ind_B^G(\chi_1, \chi_2)$ be an irreducible principal series representation, as in the introduction. Recall that we have assumed that χ_1 and $\chi_2\omega$ are distinct, and that the central character $\chi_1\chi_2$ is unitary.

Since the question of existence of integral structures is invariant under a unitary twist, we may further assume

- $\chi_1\chi_2(\pi) = 1$ and $\tilde{\chi}_2 = 1$ (χ_2 is unramified).

Write $\tilde{\chi}_1 = \varepsilon$, $\nu = n(\varepsilon)$ and

$$\chi_1(\pi) = \lambda = \chi_2(\pi)^{-1}. \tag{16}$$

Extend ε to $F^\times$ letting $\varepsilon(\pi) = 1$, so that $\varepsilon = \chi_1\chi_2 \in \Gamma$.

The *Kirillov model* of ρ is the space of functions

$$\mathcal{K} = \chi_1 C_c^\infty(F) + \chi_2\omega C_c^\infty(F) \tag{17}$$

([Bump] Theorem 4.7.2(i), watch out for the different normalization). The group G acts as follows (the action will depend on the choice of ψ). The center acts via $\varepsilon = \chi_1\chi_2$. The mirabolic subgroup acts via

$$\rho\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\phi(x) = \psi(bx)\phi(ax). \tag{18}$$

To completely specify the action of G it remains to describe $\rho(w)$, where

$$w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \tag{19}$$

This can be done explicitly, but we shall not need it.

2.4. **A criterion for the existence of an integral structure.** We now prove Proposition 1.1. Let

$$F'_k = \varepsilon\sum_{l=k}^{\infty}\lambda^{l-k}\phi_l,\ \ F''_k = \sum_{l=k}^{\infty}(\frac{1}{q\lambda})^{l-k}\phi_l. \tag{20}$$

Let Λ be the module spanned over $\mathcal{O}_C$ by

$$\psi(bx)F'_k(x) \text{ and } \psi(bx)F''_k(x) \tag{21}$$

for all $b \in F$ and $k \in \mathbb{Z}$. It is clearly stable under B, and finitely generated as an $\mathcal{O}_C[B]$-module, in fact by F'_0 and F''_0. It is also clear that Λ spans $\mathcal{K}$ over C.

Assume that $\mathcal{K}$ admits an integral structure. Then any finitely generated $\mathcal{O}_C[G]$-submodule of $\mathcal{K}$ is distinct from the whole space, and a-fortiori $\Lambda \neq \mathcal{K}$. Conversely, suppose Λ is not the whole space $\mathcal{K}$. Since $G = BK$ with $K = GL_2(\mathcal{O}_F)$ compact, and since the stabilizer in K of any non-zero ϕ is of finite index, there is a constant c such that $\mathcal{O}_C[K]\phi \subset c\Lambda$, hence $\mathcal{O}_C[G]\phi \subset c\Lambda \neq \mathcal{K}$, showing that $\mathcal{K}$ admits an integral structure.

3. A study of the module Λ

In this section we prove Theorem 1.2. We have seen that (ρ, V) admits an integral structure if and only if the module

$$\Lambda = Span_{\mathcal{O}_C}\{\psi_b F'_k,\ \psi_b F''_k\} \tag{22}$$

does *not* coincide with the whole of $\mathcal{K}$. Throughout the rest of this work we put $\mu = (q\lambda)^{-1}$, so that

$$F'_k = \varepsilon\sum_{l=k}^{\infty}\lambda^{l-k}\phi_l,\ \ F''_k = \sum_{l=k}^{\infty}\mu^{l-k}\phi_l. \tag{23}$$

3.1. **Necessity of the condition $1 \le |\lambda| \le |q^{-1}|$.** When $|\lambda| < 1$, $\Lambda = \mathcal{K}$. Indeed, for every x

$$\phi_0(x) = \frac{q-1}{q}\int_{\mathcal{O}_F}\psi(xy)dy - \frac{1}{q}\int_{\pi^{-1}U_F}\psi(xy)dy \tag{24}$$

so if $v(x) \ge -N$ $(N \ge 0)$

$$\phi_0(x) = \frac{q-1}{q^{N+1}}\sum_{y\in\mathcal{O}_F/\pi^N\mathcal{O}_F}\psi(xy) - \frac{1}{q^{N+1}}\sum_{y\in\pi^{-1}U_F/U_F^{N+1}}\psi(xy). \tag{25}$$

Since $\phi_0 = (q\lambda)^N F''_{-N}\phi_0$ and F''_{-N} is supported on $v(x) \geq -N$,

(26)
$$q\lambda^{-N}\phi_0(x) = (q-1)\sum_{y\in\mathcal{O}_F/\pi^N\mathcal{O}_F}\psi(xy)F''_{-N}(x) - \sum_{y\in\pi^{-1}U_F/U_F^{N+1}}\psi(xy)F''_{-N}(x)$$

belongs to Λ. Since N can be arbitrarily large and $|\lambda| < 1$, from the B-invariance of Λ we deduce that Λ contains $C_c^\infty(F^\times)$, hence coincides with $\mathcal{K}$. A similar computation, using F'_{-N} instead of F''_{-N}, works if $1 < |q\lambda|$.

3.2. **The unramified case.** We shall first prove Theorem 1.2 in the case $\varepsilon = 1$. Assume

$$1 \leq |\lambda| \leq |q^{-1}|. \tag{27}$$

Pick a typical function in Λ, which, after a shift by the torus, may be assumed to be of the form

$$\begin{aligned}\phi(x) &= \sum_{k=0}^{\infty}\sum_{\beta\in F/\mathcal{O}_F} c'_k(\beta)\psi_\beta(\pi^{-k}x)F'_k(x) + c''_k(\beta)\psi_\beta(\pi^{-k}x)F''_k(x)\\ &= \sum_{l=0}^{\infty}\sum_{\beta\in F/\mathcal{O}_F} C_l(\beta)\psi_\beta(\pi^{-l}x)\phi_l(x).\end{aligned}\tag{28}$$

Here all the coefficients denoted $c'_k(\beta)$ and $c''_k(\beta)$ are *integral,* and only finitely many do not vanish. By $\psi_\beta(\pi^{-k}x)$ we mean $\psi_b(\pi^{-k}x)$ for any representative b of β. Since $F'_k(x)$ is supported in $\pi^k\mathcal{O}_F$, $\psi_b(\pi^{-k}x)F'_k(x)$ depends only on $b \bmod \mathcal{O}_F$, and similarly for the double-primed terms.

In the second expression we have collected the coefficients "by annuli", so we put

$$C_l(\beta) = C'_l(\beta) + C''_l\beta) \tag{29}$$

$$C'_l(\beta) = \sum_{k=0}^{l}\lambda^{l-k}\sum_{\alpha\in F/\mathcal{O}_F,\ \pi^{l-k}\alpha=\beta} c'_k(\alpha) \tag{30}$$

and similarly

$$C''_l(\beta) = \sum_{k=0}^{l}\mu^{l-k}\sum_{\alpha\in F/\mathcal{O}_F,\ \pi^{l-k}\alpha=\beta} c''_k(\alpha).$$

We shall show that if ϕ vanishes on $\pi^l U_F$ for $0 \leq l \leq l_0$ then $|\phi(x)| \leq |q^{-1}|$ for $x \in \pi^{l_0+1}U_F$. This implies that $q^{-N}\phi_m \notin \Lambda$ if $N \geq 2$ and $m \in \mathbb{Z}$, and suffices to conclude the proof. Indeed, had $q^{-N}\phi_m$ been a member of Λ, for a large enough m', $\phi = q^{-N}\phi_{m+m'}$ would be of the above shape, violating the conclusion with $l_0 = m + m' - 1$.

First note that $\phi|_{\pi^l U_F} = 0$ if and only if $C_l(\beta)$, a-priori defined for $\beta \in F/\mathcal{O}_F$, depend only on $\beta \bmod \pi^{-1}\mathcal{O}_F$. Then consider the recursive relation

$$\begin{aligned}C'_0(\beta) &= c'_0(\beta)\\ C'_l(\beta) &= \lambda\sum_{\pi\alpha=\beta} C'_{l-1}(\alpha) + c'_l(\beta)\end{aligned}\tag{31}$$

and the corresponding relation for $C''_l(\beta)$ (with λ replaced by μ). Put $c_k(\beta) = c'_k(\beta) + c''_k(\beta)$. We assume

$$(32) \qquad \textit{(Hyp):} \quad C_l(\beta) \text{ depends only on } \beta \bmod \pi^{-1}\mathcal{O}_F \text{ for all } 0 \leq l \leq l_0,$$

and prove by induction on $l \leq l_0$ that $|C_{l+1}(\beta)| \leq |q^{-1}|$. That $|C_0(\beta)|$ and $|C_1(\beta)|$ do not exceed $|q^{-1}|$ is obvious. We now rearrange the recursive relations at level l and $l+1$, letting α, β and γ range as usual over $F/\mathcal{O}_F$. We get

$$(33) \qquad \begin{aligned} C_l(\beta) &= \lambda \sum_{\pi\alpha=\beta} C_{l-1}(\alpha) + (\mu - \lambda) \sum_{\pi\alpha=\beta} C''_{l-1}(\alpha) + c_l(\beta) \\ C_{l+1}(\gamma) &= \lambda \sum_{\pi\beta=\gamma} C_l(\beta) + (\mu - \lambda) \sum_{\pi\beta=\gamma} C''_l(\beta) + c_{l+1}(\gamma). \end{aligned}$$

We now assume that $C_{l-1}(\alpha)$ and $C_l(\beta)$ are smaller than $|q^{-1}|$ in absolute value and note that the first sum in the expression for $C_{l+1}(\gamma)$ consists, in view of *(Hyp)*, of q equal terms, hence is integral, so since $|\lambda| \leq |q^{-1}|$ we may ignore it, as well as the last term $c_{l+1}(\gamma)$. To deal with the middle sum we use the defining recursive relation

$$(34) \qquad \begin{aligned} & (\mu - \lambda) \sum_{\pi\beta=\gamma} C''_l(\beta) \\ = \; & (\mu - \lambda) \sum_{\pi\beta=\gamma} \left(\mu \sum_{\pi\alpha=\beta} C''_{l-1}(\alpha) + c''_l(\beta) \right). \end{aligned}$$

Here again, to get the desired estimate on $C_{l+1}(\gamma)$, we need only focus on the first sum inside the paranthesis, which we express, using (33) as

$$(35) \qquad \begin{aligned} & (\mu - \lambda)\mu \sum_{\pi\beta=\gamma} \sum_{\pi\alpha=\beta} C''_{l-1}(\alpha) \\ = \; & \mu \sum_{\pi\beta=\gamma} \left\{ C_l(\beta) - \lambda \sum_{\pi\alpha=\beta} C_{l-1}(\alpha) - c_l(\beta) \right\} \\ = \; & \mu q C_l(\beta_\gamma) - \mu\lambda q \sum_{\pi\beta=\gamma} C_{l-1}(\alpha_\beta) - \mu \sum_{\pi\beta=\gamma} c_l(\beta). \end{aligned}$$

Here β_γ (resp. α_β) is any β (resp. α) satisfying $\pi\beta = \gamma$ (resp. $\pi\alpha = \beta$). Since $\mu\lambda q = 1$ the induction assumption at levels $l-1$ and l implies the desired estimate at level $l+1$, namely that $|C_{l+1}(\gamma)| \leq |q^{-1}|$.

3.3. **The ramified case.** From now on assume that ε is ramified, namely $\nu \geq 1$, and $1 \leq |\lambda| \leq |q^{-1}|$. Let ϕ be given, as before, by

$$(36) \qquad \begin{aligned} \phi(x) &= \sum_{k=0}^{\infty} \sum_{\beta \in F/\mathcal{O}_F} c'_k(\beta)\psi_\beta(\pi^{-k}x)F'_k(x) + c''_k(\beta)\psi_\beta(\pi^{-k}x)F''_k(x) \\ &= \sum_{l=0}^{\infty} \sum_{\beta \in F/\mathcal{O}_F} C_l(\beta)\psi_\beta(\pi^{-l}x)\phi_l(x), \end{aligned}$$

collecting terms "by annuli". Invoking the Fourier expansion of $\varepsilon(x)\phi_l$ (15) we get the following formula

$$C_l(\beta) = \frac{\tau(\varepsilon^{-1})}{q^\nu} \sum_{u \in U_F/U_F^\nu} \varepsilon^{-1}(u) C_l'(\beta + \pi^{-\nu} u) + C_l''(\beta) \tag{37}$$

where $C_l'(\beta)$ and $C_l''(\beta)$ are defined as in the unramified case, and satisfy the same recursive relations.

3.4. **General facts on operators on functions on $F/\mathcal{O}_F$.** At this point it is helpful to introduce certain operators on finitely supported functions on $F/\mathcal{O}_F$. Let $W = F/\mathcal{O}_F$ and $W_n = \pi^{-n}\mathcal{O}_F/\mathcal{O}_F = \ker(\pi^n|W)$. If $f : W \to C$ has a finite support we define

- The *suspension* of f

$$Sf(\beta) = \sum_{\pi\alpha = \beta} f(\alpha). \tag{38}$$

- The *convolution of f with* ε and ε^{-1}

$$\begin{aligned} Ef(\beta) &= \frac{\tau(\varepsilon^{-1})}{q^\nu} \sum_{u \in U_F/U_F^\nu} \varepsilon^{-1}(u) f(\beta + \pi^{-\nu} u) \\ E'f(\beta) &= \frac{\tau(\varepsilon)}{q^\nu} \sum_{u \in U_F/U_F^\nu} \varepsilon(u) f(\beta + \pi^{-\nu} u). \end{aligned} \tag{39}$$

- The operator Π

$$\Pi f(\beta) = f(\pi\beta). \tag{40}$$

We denote by $\mathcal{C}$ the space of functions on W with finite support. We think of them as Fourier coefficients of functions on $\mathcal{O}_F$, where to f we associate its Fourier transform $\widehat{f} = \sum f(\beta)\psi_\beta$. We decompose

$$\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \tag{41}$$

where

$$\begin{aligned} \mathcal{C}_0 &= \left\{ f | \, \forall \beta, \sum_{t \in W_1} f(\beta + t) = 0 \right\} \\ \mathcal{C}_1 &= \left\{ f | \, \forall \beta \forall t \in W_1, \;\; f(\beta + t) = f(\beta) \right\}. \end{aligned} \tag{42}$$

The significance of this decomposition is the following: $f \in \mathcal{C}_1$ if and only if $\widehat{f}$ is supported on $\pi\mathcal{O}_F$, while $f \in \mathcal{C}_0$ if and only if $\widehat{f}$ is supported on U_F.

Lemma 3.1. *(i) The projection onto $\mathcal{C}_1$ is*

$$P_1 = \frac{1}{q} \Pi S. \tag{43}$$

(ii) The projection onto $\mathcal{C}_0$ is

$$P_0 = EE' = E'E. \tag{44}$$

(iii) $E'EE' = E'$ and $EE'E = E$.

Proof. Part (i) is clear, (ii) follows from the fact that $\widehat{Ef} = \varepsilon\widehat{f}$, while $\widehat{E'f} = \varepsilon^{-1}\widehat{f}$. For (iii) note that $E'f \in \mathcal{C}_0$, hence is invariant under $E'E$. ∎

Corollary 3.2. *We have* $SE = 0$*, and* $\mathcal{C}_0 = \mathrm{Im}(E) = \mathrm{Im}(E') = \ker(S)$*,* $\mathcal{C}_1 = \ker(E) = \ker(E')$*.*

Proof. Indeed, since $\Pi S = q(I - EE')$, we have $\Pi SE = q(E - EE'E) = 0$, but Π is injective so $SE = 0$. Now $\mathcal{C}_0 = \ker(q^{-1}\Pi S) = \ker(S)$ since Π is injective. As $\mathrm{Im}(E) \supset \mathrm{Im}(EE') = \mathcal{C}_0 \supset \mathrm{Im}(EE'E) = \mathrm{Im}(E)$, $\mathcal{C}_0 = \mathrm{Im}(E)$ and similarly it is equal to $\mathrm{Im}(E')$. Clearly $\ker(E)$ is contained in $\ker(E'E) = \mathcal{C}_1$. If $f \in \mathcal{C}_1$ it is invariant under W_1, so

$$Ef(\beta) = \frac{\tau(\varepsilon^{-1})}{q^\nu} \sum_{u \in U_F/U_F^\nu} \varepsilon^{-1}(u) f(\beta + \pi^{-\nu}u) = 0 \tag{45}$$

because the sum over each coset of $U_F^{\nu-1}/U_F^\nu$ vanishes. ∎

For any function $f \in \mathcal{C}$ we let $||f||$ denote its sup norm.

3.5. **Conclusion of the proof in the tamely ramified case.** Assume now that ε is tamely ramified, i.e. $\nu = 1$. To prove that Λ is not the whole of $\mathcal{K}$ we start with a function ϕ as above and define recursively elements of $\mathcal{C}$ by

$$C'_{l+1} = \lambda S C'_l + c'_{l+1}, \quad C''_{l+1} = \mu S C''_l + c''_{l+1} \tag{46}$$

and

$$\begin{aligned} C_l &= EC'_l + C''_l \\ \tilde{C}_l &= C'_l + E'C''_l. \end{aligned} \tag{47}$$

Thus

$$\phi = \sum_{l=0}^{\infty} \sum_{\beta \in F/\mathcal{O}_F} C_l(\beta)\psi_\beta(\pi^{-l}x)\phi_l(x) \tag{48}$$

and, symmetrically,

$$\varepsilon^{-1}\phi = \sum_{l=0}^{\infty} \sum_{\beta \in F/\mathcal{O}_F} \tilde{C}_l(\beta)\psi_\beta(\pi^{-l}x)\phi_l(x). \tag{49}$$

Assume that for $0 \le l \le l_0$, $\phi|_{\pi^l U_F} = 0$, so that C_l and $\tilde{C}_l$ lie in $\mathcal{C}_1$.
We claim that

$$R = \max_{0 \le l \le l_0+1} \{||C'_l||, ||C''_l||\} \le |q^{-1}|. \tag{50}$$

Note that this holds all the way up to $l = l_0 + 1$, although our assumption concerns $l \le l_0$ only.

We shall assume, to the contrary, that $R > |q^{-1}|$ and arrive at a contradiction. Clearly $||C'_0|| = ||C''_0|| = 1$. Let $0 \le l \le l_0$ be the first index such that one of $||C'_{l+1}||$ or $||C''_{l+1}||$ attains the value R. Without loss of generality we assume that $||C'_{l+1}|| = R$ (the argument in the other case being similar). From

$$\begin{aligned} \Pi C'_{l+1} &= \lambda \Pi S C'_l + \Pi c'_{l+1} \\ &= \lambda \Pi S \tilde{C}_l + \Pi c'_{l+1} \quad (\text{since } SE' = 0) \\ &= \lambda q \tilde{C}_l + \Pi c'_{l+1} \quad (\text{since } \tilde{C}_l \in \mathcal{C}_1) \end{aligned} \tag{51}$$

and from the fact that $||\Pi f|| = ||f||$ while $||c'_{l+1}|| \le 1$ we conclude that

$$||C'_{l+1}|| = |\lambda q| \cdot ||\tilde{C}_l||. \tag{52}$$

Note that for this step we only need $||C'_{l+1}|| = R > 1$. As $|\lambda q| \leq 1$, $||\tilde{C}_l|| \geq ||C'_{l+1}|| = R > ||C'_l||$, hence

$$\begin{aligned} ||\tilde{C}_l|| &= ||E'C''_l|| \\ &\leq |\frac{\tau(\varepsilon)}{q}| \cdot ||C''_l|| \text{ (here we use } \nu = 1). \end{aligned} \tag{53}$$

This imples $||C''_l|| > 1$ (as $R > |q^{-1}|$), so by the same argument that lead to (52) we now find

$$||C''_l|| = |\mu q| \cdot ||C_{l-1}||. \tag{54}$$

Taken together with (53),

$$||\tilde{C}_l|| \leq |\tau(\varepsilon)\mu| \cdot ||C_{l-1}||.$$

We conclude that

$$\begin{aligned} ||C_{l-1}|| &\geq |\frac{1}{\tau(\varepsilon)\mu}| \cdot ||\tilde{C}_l|| = |\frac{1}{\tau(\varepsilon)\mu\lambda q}| \cdot ||C'_{l+1}|| \text{ by (52)} \\ &= |\frac{1}{\tau(\varepsilon)}| \cdot ||C'_{l+1}|| \geq R > ||C''_{l-1}|| \qquad (55) \end{aligned}$$

so

$$||C_{l-1}|| = ||EC'_{l-1}|| \leq |\frac{\tau(\varepsilon^{-1})}{q}| \cdot ||C'_{l-1}||. \tag{56}$$

Putting everything together

$$||C'_{l-1}|| \geq |\frac{q}{\tau(\varepsilon^{-1})}| \cdot ||C_{l-1}|| \geq |\frac{q}{\tau(\varepsilon^{-1})\tau(\varepsilon)}| \cdot ||C'_{l+1}|| = ||C'_{l+1}|| = R. \tag{57}$$

This contradicts the choice of the index l. Thus $R \leq |q^{-1}|$.

It now follows that C_{l_0+1} is bounded, in the sup norm, by $|q^{-\nu-1}|$, hence $\phi|_{\pi^{l_0+1}U_F}$ can not be arbitrarily large. In other words, if $\phi|_{\pi^l U_F} = 0$ for $l \leq l_0$, then on the next annulus $\pi^{l_0+1}U_F$, ϕ can not be arbitrarily large. We conclude that $\Lambda \neq \mathcal{K}$ as in the unramified case.

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